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Research Article

General Univalence Criterion Associated with the *n*th Derivative

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For normalized analytic functions $f(z)$ with $f(z) \neq 0$ for $0 < |z| < 1$, we introduce a univalence criterion defined by sharp inequality associated with the nth derivative of $z/f(z)$, where $n \in \{3, 4, 5, \ldots\}$.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the following form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are normalized analytic in the open unit disk $\mathbb{U} := \{z : |z| < 1\}$.

In [1], Aksentev proved that the condition

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| \leq 1 \tag{1.2}$$

or equivalently $\mathrm{Re}(f^2(z)/z^2 f'(z)) \geq 1/2$, for $z \in \mathbb{U}$, is sufficient for $f(z) \in \mathcal{A}$ to be univalent in $\mathbb{U}$. By virtue of the aforementioned result of Aksentev, the class of functions defined by (1.2) was extensively studied by Obradović and Ponnusamy [2, 3], Ozaki and Nunokawa [4], Obradović et al. [5], and others. Afterwards, Nunokawa et al. [6] proved for $f(z) \in \mathcal{A}$ with $f(z) \neq 0$ when $0 < |z| < 1$ that

$$\left|\left(\frac{z}{f(z)}\right)''\right| \leq 1 \tag{1.3}$$

implies $|z^2 f'(z)/f^2(z) - 1| \leq 1$ for $z \in \mathbb{U}$, and hence $f(z)$ is univalent in $\mathbb{U}$. Later, Yang and Liu [7] extended this result for $f(z) \in \mathcal{A}$:

$$\left|\left(\frac{z}{f(z)}\right)''\right| \leq 2 \tag{1.4}$$

with $f(z) \neq 0$ when $0 < |z| < 1$ implies that $f(z)$ is univalent in $\mathbb{U}$ and the bound 2 is best possible for univalence. This result was also given first in the preprint of reports of the Department of Mathematics, University of Helsinki: M. Obradović, S. Ponnusamy, New criteria, and distortion theorems for univalent functions, Preprint 190, June 1998. Later, under the same name, the paper was published in Complex Variables Theory Application (see [3]). Corresponding to the functions defined by (1.4), Yang and Liu in [7] studied a class of analytic univalent functions $f(z)$ satisfying $|(z/f(z))''| \leq \beta(0 < \beta \leq 2)$ and denoted by $S(\beta)$. The class $S(\beta)$ is extensively studied in the recent years (see [2, 3, 8–10]).

In this work, we introduce a univalence criteria defined by the conditions $f(z) \neq 0$ for $0 < |z| < 1$ and

$$\sum_{k=2}^{n-1} \frac{k-1}{k!}|\beta_k| + \frac{n-1}{n!}\left|\frac{d^n}{dz^n}\left(\frac{z}{f(z)}\right)\right| \leq 1 \quad \text{for } |z| < 1, \tag{1.5}$$

where $f(z)$ is normalized analytic in $\mathbb{U}$ and $\beta_k = (d^k/dz^k)(z/f(z))|_{z=0}$, $n \in \{3, 4, \ldots\}$. The sharpness occurs for the Koebe function. Indeed, all functions satisfying the condition (1.5) are univalent in $\mathbb{U}$ and the bound 1 in the inequality is best possible for univalence. Letting $n = 2$ in (1.5) gives the univalence criterion defined by (1.4). Some special cases and examples for functions satisfying (1.5) are given.

2. Sufficient Conditions for Univalence

Let us prove the following theorem.

Theorem 2.1. *Let $f(z) \in \mathcal{A}$ with $f(z) \neq 0$ for $0 < |z| < 1$ and let $g(z) \in \mathcal{A}$ be bounded in $\mathbb{U}$ and satisfy*

$$m = \inf\left\{\left|\frac{g(z_1) - g(z_2)}{z_1 - z_2}\right| : z_1, z_2 \in \mathbb{U}\right\} > 0. \tag{2.1}$$

For any $n \in \{3, 4, \ldots\}$, if

$$\left|\frac{d^n}{dz^n}\left(\frac{z}{f(z)} - \frac{z}{g(z)}\right)\right| \leq K \quad (z \in \mathbb{U}), \tag{2.2}$$

where

$$K = \frac{n!}{n-1}\left(\frac{m}{M^2} - \sum_{k=2}^{n-1}\frac{k-1}{k!}|\alpha_k|\right), \qquad \alpha_k = \frac{\mathrm{d}^k}{\mathrm{d}z^k}\left(\frac{z}{g(z)} - \frac{z}{f(z)}\right)\bigg|_{z=0}, \tag{2.3}$$

and $M = \sup\{|g(z)| : z \in \mathbb{U}\}$, *then* $f(z)$ *is univalent in* $\mathbb{U}$.

