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Research Article

A Justification of Two-Dimensional Nonlinear Viscoelastic Shells Model

Fushan Li

School of Mathematical Sciences, Qufu Normal University, Qufu 273165, China

Correspondence should be addressed to Fushan Li, fushan99@163.com

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By applying formal asymptotic analysis and Laplace transformation, we obtain two-dimensional nonlinear viscoelastic shells model satisfied by the leading term of asymptotic expansion of the solution to the three-dimensional equations.

1. Introduction

In the case of pure nonlinear elasticity, Ciarlet and his collaborators have studied membrane shells, flexural shell and Koiter shell (see [1] and the references therein). The linear viscoelasticity was studied in [2–5], and Li [6–8] studied the global existence and uniqueness of weak solution, uniform rates of decay, and limit behavior of the solution to nonlinear viscoelastic Marguerre-von Kármán shallow shells. Xiao studied the time-dependent nonlinear elastic shells by the method of asymptotic analysis (see [9]).

Motivated by the above work, we deal with nonlinear viscoelastic shells and give the identification of two-dimensional variation problem satisfied by the leading term of of asymptotic expansion of the solution to the three-dimensional equations. The main contributions of this paper are the following: (a) the problem considered in this paper is nonlinear viscoelastic shells, to our knowledge this model has not been considered; (b) applying Laplace transformation, we overcome the difficulties caused by the integral term in the model; (c) the calculation and derivation are precise.

This paper is organized as follows. Section 2 begins with some preliminaries and then gives the main result. In Section 3, we give the proof of the main theorem.

2. Preliminaries and Main Results

We use the following conventions and notations throughout this work: *Greek* indices and exponents (except ε) belong to the set $\{1,2\}$, *Latin* indices and exponents (except when otherwise indicated, as, e.g., when they are used to index sequences) belong to the set $\{1,2,3\}$, and the *summation convention* with respect to the repeated indices and exponents is systematically used. The sign := indicates that the right-hand side defines the left-hand side.

Let $\omega \subset \mathbf{R}^2$ be a bounded connected open set with a Lipschitz boundary γ, let $\mathbf{y} = (y_\alpha)$ denote a generic point in the set $\overline{\omega}$, and let $\partial_\alpha := \partial/\partial y_\alpha$. Let $\boldsymbol{\theta} : \overline{\omega} \to \mathbf{R}^3$ be an injective mapping of C^3 such that the two vectors $\mathbf{a}_\alpha(\mathbf{y}) := \partial_\alpha \boldsymbol{\theta}(\mathbf{y})$ are linear independent at all points $\mathbf{y} \in \overline{\omega}$. They form the covariant basis of the tangent plane to the surface $\mathbf{S} = \boldsymbol{\theta}(\overline{\omega})$ at the point $\boldsymbol{\theta}(\mathbf{y})$; the two vectors $\mathbf{a}^\alpha(\mathbf{y})$ of the same tangent plane defined by the relations $\mathbf{a}^\alpha(\mathbf{y}) \cdot \mathbf{a}_\beta(\mathbf{y}) := \delta^\alpha_\beta$ constitute its contravariant basis. We also define the unit vector $\mathbf{a}_3(\mathbf{y}) = \mathbf{a}^3(\mathbf{y}) = \mathbf{a}_1(\mathbf{y}) \times \mathbf{a}_2(\mathbf{y}) / |\mathbf{a}_1(\mathbf{y}) \times \mathbf{a}_2(\mathbf{y})|$ which is normal to the $\mathbf{S}$ at the point $\boldsymbol{\theta}(\mathbf{y})$.

One then defines the first fundamental form, also known as metric tensor $(a_{\alpha\beta})$ or $(a^{\alpha\beta})$, the second fundamental form, also known as the curvature tensor $(b_{\alpha\beta})$ or (b^β_α), and the Christoffel symbols $\Gamma^\sigma_{\alpha\beta}$ of the surface $\mathbf{S}$ by setting (whenever no confusion should arise, we henceforth drop the explicit dependence on the variable $\mathbf{y} \in \overline{\omega}$)

$$\begin{gathered} a_{\alpha\beta} := \mathbf{a}_\alpha \cdot \mathbf{a}_\beta, \qquad a^{\alpha\beta} := \mathbf{a}^\alpha \cdot \mathbf{a}^\beta, \\ b_{\alpha\beta} := \mathbf{a}^3 \cdot \partial_\beta \mathbf{a}_\alpha, \qquad b^\beta_\alpha := a^{\beta\sigma} b_{\sigma\alpha}, \qquad \Gamma^\sigma_{\alpha\beta} := \mathbf{a}^\sigma \cdot \partial_\beta \mathbf{a}_\alpha. \end{gathered} \tag{2.1}$$

Note the symmetries $a_{\alpha\beta} = a_{\beta\alpha}$, $b_{\alpha\beta} = b_{\beta\alpha}$, and $\Gamma^\sigma_{\alpha\beta} = \Gamma^\sigma_{\beta\alpha}$. The area element along $\mathbf{S}$ is $\sqrt{a}dy$, where $a := \det(a_{\alpha\beta})$. All the functions defined above are at least continuous over the set $\overline{\omega}$. In particular, there exists a constant $a_0 > 0$ such that $a(\mathbf{y}) \geq a_0$, for all $\mathbf{y} \in \overline{\omega}$.

In addition, let the covariant derivatives $b^\sigma_\beta|_\alpha$ and the covariant components $c_{\alpha\beta}$ of the third form of the surface $\mathbf{S}$ be defined by

$$\begin{gathered} b^\sigma_\beta|_\alpha := \partial_\alpha b^\sigma_\beta + \Gamma^\sigma_{\alpha\tau} b^\tau_\beta - \Gamma^\tau_{\alpha\beta} b^\sigma_\tau, \\ c_{\alpha\beta} := b^\sigma_\alpha b_{\sigma\beta}. \end{gathered} \tag{2.2}$$

For each $\varepsilon > 0$, we consider a shell with thickness 2ε and middle surface $\mathbf{S}$, whose lamé relaxation modules $\lambda(t) \geq 0$ and $\mu(t) > 0$ $(t \geq 0)$ are independent of ε. We define the sets

$$\Omega^\varepsilon := \omega \times (-\varepsilon, +\varepsilon), \qquad \Gamma^\varepsilon_+ = \omega \times \{\varepsilon\}, \qquad \Gamma^\varepsilon_- = \omega \times \{-\varepsilon\}, \qquad \Gamma^\varepsilon_0 = \gamma_0 \times [-\varepsilon, \varepsilon], \tag{2.3}$$

where $\gamma_0 \subset \gamma$ and $\gamma_0 \neq \emptyset$. Note that $\Gamma^\varepsilon_+ \cup \Gamma^\varepsilon_- \cup \Gamma^\varepsilon_0$ constitutes a partition of the boundary of the set Ω^ε. Let $\mathbf{x}^\varepsilon = (x^\varepsilon_i)$ denote a generic point in the set $\overline{\Omega}^\varepsilon$, and let $\partial^\varepsilon_i := \partial/\partial x^\varepsilon_i$; hence $x^\varepsilon_\alpha = y_\alpha$ and $\partial^\varepsilon_\alpha = \partial_\alpha$.

We then define a mapping $\boldsymbol{\Theta} : \overline{\Omega}^\varepsilon \to \mathbf{R}^3$ by

$$\boldsymbol{\Theta}(\mathbf{x}^\varepsilon) := \boldsymbol{\theta}(\mathbf{y}) + x^\varepsilon_3 \mathbf{a}_3(\mathbf{y}), \quad \forall \mathbf{x}^\varepsilon = (\mathbf{y}, x^\varepsilon_3) \in \overline{\Omega}^\varepsilon, \tag{2.4}$$

then there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \le \varepsilon_0$ the mapping $\boldsymbol{\Theta} : \overline{\Omega}^{\varepsilon} \to \mathbf{R}^3$ is an injective mapping and the three vectors $\mathbf{g}_i^{\varepsilon}(\mathbf{x}^{\varepsilon}) := \partial_i^{\varepsilon}\boldsymbol{\Theta}(\mathbf{x}^{\varepsilon})(\partial_{\alpha}^{\varepsilon} = \partial/\partial x_{\alpha},\ \partial_3^{\varepsilon} = \partial/\partial x_3^{\varepsilon})$ are linear independent for each $\mathbf{x}^{\varepsilon} \in \overline{\Omega}^{\varepsilon}$. The injectivity of the mapping $\boldsymbol{\Theta} : \overline{\Omega}^{\varepsilon} \to \mathbf{R}^3$ ensures that the physical problem described below is meaningful.

The three vectors $\mathbf{g}_i^{\varepsilon}(\mathbf{x}^{\varepsilon})$ form the covariant basis at the point $\boldsymbol{\Theta}(\mathbf{x}^{\varepsilon})$, and the three vectors $\mathbf{g}^{i,\varepsilon}(\mathbf{x}^{\varepsilon})$ defined by $\mathbf{g}^{j,\varepsilon}(\mathbf{x}^{\varepsilon}) \cdot \mathbf{g}_i^{\varepsilon}(\mathbf{x}^{\varepsilon}) = \delta_i^j$ form the contravariant. We define the metric tensor (g_{ij}^{ε}) or $(g^{ij,\varepsilon})$ and the Christoffel symbols of the manifold $\boldsymbol{\Theta}(\overline{\Omega}^{\varepsilon})$ by setting (we omit the explicit dependence on $\mathbf{x}^{\varepsilon}$)

$$g_{ij}^{\varepsilon} = \mathbf{g}_i^{\varepsilon} \cdot \mathbf{g}_j^{\varepsilon}, \qquad g^{ij,\varepsilon} := \mathbf{g}^{i,\varepsilon} \cdot \mathbf{g}^{j,\varepsilon}, \qquad \Gamma_{ij}^{k,\varepsilon} := \mathbf{g}^{k,\varepsilon} \cdot \partial_i^{\varepsilon}\mathbf{g}_j^{\varepsilon}. \tag{2.5}$$

Note the symmetries

$$g_{ij}^{\varepsilon} = g_{ji}^{\varepsilon}, \qquad g^{ij,\varepsilon} = g^{ji,\varepsilon}, \qquad \Gamma_{ij}^{k,\varepsilon} = \Gamma_{ji}^{k,\varepsilon}. \tag{2.6}$$

The volume element in the set $\boldsymbol{\Theta}(\overline{\Omega}^{\varepsilon})$ is $\sqrt{g^{\varepsilon}}dx^{\varepsilon}$, where $g^{\varepsilon} := \det(g_{ij}^{\varepsilon})$.

For each $0 < \varepsilon \le \varepsilon_0$, the set $\overline{\widehat{\Omega}^{\varepsilon}} := \boldsymbol{\Theta}(\overline{\Omega}^{\varepsilon})$ is the reference configuration of a viscoelastic shell with middle surface $\mathbf{S} = \boldsymbol{\theta}(\overline{\omega})$ and thickness 2ε. We assume that the material constituting the shell is homogeneous isotropic and $\boldsymbol{\Theta}(\overline{\Omega}^{\varepsilon})$ is of a nature state, so that the material is characterized by its two lamé relaxation modules $\lambda(t) \ge 0$ and $\mu(t) > 0$ $(t \ge 0)$. Under the action of forces, the shell undergoes a displacement field.