Proof. If we put

$$h(z) = \frac{\mathrm{d}^n}{\mathrm{d}z^n}\left(\frac{z}{f(z)} - \frac{z}{g(z)}\right), \tag{2.4}$$

then the function h is analytic in $\mathbb{U}$ and, by integration from 0 to z, we get

$$\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\left(\frac{z}{f(z)} - \frac{z}{g(z)}\right) = \alpha_{n-1} + \int_0^z h(u_1)du_1. \tag{2.5}$$

Integrating both sides of the previous equation $(n-1)$-times from 0 to z gives

$$\frac{z}{f(z)} - \frac{z}{g(z)} = \sum_{k=1}^{n-1}\frac{\alpha_k}{k!}z^k + \int_0^z du_n \int_0^{u_n} du_{n-1} \cdots \int_0^{u_3} du_2 \int_0^{u_2} h(u_1)du_1. \tag{2.6}$$

Thus, we have

$$f(z) = \frac{g(z)}{1 + g(z)\sum_{k=1}^{n-1}(\alpha_k/k!)z^{k-1} + g(z)(\varphi(z)/z)}, \tag{2.7}$$

where

$$\varphi(z) = \int_0^z du_n \int_0^{u_n} du_{n-1} \cdots \int_0^{u_3} du_2 \int_0^{u_2} h(u_1)du_1. \tag{2.8}$$

Next, for $n = 3$, we have

$$z^2\left(\frac{\varphi(z)}{z}\right)' = \int_0^z u\varphi''(u)\mathrm{d}u = \int_0^z u\ \mathrm{d}u \int_0^u h(u_1)\mathrm{d}u_1, \tag{2.9}$$

and for $n = 4$,

$$z^2\left(\frac{\varphi(z)}{z}\right)' = \int_0^z u\varphi''(u)du = \int_0^z u \,\mathrm{d}u \int_0^u \mathrm{d}u_2 \int_0^{u_2} h(u_1)\mathrm{d}u_1. \tag{2.10}$$

In general, for $n \in \{3, 4, \ldots\}$,

$$\begin{aligned}
z^2\left(\frac{\varphi(z)}{z}\right)' &= \int_0^z u\varphi''(u)du \\
&= \int_0^z u\, du \int_0^u du_{n-2} \int_0^{u_{n-2}} du_{n-3} \cdots \int_0^{u_2} h(u_1)du_1 \\
&= \int_0^1 z^2 t\, dt \int_0^{zt} du_{n-2} \int_0^{u_{n-2}} du_{n-3} \cdots \int_0^{u_2} h(u_1)du_1 \quad (\text{by setting } u = zt) \\
&= \int_0^1 z^3 t^2\, dt \int_0^1 ds_1 \int_0^{u_{n-2}} du_{n-3} \cdots \int_0^{u_2} h(u_1)du_1 \quad (\text{by setting } u_{n-2} = zts_1) \\
&= \int_0^1 z^4 t^3\, dt \int_0^1 s_1 ds_1 \int_0^1 ds_2 \cdots \int_0^{u_2} h(u_1)du_1 \quad (\text{by setting } u_{n-3} = zts_1 s_2) \\
&= \int_0^1 z^n t^{n-1}\, dt \int_0^1 s_1^{n-3} ds_1 \int_0^1 s_2^{n-4} ds_2 \cdots \\
&\quad \int_0^1 s_{n-3} ds_{n-3} \int_0^1 h(zts_1 \cdot s_{n-2}) ds_{n-2} \quad (\text{by setting } u_1 = zts_1 s_2 \cdots s_{n-2}),
\end{aligned} \tag{2.11}$$

therefore

$$\left|\left(\frac{\varphi(z)}{z}\right)'\right| \leq \frac{|z|^{n-2}}{n} \cdot \frac{1}{n-2} \cdot \frac{1}{n-3} \cdots \frac{1}{2} \int_0^1 |h(zts_1 s_2 \cdots s_{n-2})|\, ds_{n-2} \leq \frac{n-1}{n!} K, \tag{2.12}$$

and so

$$\left|\frac{\varphi(z_2)}{z_2} - \frac{\varphi(z_1)}{z_1}\right| = \left|\int_{z_1}^{z_2} \left(\frac{\varphi(z)}{z}\right)' dz\right| \leq \frac{n-1}{n!} K |z_2 - z_1| \tag{2.13}$$

for $z_1, z_2 \in \mathbb{U}$ and $z_1 \neq z_2$. If $z_1 \neq z_2$, then $g(z_1) \neq g(z_2)$, and it follows, from (2.7) and (2.13), that

$$\begin{aligned}
&\left|f(z_1)-f(z_2)\right| \\
&= \frac{\left|g(z_1)-g(z_2)+g(z_1)g(z_2)\sum_{k=2}^{n-1}(\alpha_k/k!)\left(z_2^{k-1}-z_1^{k-1}\right)+g(z_1)g(z_2)(\psi(z_2)/z_2-\psi(z_1)/z_1)\right|}{\left|1+g(z_1)\sum_{k=1}^{n-1}(\alpha_k/k!)z_1^{k-1}+g(z_1)(\psi(z_1)/z_1)\right|\left|1+g(z_2)\sum_{k=1}^{n-1}(\alpha_k/k!)z_2^{k-1}+g(z_2)(\psi(z_2)/z_2)\right|} \\
&> \frac{\left|g(z_1)-g(z_2)\right|-M^2|z_1-z_2|\sum_{k=2}^{n-1}(|\alpha_k|/k!)\left|\sum_{t=0}^{k-2} z_1^t z_2^{k-1-t}\right|-((n-1)/n!)KM^2|z_1-z_2|}{\left|1+g(z_1)\sum_{k=1}^{n-1}(\alpha_k/k!)z_1^{k-1}+g(z_1)(\psi(z_1)/z_1)\right|\left|1+g(z_2)\sum_{k=1}^{n-1}(\alpha_k/k!)z_2^{k-1}+g(z_2)(\psi(z_2)/z_2)\right|} \\
&> \frac{\left|g(z_1)-g(z_2)\right|-M^2|z_1-z_2|\sum_{k=2}^{n-1}(|\alpha_k|(k-1)/k!)-((n-1)/n!)KM^2|z_1-z_2|}{\left|1+g(z_1)\sum_{k=1}^{n-1}(\alpha_k/k!)z_1^{k-1}+g(z_1)(\psi(z_1)/z_1)\right|\left|1+g(z_2)\sum_{k=1}^{n-1}(\alpha_k/k!)z_2^{k-1}+g(z_2)(\psi(z_2)/z_2)\right|} \\
&\geq 0.
\end{aligned} \tag{2.14}$$

Hence, $f(z)$ is univalent in $\mathbb{U}$. □

Corollary 2.2. *Let $f(z) \in \mathcal{A}$ with $f(z) \neq 0$ when $0 < |z| < 1$. For any $n \in \{3, 4, \ldots\}$, if*

$$\sum_{k=2}^{n-1} \frac{k-1}{k!}\left|\beta_k\right| + \frac{n-1}{n!}\left|\frac{d^n}{dz^n}\left(\frac{z}{f(z)}\right)\right| \leq 1 \quad (z \in \mathbb{U}), \tag{2.15}$$

where $\beta_k = (d^k/dz^k)(z/f(z))|_{z=0}$, then $f(z)$ is univalent in $\mathbb{U}$. The result is sharp, where equality occurs for the Koebe function $k(z) = z/(1-z)^2$ and also for functions of the following form:

$$f(z) = \frac{z}{1+az+z^2}, \quad (|a| \leq 2), \qquad f_n(z) = \frac{z}{(1 \pm (1/(n-2))z)^{n-1}}. \tag{2.16}$$

Proof. Setting $g(z) = z$ in Theorem 2.1 immediately yields (2.15). To show that the result is sharp for $n \geq 3$, we consider

$$f(z) = \frac{z}{(1+(1/(n-2))z)^{n+\epsilon-1}} \quad (\epsilon > 0). \tag{2.17}$$

A computation shows, for $1 \leq k \leq n-1$, that

$$\frac{d^k}{dz^k}\left(\frac{z}{f(z)}\right) = (n-2)^{-k}(\epsilon+n-1)(\epsilon+n-2)\cdots(\epsilon+n-k)\left(1+\frac{1}{n-2}z\right)^{\epsilon+n-k-1}. \tag{2.18}$$