Let $\widehat{\mathbf{u}}^{\varepsilon}(t) = u_i^{\varepsilon}(t)\mathbf{g}^{i,\varepsilon}$ in terms of curvilinear coordinates $\mathbf{x}^{\varepsilon}$ of the reference configuration $\boldsymbol{\Theta}(\overline{\Omega}^{\varepsilon})$. Then, the covariant displacement field $\mathbf{u}^{\varepsilon}(t) = (u_i^{\varepsilon})(t)$ satisfies the following three-dimensional equations (c.f. [1, 10]):

$$\begin{aligned}
&\mathbf{u}^{\varepsilon}(t) \in \mathbf{L}_c^{\infty}(-\infty, T; \mathbf{W}(\Omega^{\varepsilon})) \quad \text{with } \mathbf{W}(\Omega^{\varepsilon}) := \left\{\mathbf{v}^{\varepsilon} \in \mathbf{W}^{1,4}(\Omega^{\varepsilon}),\ \mathbf{v}^{\varepsilon} = \mathbf{0} \text{ on } \Gamma_0^{\varepsilon}\right\},\\
&\int_{\Omega^{\varepsilon}} u_{itt}^{\varepsilon}(t)v_i^{\varepsilon}g^{ij,\varepsilon}\sqrt{g^{\varepsilon}}dx^{\varepsilon} + \int_{\Omega^{\varepsilon}} A^{ijkl,\varepsilon}(0)E_{k\|l}^{\varepsilon}(\mathbf{u}^{\varepsilon}(t))F_{i\|j}^{\varepsilon}(\mathbf{u}^{\varepsilon}(t), \mathbf{v}^{\varepsilon})\sqrt{g^{\varepsilon}}dx^{\varepsilon}\\
&\quad + \int_{\Omega^{\varepsilon}}\int_{-\infty}^{t} A'^{ijkl,\varepsilon}(t-\tau)E_{k\|l}^{\varepsilon}(\mathbf{u}^{\varepsilon}(\tau))F_{i\|j}^{\varepsilon}(\mathbf{u}^{\varepsilon}(\tau), \mathbf{v}^{\varepsilon})\sqrt{g^{\varepsilon}}d\tau\, dx^{\varepsilon}\\
&\quad = \int_{\Omega^{\varepsilon}} f^{i,\varepsilon}(t)v_i^{\varepsilon}\sqrt{g^{\varepsilon}}dx^{\varepsilon} + \int_{\Gamma_+^{\varepsilon}\cup\Gamma_-^{\varepsilon}} h^{i,\varepsilon}(t)v_i^{\varepsilon}\sqrt{g^{\varepsilon}}d\Gamma^{\varepsilon}, \quad \forall \mathbf{v}^{\varepsilon} \in \mathbf{W}(\Omega^{\varepsilon}),
\end{aligned} \tag{2.7}$$

where the symbol L_c^{∞} denotes the subspace of $L^{\infty} > 0$ such that there exists a constant T such that the functions vanish as $s < -T$. And,

$$A^{ijkl,\varepsilon}(t) := \lambda(t)g^{ij,\varepsilon}g^{kl,\varepsilon} + \mu(t)\left(g^{ik,\varepsilon}g^{jl,\varepsilon} + g^{il,\varepsilon}g^{jk,\varepsilon}\right) \tag{2.8}$$

designate the contravariant components of the three-dimensional elasticity tensor,

$$E_{i\|j}^{\varepsilon}(v^{\varepsilon}) := \frac{1}{2}\left(v_{i\|j}^{\varepsilon} + v_{j\|i}^{\varepsilon} + g^{mn,\varepsilon}v_{m\|i}^{\varepsilon}v_{n\|j}^{\varepsilon}\right), \quad \text{where } v_{i\|j}^{\varepsilon} := \partial_j^{\varepsilon}v_i^{\varepsilon} - \Gamma_{ij}^{p,\varepsilon}v_p^{\varepsilon}, \tag{2.9}$$

designate the strains in the curvilinear coordinates associated with an arbitrary displacement field $v_i^\varepsilon g^{i,\varepsilon}$ of the manifold $\mathbf{\Theta}(\overline{\Omega})$,

$$F_{i\|j}^{\varepsilon}(u^{\varepsilon}(t),v^{\varepsilon}) := \frac{1}{2}\Big(v_{i\|j}^{\varepsilon} + v_{j\|i}^{\varepsilon} + g^{mn,\varepsilon}\Big\{u_{m\|i}^{\varepsilon}(t)v_{n\|j}^{\varepsilon} + u_{n\|j}^{\varepsilon}(t)v_{m\|i}^{\varepsilon}\Big\}\Big), \tag{2.10}$$

and, finally, $f^{i,\varepsilon} \in L^{\infty}(0,T;L^2(\Omega^{\varepsilon}))$ and $h^{i,\varepsilon} \in L^{\infty}(0,T;(\Gamma_+^{\varepsilon} \cup \Gamma_-^{\varepsilon}))$ denote the contravariant components of the applied body and surface force densities, respectively, applied to the interior $\mathbf{\Theta}(\Omega^{\varepsilon})$ of the shell and to its "uper" and "lower" faces $\mathbf{\Theta}(\Gamma_+^{\varepsilon})$ and $\mathbf{\Theta}(\Gamma_-^{\varepsilon})$, and designate the area element along $\partial\Omega^{\varepsilon}$. We thus assume that there are no surface forces applied to the portion $\mathbf{\Theta}((\gamma-\gamma_0)\times[-\varepsilon,\varepsilon])$ of the lateral face of the shell.

We record in passing the symmetries

$$A^{ijkl,\varepsilon} = A^{jikl,\varepsilon} = A^{klij,\varepsilon} \tag{2.11}$$

and the relation

$$\Gamma_{\alpha 3}^{3,\varepsilon} = \Gamma_{33}^{p,\varepsilon} = 0, \quad A^{\alpha\beta\sigma 3,\varepsilon} = A^{\alpha 333,\varepsilon} = 0 \quad \text{in } \overline{\Omega}^{\varepsilon}. \tag{2.12}$$

Our final objective consists in showing, by means of the method of formal asymptotic expansions that, if the data are of an appropriate order with respect to ε as $\varepsilon \to 0$, the above three-dimensional problems are "asymptotically equivalent" to a "two-dimensional problem posed over the middle surface of the shell." This means that the new unknown should be $\boldsymbol{\zeta}^{\varepsilon}(t) = (\zeta_i^{\varepsilon}(t))$, where $\zeta_i^{\varepsilon}(t)$ are the covariant components of the displacement $\zeta_i^{\varepsilon}(t)\mathbf{a}_i(\mathbf{y}) : \overline{\omega} \to \mathbf{R}^3$ of the middle surface $S = \boldsymbol{\theta}(\overline{\omega})$. In other words, $\zeta_i^{\varepsilon}(t,y)\mathbf{a}_i(\mathbf{y})$ is the displacement of the point $\boldsymbol{\theta}(y) \in S$.

"Asymptotic analysis" means that our objective is to study the behavior of the displacement field $u_i^{\varepsilon}(t)\mathbf{g}^{i,\varepsilon} : \overline{\Omega}^{\varepsilon} \to \mathbf{R}^3$ as $\varepsilon \to 0$, an endeavour that will be a behavior as $\varepsilon \to 0$ of the covariant components $u_i^{\varepsilon}(t) : \overline{\Omega}^{\varepsilon} \to \mathbf{R}$ of the displacement field, that is, the behavior of the unknown $\mathbf{u}^{\varepsilon}(t) = (u_i^{\varepsilon}(t)) : \overline{\Omega}^{\varepsilon} \to \mathbf{R}^3$ of the three-dimensional shell problem.

Since these fields are defined on sets $\overline{\Omega}^{\varepsilon}$ that themselves vary with ε, our first task naturally consists in transforming the three-dimensional problems into problems posed over a set that does not depend on ε.

Furthermore, we transform problem (2.7) into an equivalent problem independent of ε, posed over the domain.

Let $\Omega := \omega \times (-1,+1)$, $\Gamma_0 = \gamma_0 \times [-1,1]$, $\Gamma_+ := \omega \times \{+1\}$, and $\Gamma_- := \omega \times \{-1\}$, and let $\mathbf{x} = (x_i)$ denote a generic point in $\overline{\Omega}$. With each point $\mathbf{x} \in \overline{\Omega}$, we associate the point $\mathbf{x}^{\varepsilon}$ through the bijection $\pi^{\varepsilon} : \mathbf{x} = (x_1,x_2,x_3) \in \overline{\Omega} \to \mathbf{x}^{\varepsilon} = (x_i^{\varepsilon}) = (x_1,x_2,\varepsilon x_3) \in \overline{\Omega}^{\varepsilon}$; we thus have $\partial_{\alpha}^{\varepsilon} = \partial_{\alpha}$ and $\partial_3^{\varepsilon} = (1/\varepsilon)\partial_3$. Let $\mathbf{u}^{\varepsilon} = (u_i^{\varepsilon})$, $\Gamma_{ij}^{\varepsilon}, g^{\varepsilon}, A^{ijkl,\varepsilon} : \overline{\Omega}^{\varepsilon} \to \mathbf{R}$ and the vector fields $\mathbf{v}^{\varepsilon} = (v_i^{\varepsilon})$ appearing in the three-dimensional problem (2.7) be associated with the functions $\Gamma_{ij}^{p}(\varepsilon), g(\varepsilon), A^{ijkl}(\varepsilon) : \overline{\Omega} \to \mathbf{R}$ and the scaled vector fields $\mathbf{v} = (v_i)$ defined by

$$\begin{gathered} u_i(\varepsilon)(t,\mathbf{x}) = u_i^{\varepsilon}(t,\mathbf{x}^{\varepsilon}), \quad v_i(\mathbf{x}) = v_i^{\varepsilon}(\mathbf{x}^{\varepsilon}) \quad \forall \mathbf{x}^{\varepsilon} \in \overline{\Omega}^{\varepsilon}, \\ \Gamma_{ij}^{p}(\varepsilon)(\mathbf{x}) = \Gamma_{ij}^{p,\varepsilon}(\mathbf{x}^{\varepsilon}), \quad g(\varepsilon)(\mathbf{x}) = g^{\varepsilon}(\mathbf{x}^{\varepsilon}), \quad A^{ijkl}(\varepsilon)(\mathbf{x}) = A^{ijkl,\varepsilon}(\mathbf{x}^{\varepsilon}) \quad \forall \mathbf{x}^{\varepsilon} \in \Omega^{\varepsilon}. \end{gathered} \tag{2.13}$$

Functions $f^i(\varepsilon)(t) : \Omega \to \mathbf{R}$ and $h^i(t)(\varepsilon) : \Gamma_+ \cup \Gamma_- \to \mathbf{R}$ are defined by setting

$$\begin{aligned} f^i(\varepsilon)(t,\mathbf{x}) &= f^{i,\varepsilon}(t,\mathbf{x}^\varepsilon) \quad \forall \mathbf{x}^\varepsilon \in \Omega^\varepsilon, \\ h^i(\varepsilon)(t,\mathbf{x}) &= h^{i,\varepsilon}(t,\mathbf{x}^\varepsilon) \quad \forall \mathbf{x}^\varepsilon \in \Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon. \end{aligned} \tag{2.14}$$

Then the scaled unknown $\mathbf{u}(\varepsilon)(t)$ defined above satisfies (c.f. [1])

$$\begin{aligned} &\mathbf{u}(\varepsilon)(t) \in \mathbf{L}_c^\infty(-\infty,T;\mathbf{W}(\Omega)) \quad \text{with } \mathbf{W}(\Omega) := \left\{\mathbf{v} \in \mathbf{W}^{1,4}(\Omega),\ \mathbf{v} = \mathbf{0} \ \text{ on } \ \Gamma_0\right\}, \\ &\int_\Omega u_{itt}(\varepsilon)(t) v_i \sqrt{g(\varepsilon)}dx + \int_\Omega A^{ijkl}(\varepsilon)(0) E_{k\|l}(\varepsilon;\mathbf{u}(\varepsilon)(t)) F_{i\|j}(\varepsilon;\mathbf{u}(\varepsilon)(t),\mathbf{v})\sqrt{g(\varepsilon)}dx \\ &\quad + \int_\Omega \int_{-\infty}^t A'^{ijkl}(\varepsilon)(t-\tau) E_{k\|l}(\varepsilon;\mathbf{u}(\varepsilon)(\tau)) F_{i\|j}(\varepsilon;\mathbf{u}(\varepsilon)(\tau),\mathbf{v})\sqrt{g(\varepsilon)}d\tau\, dx \\ &\quad = \int_\Omega f^i(\varepsilon)(t) v_i \sqrt{g(\varepsilon)}dx + \frac{1}{\varepsilon}\int_{\Gamma_+\cup\Gamma_-} h^i(\varepsilon)(t) v_i \sqrt{g(\varepsilon)}d\Gamma, \quad \forall \mathbf{v} \in \mathbf{W}(\Omega), \end{aligned} \tag{2.15}$$

where

$$\begin{aligned} E_{i\|j}(\varepsilon;\mathbf{u}(\varepsilon)(t)) &:= \frac{1}{2}\left(u_{i\|j}(\varepsilon)(t) + u_{j\|i}(\varepsilon)(t) + g^{mn}(\varepsilon) u_{m\|i}(\varepsilon)(t) u_{n\|j}(\varepsilon)(t)\right), \\ F_{i\|j}(\varepsilon;\mathbf{u}(\varepsilon)(t),\mathbf{v}) &:= \frac{1}{2}\left(v_{i\|j}(\varepsilon) + v_{j\|i}(\varepsilon) + g^{mn}(\varepsilon)\left\{u_{m\|i}(\varepsilon)(t) v_{n\|j}(\varepsilon) + u_{n\|j}(\varepsilon)(t) v_{m\|i}(\varepsilon)\right\}\right), \\ u_{\beta\|\alpha}(\varepsilon)(t) &:= \partial_\alpha u_\beta(\varepsilon)(t) - \Gamma^p_{\alpha\beta}(\varepsilon) u_p(\varepsilon)(t), \qquad v_{\beta\|\alpha}(\varepsilon) = \partial_\alpha v_\beta - \Gamma^p_{\alpha\beta}(\varepsilon) v_p, \\ u_{3\|\alpha}(\varepsilon)(t) &:= \partial_\alpha u_3(\varepsilon)(t) - \Gamma^\sigma_{\alpha 3}(\varepsilon)(t) u_\sigma(\varepsilon)(t), \qquad v_{3\|\alpha}(\varepsilon) := \partial_\alpha v_3 - \Gamma^\sigma_{\alpha 3}(\varepsilon) v_\sigma, \\ u_{\alpha\|3}(\varepsilon)(t) &:= \frac{1}{\varepsilon}\partial_3 u_\alpha(\varepsilon)(t) - \Gamma^\sigma_{\alpha 3}(\varepsilon) u_\sigma(\varepsilon)(t), \qquad v_{\alpha\|3}(\varepsilon) := \frac{1}{\varepsilon}\partial_3 v_\alpha - \Gamma^\sigma_{\alpha 3}(\varepsilon) v_\sigma, \\ u_{3\|3}(\varepsilon)(t) &:= \frac{1}{\varepsilon}\partial_3 u_3(\varepsilon)(t), \qquad v_{3\|3}(\varepsilon) := \frac{1}{\varepsilon}\partial_3 v_3. \end{aligned} \tag{2.16}$$

The functions $A^{ijkl}(\varepsilon)$ are called the contravariant components of the scaled three-dimensional elasticity tensor of the shell. The functions $E_{i\|j}(\varepsilon;\mathbf{u}(\varepsilon)(t))$ are called the scaled strains in the curvilinear coordinates because they satisfy

$$E_{i\|j}(\varepsilon;\mathbf{u}(\varepsilon)(t))(x) = E^\varepsilon_{i\|j}(\mathbf{u}^\varepsilon(t))(x^\varepsilon) \quad \forall x^\varepsilon \in \overline{\Omega}^\varepsilon. \tag{2.17}$$

Note that the above definitions likewise imply that

$$\begin{aligned} F_{i\|j}(\varepsilon;\mathbf{u}(\varepsilon)(t),\mathbf{v})(x) &= F^\varepsilon_{i\|j}(\mathbf{u}^\varepsilon(t),\mathbf{v}^\varepsilon)(x^\varepsilon), \qquad u_{i\|j}(\varepsilon)(t)(x) = u^\varepsilon_{i\|j}(t)(x^\varepsilon), \\ v_{i\|j}(\varepsilon)(x) &= v^\varepsilon_{i\|j}(x^\varepsilon) \quad \forall x^\varepsilon \in \overline{\Omega}^\varepsilon. \end{aligned} \tag{2.18}$$

For notational brevity, the point $\mathbf{x}$ of some functions is suppressed where no confusion can arise.

The following two requirements constantly guide the procedures of the formal asymptotic analysis. The first requirement asserts that no restriction should be imposed on the applied forces entering the right-hand side of the equations used for determining the leading term. The second requirement asserts that, by retaining only the linear terms in any relation satisfied by terms of arbitrary order in the formal asymptotic expansion of the scaled unknown $\mathbf{u}(\varepsilon)(t)$, a relation of linear theory should be recovered. For brevity, we will call it "linearization trick" (see [1]).

Theorem 2.1. *Assume that the scaled unknown $\mathbf{u}(\varepsilon)$ satisfying problem (2.15) admits a formal asymptotic expansion of the form*

$$\mathbf{u}(\varepsilon)(t) = \mathbf{u}^0(t) + \varepsilon\mathbf{u}^1(t) + \varepsilon^2\mathbf{u}^2(t) + \cdots \tag{2.19}$$

with $\mathbf{u}^0(t) \in L^\infty(0,T;\mathbf{W}(\Omega))$ and $\mathbf{u}^1(t), \mathbf{u}^2(t) \in L^\infty(0,T;\mathbf{W}^{1,4}(\Omega))$. Then in order that no restriction be put on the applied forces and that the linearization be satisfied, the components of the applied forces must be of the form

$$f^{i,\varepsilon}(t,x^\varepsilon) = f^{i,0}(t,x), \quad x^\varepsilon \in \Omega^\varepsilon, \qquad h^{i,\varepsilon}(t,x^\varepsilon) = \varepsilon h^{i,1}(t,x), \quad x^\varepsilon \in \Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon, \tag{2.20}$$

where the functions $f^{i,0} \in L^\infty(0,T;L^2(\Omega))$ and $h^{i,1} \in Ł^\infty(0,T;L^2(\Gamma_+ \cup \Gamma_-))$ are independent of ε.

This being the case, the leading term $\mathbf{u}^0(t)$ is independent of the transverse variable x_3 and $\boldsymbol{\zeta}^0(t) = (1/2)\int_{-1}^{1}\mathbf{u}^0(t)dx_3$ satisfies the following two-dimensional variation problem:

$$\begin{aligned}
&\boldsymbol{\zeta}^0(t) \in L_c^\infty((-\infty,T];\mathbf{W}(\omega)) \quad \text{with } \mathbf{W}(\omega) := \left\{\boldsymbol{\eta} \in \mathbf{W}^{1,4}(\omega);\ \boldsymbol{\eta} = 0 \text{ on } \gamma_0\right\},\\
&\int_\omega \zeta_{itt}^0(t)\eta_j a^{ij}\sqrt{a}dy + \int_\omega a^{\alpha\beta\sigma\tau}(0)E^0_{\sigma\|\tau}(t)F^0_{\alpha\|\beta}(t,\eta)\sqrt{a}dy\\
&\quad + \int_\omega\int_{-\infty}^{t} a^{'\alpha\beta\sigma\tau}(t-\tau)E^0_{\sigma\|\tau}(\tau)F^0_{\alpha\|\beta}(\tau,\eta)\sqrt{a}dy\,d\tau\\
&= \int_\omega p^{i,0}(t)\eta_i\sqrt{a}dy, \quad \forall\boldsymbol{\eta} \in \mathbf{W}(\omega),\ \text{a.e.} -\infty < t \le T,
\end{aligned} \tag{2.21}$$

where (recall that $a^{mn} = \mathbf{a}^m \cdot \mathbf{a}^n$):

$$\begin{aligned}
E^0_{\alpha\|\beta}(t) &:= \frac{1}{2}\left(\zeta^0_{\alpha\|\beta}(t) + \zeta^0_{\beta\|\alpha}(t) + a^{mn}\zeta^0_{m\|\alpha}(t)\zeta^0_{n\|\beta}(t)\right),\\
F^0_{\alpha\|\beta}(t,\boldsymbol{\eta}) &:= \frac{1}{2}\left(\eta_{\alpha\|\beta} + \eta_{\beta\|\alpha} + a^{mn}\left\{\zeta^0_{m\|\alpha}(t)\eta_{n\|\beta} + \zeta^0_{n\|\beta}(t)\eta_{m\|\alpha}\right\}\right),\\
\eta_{\alpha\|\beta} &= \partial_\beta\eta_\alpha - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - b_{\alpha\beta}\eta_3, \qquad \eta_{3\|\beta} := \partial_\beta\eta_3 + b^\sigma_\beta\eta_\sigma,\\
a^{\alpha\beta\sigma\tau}(t) &:= a(t)a^{\alpha\beta}a^{\sigma\tau} + \mu(t)\left(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}\right),\\
a(t) &:= L^{-1}\left\{\frac{2\widehat{\lambda}(s)\widehat{\mu}(s)}{\widehat{\lambda}(s) + 2\widehat{\mu}(s)}\right\} \quad \text{with } a(0) = \frac{2\lambda(0)\mu(0)}{\lambda(0) + 2\mu(0)},\\
p^{i,0}(t) &:= \frac{1}{2}\left(\int_{-1}^{+1} f^{i,0}(t)dx_3 + h^{i,1}(\cdot,1) + h^{i,1}(\cdot,-1)\right),
\end{aligned} \tag{2.22}$$

$\widehat{\lambda}(s)$, $\widehat{\mu}(s)$ denote Laplace transformation of $\lambda(t)$, $\mu(t)$, respectively, and L^{-1} denotes the inverse Laplace transformation.

Lemma 2.2. *For small $\varepsilon > 0$, it is not difficult to verify the following relations:*

$$g^{ij}(\varepsilon) = a^{ij} + \varepsilon x_3 g^{ij,1} + O\left(\varepsilon^2\right), \tag{2.23}$$

where

$$\begin{gathered} a^{ij} := \mathbf{a}_i \cdot \mathbf{a}_j, \qquad g^{\alpha\beta,1} := 2a^{\alpha\sigma} b_\sigma^\beta, \qquad g^{i3,1} = 0, \\ \Gamma_{ij}^p(\varepsilon) = \Gamma_{ij}^{p,0} + \varepsilon x_3 \Gamma_{ij}^{p,1} + O\left(\varepsilon^2\right), \end{gathered} \tag{2.24}$$

where

$$\begin{gathered} \Gamma_{\alpha\beta}^{\sigma,0} := \Gamma_{\alpha\beta}^\sigma, \qquad \Gamma_{\alpha\beta}^{3,0} := b_{\alpha\beta}, \qquad \Gamma_{\alpha 3}^{\beta,0} := -b_\alpha^\beta, \qquad \Gamma_{\alpha 3}^{3,0} = \Gamma_{33}^{p,0} := 0, \\ \Gamma_{\alpha\beta}^{\sigma,1} := -b_\beta^\sigma|_\alpha, \qquad \Gamma_{\alpha\beta}^{3,1} := -b_\alpha^\sigma b_{\sigma\beta}, \qquad \Gamma_{\alpha 3}^{\sigma,1} := -b_\alpha^\tau b_\tau^\sigma, \qquad \Gamma_{\alpha 3}^{3,1} = \Gamma_{33}^{p,1} := 0, \\ A^{ijkl}(\varepsilon)(t)\sqrt{g(\varepsilon)} = A^{ijkl}(t)\sqrt{a} + \varepsilon B^{ijkl,1}(t) + \varepsilon^2 B^{ijkl,2}(t) + o\left(\varepsilon^2\right), \end{gathered} \tag{2.25}$$

where

$$\begin{gathered} A^{\alpha\beta\sigma\tau}(t) := \lambda(t) a^{\alpha\beta} a^{\sigma\tau} + \mu(t)\left(a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma}\right), \\ A^{\alpha\beta 33}(t) := \lambda(t) a^{\alpha\beta}, \qquad A^{\alpha 3\sigma 3}(t) := \mu(t) a^{\alpha\sigma}, \\ A^{3333}(t) := \lambda(t) + 2\mu(t), \qquad A^{\alpha\beta\sigma 3}(t) = A^{\alpha 333}(t) := 0. \end{gathered} \tag{2.26}$$

Lemma 2.3 (see [1]). *Let ω be a domain in $\mathbf{R}^2$, and let $\boldsymbol{\theta} \in C^2(\overline{\omega}; \mathbf{R}^3)$ be an injective mapping such that the two vectors $\mathbf{a}_\alpha$ are linear independent at all points of $\overline{\omega}$. The derivatives of the vectors of the covariant and contravariant basis are given by the formulas of Gauss*

$$\partial_\alpha \mathbf{a}_\beta = \Gamma_{\alpha\beta}^\sigma \mathbf{a}_\sigma + b_{\alpha\beta}\mathbf{a}_3, \qquad \partial_\alpha \mathbf{a}^\beta = -\Gamma_{\alpha\sigma}^\beta \mathbf{a}^\sigma + b_\alpha^\beta \mathbf{a}^3 \tag{2.27}$$

and Weingarten

$$\partial_\alpha \mathbf{a}_3 = \partial_\alpha \mathbf{a}^3 = -b_{\alpha\beta}\mathbf{a}^\beta = -b_\alpha^\sigma \mathbf{a}_\sigma. \tag{2.28}$$

Lemma 2.4. *Let $\omega \in L^2(\Omega)$ be a function such that $\int_\Omega \omega \partial_3 v = 0$ for all $v \in C^\infty(\overline{\Omega})$ satisfing $v = 0$ on $\gamma \times [-1, +1]$. Then, $\omega = 0$.*

Proof. Thanks to Theorem 3.4-1 in [1]. □

Lemma 2.5. *Assume that the scaled unknown satisfying* (2.15) *admits for each* $0 < \varepsilon < \varepsilon_0$ *a formal asymptotic expansion of the form*

$$\mathbf{u}(\varepsilon)(t) = \frac{1}{\varepsilon^N}\mathbf{u}^{-N}(t) + \frac{1}{\varepsilon^{N-1}}\mathbf{u}^{-N+1}(t) + \cdots \tag{2.29}$$

with

$$\mathbf{u}^{-N}(t), \mathbf{u}^{-N+1}(t) \in L^\infty(0,T;\mathbf{W}(\Omega)), \quad \mathbf{W}(\Omega) = \left\{\mathbf{v} \in \mathbf{W}^{1,4}(\Omega),\ \mathbf{v} = 0 \text{ on } \Gamma_0\right\},\ \mathbf{u}^{-N}(t) \neq 0, \tag{2.30}$$

for some integer $N \in \mathbf{Z}$. *Then,* $N = 0$.