Letting $\epsilon = 0$ in (2.17) and (2.18) implies, respectively, that $(d^n/dz^n)(z/f(z)) = 0$ and

$$|\beta_k| = \frac{(n-1)!}{(n-k-1)!(n-2)^k}. \tag{2.19}$$

This satisfies the equality in (2.15), because for $x \in \mathbb{R}$ and $n \geq 3$, an application of the binomial theorem gives

$$(1+x)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} x^k, \tag{2.20}$$

and so

$$\begin{aligned}\sum_{k=2}^{n-1}(k-1)\binom{n-1}{k} x^k &= 1 + (n-1)(1+x)^{n-2}x - (1+x)^{n-1} \\ &= 1 + (1+x)^{n-2}[x(n-2) - 1].\end{aligned} \tag{2.21}$$

Choosing $x = 1/(n-2)$ in assertion (2.21) gives the equality. However, for every $\epsilon > 0$, we have

$$f'\left(\frac{n-2}{n-2+\epsilon}\right) = 0. \tag{2.22}$$

Hence f is not univalent in $\mathbb{U}$ and the result is sharp. Moreover it can be easily checked that the equality in (2.15) holds for the given functions and the proof is complete. □

3. Special Cases and Examples

Letting $n = 2$ in inequality (2.15) gives the univalence criterion defined by (1.4), which is due to Yang and Liu [7]. Next, we reduce the result for some values of n by computing the corresponding values of β_k in terms of the coefficients. More precisely, for $n = 3$ and $n = 4$, Corollary 2.2 reduces at once to the following two remarks.

Remark 3.1. Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ with $f(z) \neq 0$ when $0 < |z| < 1$ satisfy

$$\left|\left(\frac{z}{f(z)}\right)'''\right| \leq 3 - 3\left|a_2^2 - a_3\right| \quad (z \in \mathbb{U}). \tag{3.1}$$

Then $f(z)$ is univalent in $\mathbb{U}$. The bound in (3.1) is best possible, where equality occurs for the Koebe function and for functions of the following form:

$$f(z) = \frac{z}{1 + az + z^2} \qquad (|a| \leq 2). \tag{3.2}$$

Proof. The result follows from taking $n = 3$ in Corollary 2.2 and that $|\beta_2| = 2|a_2^2 - a_3|$. □

Remark 3.2. Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ with $f(z) \neq 0$ for $0 < |z| < 1$ satisfy

$$\left|\frac{d^4}{dz^4}\left(\frac{z}{f(z)}\right)\right| \leq 8 - 8\left|a_2^2 - a_3\right| - 16\left|a_4 - 2a_2a_3 + a_2^3\right| \quad (z \in \mathbb{U}). \tag{3.3}$$

Then $f(z)$ is univalent in $\mathbb{U}$. The bound in (3.3) is best possible, where equality occurs for the Koebe function and also for functions of the following form:

$$f(z) = \frac{z}{1 + az + z^2} \quad (|a| \leq 2), \qquad f(z) = \frac{z}{(1 \pm (1/2)z)^3}. \tag{3.4}$$

Proof. The result follows from taking $n = 4$ in Corollary 2.2 and that $|\beta_3| = 6|a_4 - 2a_2a_3 + a_2^3|$, and $|\beta_2| = 2|a_2^2 - a_3|$. □

To understand the behavior of the extremal functions for our criterion (2.15), let us consider, for example, $f(z) = z/(1 - (1/2)z)^3$, which is an extremal function for the case $n = 4$. Figures 1(a) and 1(b) show the images of the unit circle under the functions $f(z)$ and $g(z) = z/(1 - (1/2)z)^{3.05}$, respectively. If we restrict the images around the cusps as shown in Figures 1(c) and 1(d), we see that the image of g is a curve that intersects itself in some purely real point u. This means that there are two different points z_1 and z_2 that lie on the unit circle such that $g(z_1) = g(z_2) = u$. In fact, each purely real point lies inside the closed curve of Figures 1(c) and 1(d) which is an image for two different points in $\mathbb{U}$ having the same modulus but different arguments. However, we cannot find such points for the function f, and this interprets why f is an extremal function for univalence, since the closed curve of Figure 1(d) vanishes whenever the power in the function g approaches to 3 as shown in Figure 1(c).

From Corollary 2.2, we have the following.

Corollary 3.3. *Let*

$$f(z) = \frac{z}{1 + \sum_{k=1}^{\infty} b_k z^k} \in \mathcal{A} \tag{3.5}$$

with $f(z) \neq 0$ *for* $0 < |z| < 1$ *and*

$$\sum_{k=2}^{n}(k-1)|b_k| + (n-1)\sum_{k=n+1}^{\infty}\binom{k}{n}|b_k| \leq 1, \tag{3.6}$$

for some $n \in \{2, 3, \ldots\}$. *Then* $f(z)$ *is univalent in* $\mathbb{U}$.