Proof. The proof is broken into seven parts. Before beginning the proper induction in (iv), we record several useful preliminaries.

(i) Let the functions $A^{ijkl}(0)$ be defined as in Lemma 2.2. Then, for any symmetric matrices (s_{kl}) and (t_{ij}),

$$\begin{aligned} A^{ijkl}(0)s_{kl}t_{ij} = {} & \left(\lambda(0)a^{\alpha\beta}a^{\sigma\tau} + \mu(0)\left(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}\right)\right)s_{\sigma\tau}t_{\alpha\beta} + 4\mu(0)a^{\alpha\sigma}s_{\alpha 3}t_{\sigma 3} \\ & + \lambda(0)a^{\alpha\beta}s_{33}t_{\alpha\beta} + \lambda(0)a^{\sigma\tau}s_{\sigma\tau}t_{33} + \left(\lambda(0) + 2\mu(0)\right)s_{33}t_{33}. \end{aligned} \tag{2.31}$$

This formula, which immediately follows from the definitions, will be constantly put to use in the ensuing arguments.

(ii) Let $a^{ij} := \mathbf{a}^i \cdot \mathbf{a}^j$. Then, for any $y \in \overline{\omega}$ and any matrix (t_{ij}),

$$a^{ij}(y)a^{mn}(y)t_{im}t_{jn} \geq 0, \qquad a^{ij}(y)a^{mn}(y)t_{im}t_{jn} = 0 \Longleftrightarrow t_{ij} = 0. \tag{2.32}$$

Given any $y \in \overline{\omega}$ and any matrix (t_{ij}), let $\mathbf{t}_i(y) := t_{im}\mathbf{a}^m(y)$ and let $[\mathbf{t}_i(y)]^q$ denote the qth Cartesian component of the vector $\mathbf{t}_i(y)$. We thus have

$$\begin{aligned} a^{ij}(y)a^{mn}(y)t_{im}t_{jn} &= a^{ij}(y)\{(t_{im}\mathbf{a}^m(y)) \cdot (t_{jn}\mathbf{a}^n(y))\} = a^{ij}(y)\{\mathbf{t}_i(y) \cdot \mathbf{t}_j(y)\} \\ &= a^{ij}(y)[\mathbf{t}_i(y)]^p[\mathbf{t}_j(y)]^p = \left([\mathbf{t}_i(y)]^p\mathbf{a}^i(y)\right) \cdot \left([\mathbf{t}_j(y)]^p\mathbf{a}^j(y)\right) \\ &= \sum_{p=1}^{3}\left|[\mathbf{t}_i(y)]^p\mathbf{a}^i(y)\right|^2. \end{aligned} \tag{2.33}$$

Hence, $a^{ij}(y)a^{mn}(y)t_{im}t_{jn} \geq 0$ and

$$\begin{aligned} a^{ij}a^{mn}(y)t_{im}t_{jn} = 0 &\Longrightarrow [\mathbf{t}_i(y)]^p\mathbf{a}^i(y) = 0, \quad p = 1,2,3, \\ &\Longrightarrow \mathbf{t}_i(y) = t_{im}\mathbf{a}^m(y) = 0, \quad i = 1,2,3, \\ &\Longrightarrow t_{im} = 0, \quad i,m = 1,2,3, \end{aligned} \tag{2.34}$$

for the three vectors $\mathbf{a}^i(y)$ are linear independent.

(iii) Assume that the formal asymptotic expansion of the scaled unknown is of the form

$$\mathbf{u}(\varepsilon)(t) = \frac{1}{\varepsilon^N}\mathbf{u}^{-N}(t) + \frac{1}{\varepsilon^{N-1}}\mathbf{u}^{-N+1}(t) + \cdots \quad \text{for some integer } N \geq 0, \tag{2.35}$$

with $\mathbf{u}^{-N} \in L^\infty(0,T;\mathbf{W}(\Omega))$ and $\mathbf{u}^{-N+1} \in L^\infty(0,T;\mathbf{W}(\Omega))$.

Together with the asymptotic behavior of the functions $g^{ij}(\varepsilon)$ and $\Gamma^p_{ij}(\varepsilon)$ as $\varepsilon \to 0$, such an expansion induces specific formal asymptotic expansions of the various functions appearing in the formulation of problem (2.15)

$$\begin{aligned}
u_{m\|\alpha}(\varepsilon)(t) &= \frac{1}{\varepsilon^N}u^{-N}_{m\|\alpha}(t) + \cdots, \qquad u_{m\|3}(\varepsilon)(t) = \frac{1}{\varepsilon^{N+1}}u^{-N-1}_{m\|3}(t) + \frac{1}{\varepsilon^N}u^{-N}_{m\|3}(t) + \cdots,\\
E_{\alpha\|\beta}(\varepsilon;\mathbf{u}(\varepsilon)(t)) &= \frac{1}{\varepsilon^{2N}}E^{-2N}_{\alpha\|\beta}(t) + \cdots, \qquad E_{\alpha\|3}(\varepsilon;\mathbf{u}(\varepsilon)(t)) = \frac{1}{\varepsilon^{2N+1}}E^{-2N-1}_{\alpha\|3}(t) + \cdots,\\
E_{3\|3}(\varepsilon;\mathbf{u}(\varepsilon)(t)) &= \frac{1}{\varepsilon^{2N+2}}E^{-2N-2}_{3\|3}(t) + \cdots, \qquad F_{\alpha\|\beta}(\varepsilon;\mathbf{u}(\varepsilon)(t),\mathbf{v}) = \frac{1}{\varepsilon^N}F^{-N}_{\alpha\|\beta}(t,\mathbf{v}) + \cdots,\\
F_{\alpha\|3}(\varepsilon;\mathbf{u}(\varepsilon)(t),\mathbf{v}) &= \frac{1}{\varepsilon^{N+1}}F^{-N-1}_{\alpha\|3}(t,\mathbf{v}) + \cdots, \qquad F_{3\|3}(\varepsilon;\mathbf{u}(\varepsilon)(t),\mathbf{v}) = \frac{1}{\varepsilon^{N+2}}F^{-N-2}_{3\|3}(t,\mathbf{v}) + \cdots,
\end{aligned} \tag{2.36}$$

where, by definition, $u^q_{i\|j}(t), E^q_{i\|j}(t)$, and $F^q_{i\|j}(t,\mathbf{v})$ designate for each $q \in \mathbf{Z}$ the coefficient of ε^q in the induced expansions of $u_{i\|j}(\varepsilon)(t), E_{i\|j}(\varepsilon;\mathbf{u}(\varepsilon)(t))$, and $F_{i\|j}(\varepsilon;\mathbf{u}(\varepsilon)(t),\mathbf{v})$.

Note in passing that, while the functions factorizing the powers of ε are by definition independent of ε, they are dependent on one or several terms $\mathbf{u}^q(t), q \geq -N$. In this respect, particular caution should be exercised as regards this dependence. For instance,

$$u^{-N}_{m\|\alpha}(t) = \partial_\alpha u^{-N}_m(t) - \Gamma^{p,0}_{\alpha m}u^{-N}_p(t), \qquad u^{-N}_{m\|3}(t) = \partial_3 u^{-N+1}_m(t) - \Gamma^{p,0}_{m3}u^{-N}_p(t), \tag{2.37}$$

that is, the factor of $1/\varepsilon^N$ in $u_{m\|\alpha}(\varepsilon)(t)$ depends on $\mathbf{u}^{-N}(t)$ but the one in $u_{m\|3}(\varepsilon)(t)$ depends also on $\mathbf{u}^{-N+1}(t)$.

Likewise, it should be remembered that the expression of some factor may differ according to which value of N is considered, for instance,

$$\begin{aligned}
E^{-2N}_{\alpha\|\beta}(t) &= \begin{cases} \dfrac{1}{2}a^{mn}u^{-N}_{m\|\alpha}(t)u^{-N}_{n\|\beta}(t) & \text{if } N \geq 1,\\[2mm] \dfrac{1}{2}\left(u^0_{\alpha\|\beta}(t) + u^0_{\beta\|\alpha}(t) + a^{mn}u^0_{m\|\alpha}(t)u^0_{n\|\beta}(t)\right) & \text{if } N = 0,\end{cases}\\
F^{-N}_{\alpha\|\beta}(t,\mathbf{v}) &= \begin{cases} \dfrac{1}{2}a^{mn}\left\{u^{-N}_{m\|\alpha}(t)v_{n\|\beta} + u^{-N}_{n\|\beta}(t)v_{m\|\alpha}\right\} & \text{if } N \geq 1,\\[2mm] \dfrac{1}{2}\left(v_{\alpha\|\beta} + v_{\beta\|\alpha} + a^{mn}\left\{u^0_{m\|\alpha}(t)v_{n\|\beta} + u^0_{n\|\beta}(t)v_{m\|\alpha}\right\}\right) & \text{if } N = 0,\end{cases}
\end{aligned} \tag{2.38}$$

where

$$v_{m\|\alpha} := \partial_\alpha v_m - \Gamma^{p,0}_{\alpha m} v_p. \tag{2.39}$$

We are now in a position to start the cancellation of the factors of the successive powers of ε found in the variational equations of problem (2.15) when $\mathbf{u}(\varepsilon)(t)$ is replaced by its formal expansion. In what follows, L^r designates for any integer $r \geq -3N-4$ the linear form defined by

$$L^r(t, \mathbf{v}) := \int_\Omega f^{i,r}(t) v_i \sqrt{a} dx + \int_{\Gamma_+ \cup \Gamma_-} h^{i,r+1}(t) v_i \sqrt{a} d\Gamma. \tag{2.40}$$

(iv) Assume that $N \geq 0$. Since the lowest power of ε in the left-hand side is ε^{-3N-4}, we are naturally led to first try

$$f^i(\varepsilon)(t) = \frac{1}{\varepsilon^{-3N-4}} f^{i,-3N-4}(t), \qquad h^i(\varepsilon)(t) = \frac{1}{\varepsilon^{3N+3}} h^{i,-3N-3}(t). \tag{2.41}$$

Comparing the coefficients of ε^{-3N-4} in (2.15) and using Lemma 2.2 and (2.36), we get the equations

$$\begin{aligned} &\int_\Omega (\lambda(0) + 2\mu(0)) E^{-2N-2}_{3\|3}(t) F^{-N-2}_{3\|3}(t, \mathbf{v}) \sqrt{a} dx \\ &\quad + \int_\Omega \int_{-\infty}^t (\lambda'(t-\tau) + 2\mu'(t-\tau)) E^{-2N-2}_{3\|3}(\tau) F^{-N-2}_{3\|3}(\tau, \mathbf{v}) \sqrt{a} dx \, d\tau = L^{-3N-4}(t, \mathbf{v}) \end{aligned} \tag{2.42}$$

for all $\mathbf{v} \in \mathbf{W}(\Omega)$. Since

$$E^{-2N-2}_{3\|3}(t) = \frac{1}{2} a^{mn} \partial_3 u^{-N}_m(t) \partial_3 u^{-N}_n(t), \qquad F^{-N-2}_{3\|3}(t, \mathbf{v}) = a^{mn} \partial_3 u^{-N}_m(t) \partial_3 v_n, \tag{2.43}$$

we must have

$$L^{-3N-4}(t, \mathbf{v}) = \int_\Omega f^{i,-3N-4}(t) v_i \sqrt{a} dx + \int_{\Gamma_+ \cup \Gamma_-} h^{i,-3N-3}(t) v_i \sqrt{a} d\Gamma = 0 \tag{2.44}$$

for all $\mathbf{v} \in \mathbf{W}$ that are independent of x_3. Consequently, the first requirement (that there be no restriction on the applied forces) implies that we must let

$$f^{i,-3N-4}(t) = 0, \qquad h^{i,-3N-3}(t) = 0. \tag{2.45}$$

By recalling (2.42)–(2.45), we have

$$\frac{1}{2}\int_{\Omega}(\lambda(0)+2\mu(0))\left[\left(a^{mn}\partial_3 u_m^{-N}(t)\partial_3 u_n^{-N}(t)\right)\left(a^{mn}\partial_3 u_m^{-N}(t)\partial_3 v_n\right)\right]\sqrt{a}dx$$
$$+\frac{1}{2}\int_{\Omega}\int_{-\infty}^{t}(\lambda'(t-\tau)+2\mu'(t-\tau))\left[\left(a^{mn}\partial_3 u_m^{-N}(\tau)\partial_3 u_n^{-N}(\tau)\right)\left(a^{mn}\partial_3 u_m^{-N}(\tau)\partial_3 v_n\right)\right]\sqrt{a}dx\,d\tau=0, \tag{2.46}$$

that is,

$$\frac{d}{dt}\int_{-\infty}^{t}\int_{\Omega}(\lambda(t-\tau)+2\mu(t-\tau))\left[\left(a^{mn}\partial_3 u_m^{-N}(\tau)\partial_3 u_n^{-N}(\tau)\right)\left(a^{mn}\partial_3 u_m^{-N}(\tau)\partial_3 v_n\right)\right]dx\,d\tau=0. \tag{2.47}$$

Therefore,

$$\int_{-\infty}^{t}\int_{\Omega}(\lambda(t-\tau)+2\mu(t-\tau))\left[\left(a^{mn}\partial_3 u_m^{-N}(\tau)\partial_3 u_n^{-N}(\tau)\right)\left(a^{mn}\partial_3 u_m^{-N}(\tau)\partial_3 v_n\right)\right]dx\,d\tau=\text{const}, \tag{2.48}$$

which implies

$$\int_{-\infty}^{t}\int_{\Omega}(\lambda(t-\tau)+2\mu(t-\tau))\left[\left(a^{mn}\partial_3 u_m^{-N}(\tau)\partial_3 u_n^{-N}(\tau)\right)\left(a^{mn}\partial_3 u_m^{-N}(\tau)\partial_3 v_n\right)\right]dx\,d\tau=0. \tag{2.49}$$

Letting $\mathbf{v}=\mathbf{u}^{-N}(\tau)$ in (2.49) shows that

$$\int_{-\infty}^{t}\int_{\Omega}(\lambda(t-\tau)+2\mu(t-\tau))\left[a^{mn}\partial_3 u_m^{-N}(\tau)\partial_3 u_n^{-N}(\tau)\right]^2dx\,d\tau=0. \tag{2.50}$$

Since the symmetric (a^{ij}) is positive definite, we conclude that

$$\partial_3\mathbf{u}^{-N}(t)=\left(\partial_3 u_m^{-N}(t)\right)=0\quad\text{in }\Omega, \tag{2.51}$$

that is, $\mathbf{u}^{-N}(t)$ is independent of x_3. Inserting (2.51) into (2.43) yields

$$E_{3\|3}^{-2N-2}(t)=0,\qquad F_{3\|3}^{-N-2}(t,\mathbf{v})=0\quad\forall\mathbf{v}\in\mathbf{W}(\Omega). \tag{2.52}$$

A usual, any function defined on $\overline{\Omega}$ that is independent of x_3 is identified with a function defined on $\overline{\omega}$, and (2.36) and (2.52) imply

$$\mathbf{u}^{-N}(t) \in L^{\infty}(0,T;\mathbf{W}(\omega)), \quad \mathbf{W}(\omega) := \left\{\boldsymbol{\eta} \in \mathbf{W}^{1,4}(\omega);\ \boldsymbol{\eta} = 0 \text{ on } \gamma_0\right\}, \tag{2.53}$$

$$E_{i\|j}^{-2N-2}(t) = 0 \quad \text{in } \Omega, \qquad F_{i\|j}^{-N-2}(t,\mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathbf{W}(\Omega) \tag{2.54}$$

Noting (2.36) and (2.51), we also have

$$E_{i\|j}^{-2N-1}(t) = 0 \quad \text{in } \Omega. \tag{2.55}$$

Since $E_{\alpha\|\beta}^{-2N-1}(t) = 0$ (the leading term in the formal expansion of $E_{\alpha\|\beta}(\varepsilon;\mathbf{u}(\varepsilon)(t))$ is order of $-2N$) and $E_{i\|3}^{-2N-1}(t) = 0$ (since $\partial_3\mathbf{u}^{-N}(t) = 0$, each factor of $1/\varepsilon^{2N+1}$ in the expansion of $E_{\alpha\|3}(\varepsilon;\mathbf{u}(\varepsilon)(t))$ vanishes because it contains some derivative $\partial_3 u_m^{-N}$ and the leading term in the expansion of $E_{3\|3}(\varepsilon;\mathbf{u}(\varepsilon)(t))$ is of order strictly higher than $(-2N-1)$), our next try is thus

$$f^i(\varepsilon)(t) = \frac{1}{\varepsilon^{3N+3}} f^{i,-3N-3}(t), \qquad h^i(\varepsilon)(t) = \frac{1}{\varepsilon^{3N+2}} h^{i,-3N-2}(t). \tag{2.56}$$

Comparing the coefficient of ε^{-3N-3} in (2.15) then yields equations (the functions $A^{ijkl}(0)$ are defined in Lemma 2.2)

$$\begin{aligned}&\int_{\Omega} A^{ijkl}(0)E_{k\|l}^{-2N-1}(t)F_{i\|j}^{-N-2}(t,\mathbf{v})\sqrt{a}dx \\ &\quad + \int_{\Omega}\int_{-\infty}^{t} A^{ijkl}(0)E_{k\|l}^{-2N-1}(\tau)F_{i\|j}^{-N-2}(\tau,\mathbf{v})\sqrt{a}dx\,d\tau = L^{-3N-3}(t,\mathbf{v})\end{aligned} \tag{2.57}$$

for all $\mathbf{v} \in \mathbf{W}(\Omega)$. But since (2.55), we must let $f^{i,-3N-3}(t) = 0$ and $h^{i,-3N-2}(t) = 0$ (first requirement) and accordingly try

$$f^i(\varepsilon)(t) = \frac{1}{\varepsilon^{3N+2}} f^{i,-3N-2}(t), \qquad h^i(\varepsilon)(t) = \frac{1}{\varepsilon^{3N+2}} h^{i,-3N-1}(t). \tag{2.58}$$

In which case the cancellation of the coefficient of ε^{-3N-2} in (2.15) yields the equations

$$\begin{aligned}&\int_{\Omega} A^{ijkl}(0)E_{k\|l}^{-2N}(t)F_{i\|j}^{-N-2}(t,\mathbf{v})\sqrt{a}dx \\ &\quad + \int_{\Omega}\int_{-\infty}^{t} A'^{ijkl}(t-\tau)E_{k\|l}^{-2N}(\tau)F_{i\|j}^{-N-2}(\tau,\mathbf{v})\sqrt{a}dx\,d\tau = L^{-3N-2}(t,\mathbf{v})\end{aligned} \tag{2.59}$$

for all $\mathbf{v} \in \mathbf{W}(\Omega)$. But since (2.54), we must let $f^{i,-3N-2}(t) = 0$ and $h^{i,-3N-1}(t) = 0$ (first requirement).

(v) Assume that $N \geq 1$. Our next try being thus

$$f^i(\varepsilon)(t) = \frac{1}{\varepsilon^{3N+1}} f^{i,-3N-1}, \qquad h^i(\varepsilon) = \frac{1}{\varepsilon^{3N+2}} h^{i,-3N}, \tag{2.60}$$

the cancellation of the coefficient of ε^{-3N-1} in the variational equations of problem (2.15) then yields the equations

$$\begin{aligned}&\int_\Omega A^{ijkl}(0) E_{k\|l}^{-2N}(t) F_{i\|j}^{-N-1}(t,\mathbf{v})\sqrt{a}dx \\ &\quad + \int_\Omega \int_{-\infty}^t A'^{ijkl}(t-\tau) E_{k\|l}^{-2N}(\tau) F_{i\|j}^{-N-1}(\tau,\mathbf{v})\sqrt{a}dx\, d\tau = L^{-3N-1}(t,\mathbf{v})\end{aligned} \tag{2.61}$$

for all $\mathbf{v} \in \mathbf{W}(\Omega)$, where

$$\begin{aligned}E_{\alpha\|\beta}^{-2N}(t) &= \frac{1}{2} a^{mn} u_{m\|\alpha}^{-N}(t) u_{n\|\beta}^{-N}(t), \qquad F_{\alpha\|\beta}^{-N-1}(t,\mathbf{v}) = 0, \\ E_{\alpha\|3}^{-2N}(t) &= \frac{1}{2} a^{mn} u_{m\|\alpha}^{-N}(t) u_{n\|3}^{-N}(t), \qquad F_{\alpha\|3}^{-N-1}(t,\mathbf{v}) = \frac{1}{2} a^{mn} u_{m\|\alpha}^{-N}(t)\partial_3 v_n, \\ E_{3\|3}^{-2N}(t) &= \frac{1}{2} a^{mn} u_{m\|3}^{-N}(t) u_{n\|3}^{-N}(t), \qquad F_{3\|3}^{-N-1}(t,\mathbf{v}) = a^{mn} u_{m\|3}^{-N}(t)\partial_3 v_n,\end{aligned} \tag{2.62}$$

the functions $u_{m\|i}^{-N}(t)$ being those defined in (2.36).

Letting $\mathbf{v} \in \mathbf{W}(\Omega)$ be independent of x_3 then shows that we must let $f^{i,-3N-1}(t) = 0$ and $h^{i,-3N} = 0$; hence,

$$\begin{aligned}&\int_\Omega A^{ijkl}(0) E_{k\|l}^{-2N}(t) F_{i\|j}^{-N-1}(t,\mathbf{v})\sqrt{a}dx \\ &\quad + \int_\Omega \int_{-\infty}^t A'^{ijkl}(t-\tau) E_{k\|l}^{-2N}(\tau) F_{i\|j}^{-N-1}(\tau,\mathbf{v})\sqrt{a}dx\, d\tau = 0\end{aligned} \tag{2.63}$$

for all $\mathbf{v} \in \mathbf{W}(\Omega)$. Let the field $\mathbf{w}^N = (w_m^N)$ be defined for all $(y, x_3) \in \Omega$ by

$$w_m^N(t) := u_m^{-N+1}(t) - (1 + x_3)\Gamma_{m3}^{p,0} u_p^{-N}(t). \tag{2.64}$$

Then, $\mathbf{w}^N(t) \in L^\infty(0,T;\mathbf{W}(\Omega))$ because both $\mathbf{u}^{-N}(t)$ and $\mathbf{u}^{-N+1}(t)$ are assumed to be in the space $L^\infty(0,T;\mathbf{W}(\Omega))$.

Furthermore, $\partial_3 w_m^N(t) = u_{m\|3}^{-N}(t)$, so that

$$F_{\alpha\|3}^{-N-1}\left(t,\mathbf{w}^N(t)\right) = E_{\alpha\|3}^{-2N}(t), \qquad F_{3\|3}^{-N-1}\left(t,\mathbf{w}^N(t)\right) = 2E_{3\|3}^{-2N}(t). \tag{2.65}$$

Using Lemma 2.2 and (2.65), we get

$$A^{ijkl}(0)E_{k\|l}^{-2N}(t)F_{i\|j}^{-N-1}\left(t,\mathbf{w}^N\right) = 2\lambda(0)a^{mn}E_{m\|n}^{-2N}(t)E_{3\|3}^{-2N}(t) + 4\mu(0)a^{mn}E_{m\|3}^{-2N}(t)E_{n\|3}^{-2N}(t). \quad (2.66)$$

Since

$$a^{mn}E_{m\|n}^{-2N}(t) = \frac{1}{2}a^{ij}a^{mn}u_{i\|m}^{-N}(t)u_{j\|n}^{-N}(t) \geq 0 \quad \text{in } \Omega, \quad (2.67)$$

(by (ii))

$$E_{3\|3}^{-2N}(t) = \frac{1}{2}a^{mn}u_{m\|3}^{-N}(t)u_{n\|3}^{-N}(t) \geq 0, \qquad a^{mn}E_{m\|3}^{-2N}(t)E_{n\|3}^{-2N}(t) \geq 0 \quad \text{in } \Omega \quad (2.68)$$

(the matrix (a^{mn}) is positive definite), in a similar way as in (iv), we can obtain from (2.63) that

$$\int_{-\infty}^{t}\int_{\Omega} A^{ijkl}(t-\tau)E_{k\|l}^{-2N}(\tau)F_{i\|j}^{-N-1}(\tau,\mathbf{v})dx\,d\tau = 0 \quad (2.69)$$

for all $\mathbf{v} \in \mathbf{W}(\Omega)$.