Proof. In view of (3.5) and by simple computation we have

$$\frac{d^n}{dz^n}\left(\frac{z}{f(z)}\right) = n! b_n + \sum_{k=n+1}^{\infty}\frac{k!}{(k-n)!}b_k z^{k-n}, \tag{3.7}$$

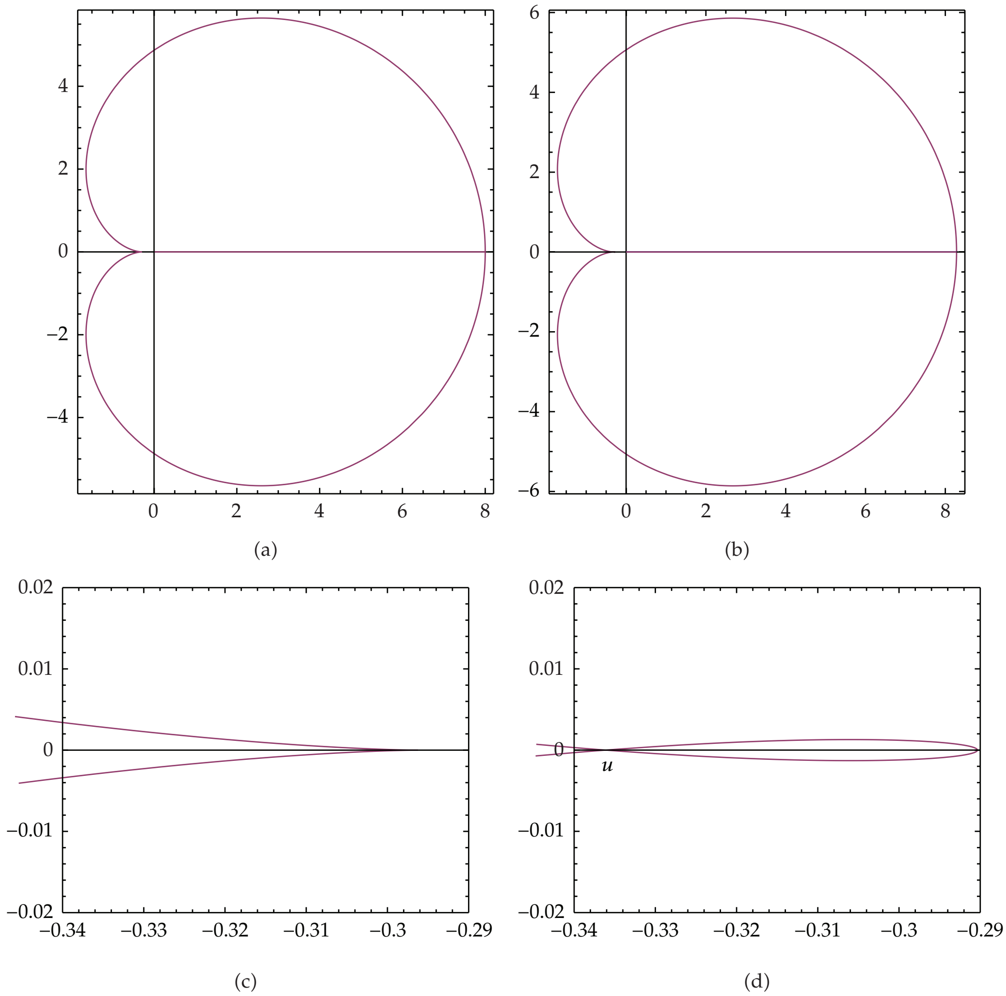

Figure 1: Geometric description for the sharpness of the case $n = 4$.

and so $\beta_m = m! b_m$, for $1 \le m \le n-1$. It follows that

$$\left| \frac{d^n}{dz^n} \left(\frac{z}{f(z)} \right) \right| \le \sum_{k=n}^{\infty} \frac{k! |b_k|}{(k-n)!}. \tag{3.8}$$

Hence, by applying Corollary 2.2, we get the desired result. □

Remark 3.4. Taking $n = 2$ in Corollary 3.3 gives a result of Yang and Liu [7].

Example 3.5. From Corollary 3.3, it can be easily seen that the functions

$$f(z) = \frac{z}{1 + \sum_{k=1}^{n} b_k z^k}, \tag{3.9}$$

with $f(z) \neq 0$ for $0 < |z| < 1$ and $\sum_{k=2}^{n} (k-1) |b_k| \le 1$, are univalent in $\mathbb{U}$.

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