Letting $\mathbf{v} = \mathbf{w}^N(\tau)$ in (2.69) and noting (2.66)–(2.68), we conclude that

$$a^{mn}E_{m\|3}^{-2N}(t)E_{n\|3}^{-2N}(t) = 0 \quad \text{in } \Omega; \quad (2.70)$$

hence (the matrix (a^{mn}) is positive definite)

$$E_{m\|3}^{-2N}(t) = 0 \quad \text{in } \Omega. \quad (2.71)$$

In particular then, $E_{3\|3}^{-2N}(t) = (1/2)a^{mn}u_{m\|3}^{-N}(t)u_{n\|3}^{-N}(t) = 0$ (by (2.62)) and thus (the matrix (a^{mn}) is positive definite)

$$u_{m\|3}^{-N}(t) = 0. \quad (2.72)$$

(vi) Assume that $N \geq 2$ (the case $N = 1$ is considered separately, c.f. (viii)). Our next try being thus

$$f^i(\varepsilon)(t) = \frac{1}{\varepsilon^{3N}}f^{i,-3N}(t), \qquad h^i(\varepsilon)(t) = \frac{1}{\varepsilon^{3N-1}}h^{i,-3N+1}(t), \quad (2.73)$$

the cancellation of the coefficient of ε^{-3N} in the variational equations of problem (2.15) then yields the equations (note that two terms are needed here from the expansions of the functions $A^{ijkl}(\varepsilon)\sqrt{g(\varepsilon)}$, c.f. Lemma 2.2))

$$\begin{aligned}
&\int_\Omega A^{ijkl}(0)\left\{E^{-2N}_{k\|l}(t)F^{-N}_{i\|j}(t,\mathbf{v})+E^{-2N+1}_{k\|l}(t)F^{-N-1}_{i\|j}(t,\mathbf{v})\right\}\sqrt{a}dx\\
&\quad+\int_\Omega B^{ijkl,1}E^{-2N}_{k\|l}(t)F^{-N-1}_{i\|j}(t,\mathbf{v})dx\\
&\quad+\int_\Omega\int_{-\infty}^{t} A'^{ijkl}(t-\tau)\left\{E^{-2N}_{k\|l}(\tau)F^{-N}_{i\|j}(\tau,\mathbf{v})+E^{-2N+1}_{k\|l}(\tau)F^{-N-1}_{i\|j}(\tau,\mathbf{v})\right\}\sqrt{a}dx\,d\tau\\
&\quad+\int_\Omega B'^{ijkl,1}(t-\tau)E^{-2N}_{k\|l}(\tau)F^{-N-1}_{i\|j}(\tau,\mathbf{v})dx\,d\tau=L^{-3N}(t,\mathbf{v})
\end{aligned}\tag{2.74}$$

for all $\mathbf{v}\in\mathbf{W}(\Omega)$, where (by (2.62) and (2.72))

$$\begin{gathered}
E^{-2N}_{\alpha\|\beta}(t)=\frac{1}{2}a^{mn}u^{-N}_{m\|\alpha}(t)u^{-N}_{m\|\beta}(t),\qquad E^{-2N}_{i\|3}(t)=0,\\
F^{-N-1}_{\alpha\|\beta}(t,\mathbf{v})=0,\qquad F^{-N-1}_{\alpha\|3}(t,\mathbf{v})=\frac{1}{2}a^{mn}u^{-N}_{m\|\alpha}(t)\partial_3 v_n,\qquad F^{-N-1}_{3\|3}(t,\mathbf{v})=0,\\
F^{-N}_{\alpha\|\beta}(t,\mathbf{v})=\frac{1}{2}a^{mn}\left(u^{-N}_{m\|\alpha}(t)v_{n\|\beta}+u^{-N}_{n\|\beta}(t)v_{m\|\alpha}\right),\qquad F^{-N}_{3\|3}=a^{mn}u^{-N+1}_{m\|3}(t)\partial_3 v_n,
\end{gathered}\tag{2.75}$$

the last expression of $F^{-N}_{3\|3}(t,\mathbf{v})$ being valid only if $N\geq 2$ (the expressions of $F^{-N}_{\alpha\|3}(t,\mathbf{v})$ are not needed since $A^{\alpha 3\sigma\tau}(0)=0$ by Lemma 2.2 and $E^{-2N}_{\alpha\|3}(t)=0$ by (2.71)).

Noting that $F^{-N-1}_{\alpha\|3}(\mathbf{v})=F^{-N}_{3\|3}=0$ if $\partial_3\mathbf{v}=0$, we thus conclude that the variational equations (2.74) reduce to

$$\begin{aligned}
&\int_\Omega A^{\alpha\beta\sigma\tau}(0)E^{-2N}_{\sigma\|\tau}(t)F^{-N}_{\alpha\|\beta}(t,\mathbf{v})\sqrt{a}dx\\
&\quad+\int_\Omega\int_{-\infty}^{t} A'^{\alpha\beta\sigma\tau}(t-\tau)E^{-2N}_{\sigma\|\tau}(\tau)F^{-N}_{\alpha\|\beta}(\tau,\mathbf{v})\sqrt{a}dx\,d\tau=L^{-3N}(t,\mathbf{v})
\end{aligned}\tag{2.76}$$

for all $\mathbf{v}\in\mathbf{W}(\Omega)$ that are independent of x_3.

Since each term in the sum $A^{\alpha\beta\sigma\tau}(0)E^{-2N}_{\sigma\|\tau}(t)F^{-N}_{\alpha\|\beta}(t,\mathbf{v})$ is cubic with respect to the functions $u^{-N}_{m\|\alpha}(t)$, the linearization trick (second requirement) implies that $L^{-3N}(t,\mathbf{v})=0$ for all $\mathbf{v}\in\mathbf{W}(\Omega)$ that are independent of x_3. Hence, we must let $f^{i,-3N}(t)=0$ and $h^{i,-3N+1}(t)=0$. Hence

$$\int_\Omega A^{\alpha\beta\sigma\tau}(0)E^{-2N}_{\sigma\|\tau}(t)F^{-N}_{\alpha\|\beta}(t,\mathbf{v})\sqrt{a}dx+\int_\Omega\int_{-\infty}^{t} A'^{\alpha\beta\sigma\tau}(t-\tau)E^{-2N}_{\sigma\|\tau}(\tau)F^{-N}_{\alpha\|\beta}(\tau,\mathbf{v})\sqrt{a}dx\,d\tau=0.\tag{2.77}$$

In a similar way as in (iv), we can obtain from (2.77) that

$$\int_{-\infty}^{t}\int_{\Omega} A^{\alpha\beta\sigma\tau}(t-\tau)E_{\sigma\|\tau}^{-2N}(\tau)F_{\alpha\|\beta}^{-N}(\tau,\mathbf{v})dx\,d\tau = 0. \tag{2.78}$$

Recalling that $\mathbf{u}^{-N}(t)$ is independent of x_3 by (2.51), we may let $\mathbf{v} = \mathbf{u}^{-N}(\tau)$ in (2.78). This gives

$$\int_{-\infty}^{t}\int_{\Omega} A^{\alpha\beta\sigma\tau}(t-\tau)E_{\sigma\|\tau}^{-2N}(\tau)E_{\alpha\|\beta}^{-2N}(\tau)dx\,d\tau = 0 \tag{2.79}$$

since $F_{\alpha\|\beta}^{-N}(\tau,\mathbf{u}^{-N}(\tau)) = 2E_{\alpha\|\beta}^{-2N}(t)$ (by (2.75)). But (see Lemma 2.2)

$$A^{\alpha\beta\sigma\tau}(0) = \lambda(0)a^{\alpha\beta}a^{\sigma\tau} + \mu(0)\left(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}\right) \tag{2.80}$$

and thus (the matrix (a^{mn}) is positive definite)

$$E_{\alpha\|\beta}^{-2N}(t) = \frac{1}{2}a^{mn}u_{m\|\alpha}^{-N}(t)u_{n\|\beta}^{-N}(t) = 0 \quad \text{in } \Omega \tag{2.81}$$

(to reach this conclusion, observe that $a^{\alpha\sigma}a^{\beta\tau}t_{\sigma\tau}t_{\alpha\beta} \geq 0$ and that $a^{\alpha\sigma}a^{\beta\tau}t_{\sigma\tau}t_{\alpha\beta} = 0$ only if $t_{\alpha\beta} = 0$ by (ii)); these relations in turn imply that

$$u_{m\|\alpha}^{-N}(t) = 0. \tag{2.82}$$

By definition (see (2.36) and Lemma 2.2),

$$\begin{aligned} u_{\beta\|\alpha}^{-N}(t) &= \partial_\alpha u_\beta^{-N}(t) - \Gamma_{\alpha\beta}^{p,0}u_p^{-N}(t) = \partial_\alpha u_\beta^{-N}(t) - \Gamma_{\alpha\beta}^{\sigma}u_\sigma^{-N}(t) - b_{\alpha\beta}u_3^{-N}(t), \\ u_{3\|\alpha}^{-N}(t) &= \partial_\alpha u_3^{-N}(t) - \Gamma_{\alpha 3}^{p,0}u_p^{-N}(t) = \partial_\alpha u_3^{-N}(t) + b_\alpha^\sigma u_\sigma^{-N}(t). \end{aligned} \tag{2.83}$$

Let $\zeta_i(t) = u_i^{-N}(t)|_{x_3=0}$. Then, $\zeta_i(t) \in L^\infty(0,T;\mathbf{W}(\omega))$ since $u_i^{-N}(t) \in L^\infty(0,T;\mathbf{W}(\Omega))$ and $\partial_3 u_i^{-N}(t) = 0$ in Ω and $\zeta_i(t) = 0$ on γ_0 since $u_i^{-N}(t) = 0$ on Γ_0. The above relations combined with the Gauss and Weingarten formulas (Lemma 2.3) then imply that $\partial_\alpha(\zeta_i(t)\mathbf{a}^i) = 0$ in ω and hence that $\zeta_i(t) = 0$. We have thus shown that

$$\mathbf{u}^{-N}(t) = 0 \quad \forall N \geq 2. \tag{2.84}$$

(vii) Finally, assume that $N = 1$. The only difference from (vi) is that now

$$F_{3\|3}^{-1}(t,\mathbf{v}) = \partial_3 v_3 + a^{mn}u_{m\|3}^{0}(t)\partial_3 v_n. \tag{2.85}$$

But since the arguments that led in (vi) to the conclusion that $\mathbf{u}^{-N} = 0$ for $N \geq 2$ only required that consideration of functions $\mathbf{v} \in \mathbf{W}$ that are independent of x_3, in which case $F^{-1}_{3\|3}(t, \mathbf{v}) = 0$, they can be reproduced verbatim for $N = 1$, thus showing that

$$\mathbf{u}^{-1}(t) = 0. \tag{2.86}$$

The proof is complete. □

3. The Proof of the Main Result

Proof. The proof comprises three parts.

(i) Using Lemma 2.5, $\mathbf{u}(\varepsilon)(t)$ can be expanded as

$$\mathbf{u}(\varepsilon)(t) = \mathbf{u}^{-N}(t) + \mathbf{u}^{-N+1}(t) + \cdots, \tag{3.1}$$

with $\mathbf{u}^{-N} \in L^\infty(0, T; \mathbf{W}(\Omega))$. Letting $N = 0$, we thus infer that $\partial_3 \mathbf{u}^0(t) = 0$ in Ω, that

$$\boldsymbol{\zeta}^0(t) := \frac{1}{2}\int_{-1}^{1} \mathbf{u}^0(t) dx_3 \in L^\infty(0, T; \mathbf{W}(\omega)), \quad \mathbf{W}(\omega) = \left\{\boldsymbol{\eta} \in \mathbf{W}^{1,4}(\omega);\ \boldsymbol{\eta} = 0 \ \text{ on } \ \gamma_0\right\}, \tag{3.2}$$

and also that (see (2.54) and (2.55))

$$\begin{gathered} E^q_{i\|j}(t) = 0 \quad \forall \text{ integers } q \leq -1, \\ F^q_{i\|j}(t, \mathbf{v}) = 0 \quad \forall \text{ integers } q \leq -2 \ \forall \mathbf{v} \in \mathbf{W}(\Omega), \end{gathered} \tag{3.3}$$

and, finally, that we must let $f^{i,-2}(t) = 0$ and $h^{i,-1}(t) = 0$.

(ii) Our next try is thus

$$f^i(\varepsilon)(t) = \frac{1}{\varepsilon} f^{i,-1}(t), \qquad h^i(\varepsilon)(t) = h^{i,0}(t), \tag{3.4}$$

where it is understood as in the proof of Lemma 2.5 that each function $f^{i,r}(t) \in L^\infty(0, T; L^2(\Omega))$ and each function $h^{i,r+1}(t) \in L^\infty(0, T; L^2(\Gamma_+ \cup \Gamma_-)), r \geq -1$, appearing here and subsequently is independent of ε; likewise, we again let

$$L^r(t, \mathbf{v}) := \int_\Omega f^{i,r}(t) v_i \sqrt{a} dx + \int_{\Gamma_+ \cup \Gamma_-} h^{i,r+1}(t) v_i \sqrt{a} d\Gamma, \quad r \geq -1. \tag{3.5}$$

The cancellation of the coefficient of ε^{-1} in the variational equations of problem (2.15) then yields the equations:

$$\int_\Omega A^{ijkl}(0) E^0_{k\|l}(t) F^{-1}_{i\|j}(t, \mathbf{v}) \sqrt{a} dx + \int_\Omega \int_{-\infty}^{t} A'^{ijkl}(t-\tau) E^0_{k\|l}(\tau) F^{-1}_{i\|j}(\tau, \mathbf{v}) \sqrt{a} dx\, dt = L^{-1}(t, \mathbf{v}) \tag{3.6}$$

for all $\mathbf{v} \in \mathbf{W}(\Omega)$, where

$$E^0_{\alpha\|\beta}(t) = \frac{1}{2}\Big(u^0_{\alpha\|\beta}(t) + u^0_{\beta\|\alpha}(t) + a^{mn}u^0_{m\|\alpha}(t)u^0_{n\|\beta}(t)\Big),$$

$$E^0_{\alpha\|3}(t) = \frac{1}{2}\Big(u^{(0)}_{\alpha\|3}(t) + u^0_{3\|\alpha}(t) + a^{mn}u^0_{m\|\alpha}(t)u^{(0)}_{n\|3}(t)\Big),$$

$$E^0_{3\|3}(t) = u^{(0)}_{3\|3}(t) + \frac{1}{2}a^{mn}u^{(0)}_{m\|3}(t)u^{(0)}_{n\|3}(t), \tag{3.7}$$

$$F^{-1}_{\alpha\|\beta}(t,\mathbf{v}) = 0, \qquad F^{-1}_{\alpha\|3}(t,\mathbf{v}) = \frac{1}{2}\Big(\partial_3 v_\alpha + a^{mn}u^0_{m\|\alpha}(t)\partial_3 v_n\Big),$$

$$F^{-1}_{3\|3}(t,\mathbf{v}) = \partial_3 v_3 + a^{mn}u^{(0)}_{m\|3}(t)\partial_3 v_n,$$

$$u^0_{m\|\alpha}(t) := \partial_\alpha u^0_m(t) - \Gamma^{p,0}_{\alpha m}u^0_p(t), \qquad u^{(0)}_{m\|3}(t) := \partial_3 u^1_m(t) - \Gamma^{p,0}_{m3}u^0_p(t). \tag{3.8}$$

The special notation $u^{(0)}_{m\|3}(t)$ emphasizes that, by contrast with the functions $u^0_{m\|\alpha}(t)$, which only depend on $\mathbf{u}^0(t)$, the functions $u^{(0)}_{m\|3}(t)$ also depend on $\mathbf{u}^1(t)$.

The expressions of the functions $F^{-1}_{i\|j}(t,\mathbf{v})$ imply that $L^{-1}(t,\mathbf{v}) = 0$ for all for all $\mathbf{v} \in \mathbf{W}(\Omega)$ that are independent of x_3. Hence, we must let $f^{i,-1}(t) = 0$ and $h^{i,0}(t) = 0$ (first requirement), so that we are left with the equations

$$\int_\Omega A^{ijkl}(0)E^0_{k\|l}(t)F^{-1}_{i\|j}(t,\mathbf{v})\sqrt{a}dx + \int_\Omega\int_{-\infty}^{t} A'^{ijkl}(t-\tau)E^0_{k\|l}(\tau)F^{-1}_{i\|j}(\tau,\mathbf{v})\sqrt{a}dx\,dt = 0 \tag{3.9}$$

for all $\mathbf{v} \in \mathbf{W}(\Omega)$. When the functions $F^{-1}_{i\|j}(t,\mathbf{v})$ are replaced by their expression given in (3.7), the integrand in (3.9) takes the form $(w^\tau\partial_3 v_\tau + w^3\partial_3 v_3)$.

Then, Lemma 2.4 shows that the functions w^τ and w^3 vanish in Ω, that is,

$$\begin{aligned}
&\Big(\lambda(0)a^{\alpha\beta}E^0_{\alpha\|\beta}(t) + (\lambda(0)+2\mu(0))E^0_{3\|3}(t)\Big)a^{\sigma\tau}u^{(0)}_{\sigma\|3}(t) + 2\mu(0)E^0_{\alpha\|3}(t)\Big(a^{\alpha\tau} + a^{\alpha\sigma}a^{\beta\tau}u^0_{\beta\|\sigma}(t)\Big)\\
&\quad+\int_{-\infty}^{t}\Big[\Big(\lambda'(t-s)a^{\alpha\beta}E^0_{\alpha\|\beta}(s) + (\lambda'(t-s)+2\mu'(t-s))E^0_{3\|3}(s)\Big)a^{\sigma\tau}u^{(0)}_{\sigma\|3}(s)\\
&\qquad+2\mu'(t-s)E^0_{\alpha\|3}(s)\Big(a^{\alpha\tau} + a^{\alpha\sigma}a^{\beta\tau}u^0_{\beta\|\sigma}(s)\Big)\Big]ds = 0 \quad \text{in } \Omega, \tau = 1,2,\\
&\Big(\lambda(0)a^{\alpha\beta}E^0_{\alpha\|\beta}(t) + (\lambda(0)+2\mu(0))E^0_{3\|3}(t)\Big)\Big(1+u^{(0)}_{3\|3}(t)\Big) + 2\mu(0)a^{\alpha\sigma}E^0_{\alpha\|3}(t)u^0_{3\|\sigma}(t)\\
&\quad+\int_{-\infty}^{t}\Big[\Big(\lambda'(t-s)a^{\alpha\beta}E^0_{\alpha\|\beta}(s) + (\lambda'(t-s)+2\mu'(t-s))E^0_{3\|3}(s)\Big)\Big(1+u^{(0)}_{3\|3}(s)\Big)\\
&\qquad+2\mu'(t-s)a^{\alpha\sigma}E^0_{\alpha\|3}(s)u^0_{3\|\sigma}(s)\Big]ds = 0 \quad \text{in } \Omega,
\end{aligned} \tag{3.10}$$

that is,

$$\begin{aligned}
&\int_{-\infty}^{t}\Big[\Big(\lambda(t-\tau)a^{\alpha\beta}E^0_{\alpha\|\beta}(\tau)+\big(\lambda(t-s)+2\mu(t-s)\big)E^0_{3\|3}(s)\Big)a^{\sigma\tau}u^{(0)}_{\sigma\|3}(s)\\
&\qquad+2\mu(t-s)E^0_{\alpha\|3}(s)\Big(a^{\alpha\tau}+a^{\alpha\sigma}a^{\beta\tau}u^0_{\beta\|\sigma}(s)\Big)\Big]ds=0\quad\text{in }\Omega,\tau=1,2,\\
&\int_{-\infty}^{t}\Big[\Big(\lambda(t-s)a^{\alpha\beta}E^0_{\alpha\|\beta}(s)+\big(\lambda(t-s)+2\mu(t-s)\big)E^0_{3\|3}(s)\Big)\Big(1+u^{(0)}_{3\|3}(s)\Big)\\
&\qquad+2\mu(t-s)a^{\alpha\sigma}E^0_{\alpha\|3}(s)u^0_{3\|\sigma}(s)\Big]ds=0\quad\text{in }\Omega.
\end{aligned}\tag{3.11}$$

Under the conditions of integral mean value theorem, one obvious solution to this system of three equations is

$$\begin{gathered}
E^0_{\alpha\|3}(t)=0\quad\text{in }\Omega,\\
\widehat{\lambda}(s)a^{\alpha\beta}\widehat{E}^0_{\alpha\|\beta}(s)+\Big(\widehat{\lambda}(s)+2\widehat{\mu}(s)\Big)\widehat{E}^0_{3\|3}(s)=0\quad\text{in }\Omega,
\end{gathered}\tag{3.12}$$

$\widehat{\lambda}(s)$, $\widehat{\mu}(s)$, and $\widehat{E}^0_{\alpha\|\beta}(s)$ denote the Laplace transformation of $\lambda(t)$, $\mu(t)$, and $E^0_{\alpha\|\beta}(t)$ (c.f. [11]). But there may be other solutions to this nonlinear system. Denoting by $[\cdots]^{\text{lin}}$ the linear part with respect to (any component of) $\mathbf{u}^0(t)$ or $\mathbf{u}^1(t)$ in the expression $[\cdots]$, we have

$$\begin{gathered}
\Big[E^0_{\alpha\|3}(t)\Big]^{\text{lin}}=e^0_{\alpha\|3}(s),\\
\Big[\widehat{\lambda}(s)a^{\alpha\beta}\widehat{E}^0_{\alpha\|\beta}(s)+\Big(\widehat{\lambda}(s)+2\mu(t-s)\Big)\widehat{E}^0_{3\|3}(s)\Big]^{\text{lin}}=\widehat{\lambda}(s)a^{\alpha\beta}\widehat{e}^0_{\alpha\|\beta}(s)+\Big(\widehat{\lambda}(s)+2\widehat{\mu}(s)\Big)\widehat{e}^0_{3\|3}(s),
\end{gathered}\tag{3.13}$$

by definition of the functions $E^0_{i\|3}(t)$ and $e^0_{i\|3}(t)$ as the coefficient of ε^0 in the formal expansions of the functions $E_{i\|3}(\varepsilon,\mathbf{u}(\varepsilon))(t)$ and $e_{i\|3}(\varepsilon,\mathbf{u}(\varepsilon))(t)$, the latter being precisely the linear part in [2].

Since it was found in the linear case (see [2]) that

$$\begin{gathered}
e^0_{\alpha\|3}(t)=0\quad\text{in }\Omega,\\
\widehat{\lambda}(s)a^{\alpha\beta}\widehat{e}^0_{\alpha\|\beta}(s)+\Big(\widehat{\lambda}(s)+2\widehat{\mu}(s)\Big)\widehat{e}^0_{3\|3}(s)=0\quad\text{in }\Omega,
\end{gathered}\tag{3.14}$$

the linearization trick (second requirement) suggests that we only retain the "obvious" solution found above.

(iii) Our next try is thus

$$f^i(\varepsilon)(t)=f^{i,0}(t),\qquad h^i(\varepsilon)(t)=\varepsilon h^{i,1}(t).\tag{3.15}$$

The cancellation of the coefficient of ε^0 in the variational equations of problem (2.15) then leads to the equations

$$\begin{aligned}&\int_\Omega u^0_{itt}(t)v_j a^{ij}\sqrt{a}dx+\int_\Omega A^{ijkl}(0)\left\{E^0_{k\|l}(t)F^0_{i\|j}(t,\mathbf{v})+E^1_{k\|l}F^{-1}_{i\|j}(t,\mathbf{v})\right\}\sqrt{a}dx\\&\quad+\int_\Omega\int_{-\infty}^t A'^{ijkl}(t-\tau)\left\{E^0_{k\|l}(\tau)F^0_{i\|j}(\tau,\mathbf{v})+E^1_{k\|l}(\tau)F^{-1}_{i\|j}(\tau,\mathbf{v})\right\}\sqrt{a}dx\,d\tau\\&\quad+\int_\Omega B^{ijkl,1}(0)E^0_{k\|l}(t)F^{-1}_{i\|j}(t,\mathbf{v})dx+\int_\Omega\int_{-\infty}^t B'^{ijkl,1}(t-\tau)E^0_{k\|l}(\tau)F^{-1}_{i\|j}(\tau,\mathbf{v})dx\,d\tau=L^0(t,\mathbf{v})\end{aligned}\tag{3.16}$$

for all $\mathbf{v}\in\mathbf{W}(\Omega)$, where the functions $E^1_{i\|j}(t)$ and $F^0_{i\|j}(t)$ are defined by means of formal expansions

$$E_{i\|j}(\varepsilon;\mathbf{u}(\varepsilon)(t))=E^0_{i\|j}(t)+\varepsilon E^1_{i\|j}(t)+\cdots,\qquad F_{i\|j}(\varepsilon;\mathbf{u}(\varepsilon)(t),\mathbf{v})=\frac{1}{\varepsilon}F^{-1}_{i\|j}(t,\mathbf{v})+F^0_{i\|j}(t,\mathbf{v})+\cdots.\tag{3.17}$$

Note that, while the functions $E^0_{i\|j}(t)$, $F^{-1}_{i\|j}(t,\mathbf{v})$, and $F^0_{i\|j}(t,\mathbf{v})$ depend only on $\mathbf{u}^0(t)$ and $\mathbf{u}^1(t)$, the functions $E^1_{i\|j}(t)$ depend also on $\mathbf{u}^2(t)$ (but not on $\mathbf{u}^3(t)$; each term involving $\mathbf{u}^3(t)$ vanishes because it contains some derivative $\partial_3 u^0_m(t)$ as a factor). For this reason the formal asymptotic expansion of $\mathbf{u}(\varepsilon)(t)$ must be "at least" of the form

$$\mathbf{u}(\varepsilon)(t)=\sum_{q=0}^{2}\varepsilon^q\mathbf{u}^q(t)+\cdots.\tag{3.18}$$

In particular then, we must have (by (3.7))

$$\begin{aligned}&\int_\Omega u^0_{itt}(t)v_j a^{ij}\sqrt{a}dx+\int_\Omega A^{ijkl}(0)E^0_{k\|l}(t)F^0_{i\|j}(t,\mathbf{v})\sqrt{a}dx\\&\quad+\int_\Omega\int_{-\infty}^t A'^{ijkl}(t-\tau)E^0_{k\|l}(\tau)F^0_{i\|j}(\tau,\mathbf{v})\sqrt{a}dx\,d\tau=L^0(t,\mathbf{v})\end{aligned}\tag{3.19}$$

for all $\mathbf{v}\in\mathbf{W}(\Omega)$ that are independent of x_3 since $F^{-1}_{i\|j}(t,\mathbf{v})=0$ for such functions; equivalently, after performing the usual identification, we must have

$$\begin{aligned}&\int_\Omega u^0_{itt}(t)\eta_j a^{ij}\sqrt{a}dx+\int_\Omega A^{ijkl}(0)E^0_{k\|l}(t)F^0_{i\|j}(t,\boldsymbol{\eta})\sqrt{a}dx\\&\quad+\int_\Omega\int_{-\infty}^t A'^{ijkl}(t-\tau)E^0_{k\|l}(\tau)F^0_{i\|j}(\tau,\boldsymbol{\eta})\sqrt{a}dx\,d\tau=L^0(t,\mathbf{v})\end{aligned}\tag{3.20}$$

for all $\boldsymbol{\eta}\in\mathbf{W}(\omega)=\{\boldsymbol{\eta}\in\mathbf{W}^{1,4}(\omega);\ \boldsymbol{\eta}=0\ \text{ on }\ \gamma_0\}$.

Using Lemma 2.2 and (3.12), (3.20) can be written as

$$\begin{aligned}
&\int_\Omega u^0_{itt}\eta_j a^{ij}\sqrt{a}dx + \int_\Omega \left\{A^{\alpha\beta\sigma\tau}(0)E^0_{\sigma\|\tau}(t) + A^{\alpha\beta 33}(0)E^0_{3\|3}(t)\right\}F^0_{\alpha\|\beta}(t,\boldsymbol{\eta})\sqrt{a}dx \\
&\quad + \int_\Omega\int_{-\infty}^t \left\{A'^{\alpha\beta\sigma\tau}(t-\tau)E^0_{\sigma\|\tau}(\tau) + A'^{\alpha\beta 33}(t-\tau)E^0_{3\|3}(\tau)\right\}F^0_{\alpha\|\beta}(\tau,\boldsymbol{\eta})\sqrt{a}d\tau\, dx \\
&\quad + \int_\Omega \left\{A^{33\sigma\tau}(0)E^0_{\sigma\|\tau}(t) + A^{3333}(0)E^0_{3\|3}(t)\right\}F^0_{3\|3}(t,\boldsymbol{\eta})\sqrt{a}dx \\
&\quad + \int_\Omega\int_{-\infty}^t \left\{A'^{33\sigma\tau}(t-\tau)E^0_{\sigma\|\tau}(\tau) + A'^{3333}(t-\tau)E^0_{3\|3}(\tau)\right\}F^0_{3\|3}(\tau,\boldsymbol{\eta})\sqrt{a}d\tau\, dx \\
&= \int_\Omega f^{i,0}(t)\eta_i\sqrt{a}dx + \int_{\Gamma_+\cup\Gamma_-} h^{i,1}(t)\eta_i\sqrt{a}d\Gamma
\end{aligned} \tag{3.21}$$

for all $\boldsymbol{\eta} \in \mathbf{W}(\omega) = \{\boldsymbol{\eta} \in \mathbf{W}^{1,4}(\omega); \boldsymbol{\eta} = 0 \text{ on } \gamma_0\}$, that is,

$$\begin{aligned}
&\int_\Omega u^0_{itt}\eta_j a^{ij}\sqrt{a}dx \\
&\quad + \int_\Omega \frac{\partial}{\partial t}\left(\int_{-\infty}^t \left\{A^{\alpha\beta\sigma\tau}(t-\tau)E^0_{\sigma\|\tau}(\tau) + A^{\alpha\beta 33}(t-\tau)E^0_{3\|3}(\tau)\right\}F^0_{\alpha\|\beta}(\tau,\boldsymbol{\eta})d\tau\right)\sqrt{a}dx \\
&\quad + \int_\Omega \frac{\partial}{\partial t}\left(\int_{-\infty}^t \left\{A^{33\sigma\tau}(t-\tau)E^0_{\sigma\|\tau}(\tau) + A^{3333}(t-\tau)E^0_{3\|3}(\tau)\right\}F^0_{3\|3}(\tau,\boldsymbol{\eta})d\tau\right)\sqrt{a}dx \\
&= \int_\Omega f^{i,0}(t)\eta_i\sqrt{a}dx + \int_{\Gamma_+\cup\Gamma_-} h^{i,1}(t)\eta_i\sqrt{a}d\Gamma.
\end{aligned} \tag{3.22}$$

Setting

$$\begin{aligned}
C(t) :=& \int_{-\infty}^t \left\{A^{\alpha\beta\sigma\tau}(t-\tau)E^0_{\sigma\|\tau}(\tau) + A^{\alpha\beta 33}(t-\tau)E^0_{3\|3}(\tau)\right\}F^0_{\alpha\|\beta}(\tau,\boldsymbol{\eta})d\tau \\
&+ \int_{-\infty}^t \left\{A^{33\sigma\tau}(t-\tau)E^0_{\sigma\|\tau}(\tau) + A^{3333}(t-\tau)E^0_{3\|3}(\tau)\right\}F^0_{3\|3}(\tau,\boldsymbol{\eta})d\tau,
\end{aligned} \tag{3.23}$$

we have

$$\begin{aligned}
\widehat{C}(s) =& \widehat{A}^{\alpha\beta\sigma\tau}(s)\widehat{E}^0_{\sigma\|\tau}(s) * \widehat{F}^0_{\alpha\|\beta}(s,\boldsymbol{\eta}) + \widehat{A}^{\alpha\beta 33}(s)\widehat{E}^0_{3\|3}(s) * \widehat{F}^0_{\alpha\|\beta}(s,\boldsymbol{\eta}) \\
&+ \widehat{A}^{33\sigma\tau}(s)\widehat{E}^0_{\sigma\|\tau}(s) * \widehat{F}^0_{3\|3}(s,\boldsymbol{\eta}) + \widehat{A}^{3333}(s)\widehat{E}^0_{3\|3} * \widehat{F}^0_{3\|3}(s,\boldsymbol{\eta}),
\end{aligned} \tag{3.24}$$

where $*$ denotes convolution. Substituting (3.12) into (3.24), we get

$$\begin{aligned}\widehat{C}(s) &= \widehat{A}^{\alpha\beta\sigma\tau}(s)\widehat{E}^0_{\sigma\|\tau}(s) * \widehat{F}^0_{\alpha\|\beta}(s,\boldsymbol{\eta}) - \frac{\widehat{\lambda}(s)}{\widehat{\lambda}(s)+2\widehat{\mu}(s)}\widehat{A}^{\alpha\beta 33}(s)a^{\sigma\tau}\widehat{E}^0_{\sigma\|\tau}(s) * \widehat{F}^0_{\alpha\|\beta}(s,\boldsymbol{\eta}) \\ &\quad + \widehat{A}^{33\sigma\tau}(s)\widehat{E}^0_{\sigma\|\tau}(s) * \widehat{F}^0_{3\|3}(s,\boldsymbol{\eta}) - \frac{\widehat{\lambda}(s)}{\widehat{\lambda}(s)+2\widehat{\mu}(s)}\widehat{A}^{3333}(s)a^{\sigma\tau}\widehat{E}^0_{\sigma\|\tau}(s) * \widehat{F}^0_{3\|3}(s,\boldsymbol{\eta}) \\ &= \widehat{\mu}(s)\left(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}\right)\widehat{E}^0_{\sigma\|\tau}(s) * \widehat{F}^0_{\alpha\|\beta}(s,\boldsymbol{\eta}) + \frac{2\widehat{\mu}(s)\widehat{\lambda}(s)}{\widehat{\lambda}(s)+2\widehat{\mu}(s)}a^{\alpha\beta}a^{\sigma\tau}\widehat{E}^0_{\sigma\|\tau}(s) * \widehat{F}^0_{\alpha\|\beta}(s,\boldsymbol{\eta}).\end{aligned} \tag{3.25}$$

Applying the inverse Laplace transformation to (3.25), we obtain

$$C(t) = \int_{-\infty}^{t} a^{\alpha\beta\sigma\tau}(t-\tau)E^0_{\sigma\|\tau}(\tau)F^0_{\alpha\|\beta}(\tau,\boldsymbol{\eta})d\tau, \tag{3.26}$$

where

$$\begin{aligned} a(t) &:= L^{-1}\left\{\frac{2\widehat{\lambda}(s)\widehat{\mu}(s)}{\widehat{\lambda}(s)+2\widehat{\mu}(s)}\right\}, \\ a^{\alpha\beta\sigma\tau}(t) &:= a(t)a^{\alpha\beta}a^{\sigma\tau} + \mu(t)\left(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}\right). \end{aligned} \tag{3.27}$$

Inserting (3.26) into (3.22), we get the equation in Theorem 2.1.

Since $\mathbf{u}^0(t)$ is independent of x_3, it may be identified with a function $\boldsymbol{\zeta}^0(t) \in L^\infty(0,T;\mathbf{W}(\Omega))$. Consequently, the functions

$$\begin{aligned} E^0_{\alpha\|\beta}(t) &:= \frac{1}{2}\left(\mathbf{u}^0_{\beta\|\alpha}(t) + \mathbf{u}^0_{\beta\|\alpha}(t) + a^{mn}u^0_{m\|\alpha}(t)u^0_{n\|\beta}(t)\right) \in L^\infty\left(0,T;L^2(\Omega)\right), \\ F^0_{\alpha\|\beta}(t,\boldsymbol{\eta}) &:= \frac{1}{2}\left(\eta_{\alpha\|\beta} + \eta_{\beta\|\alpha} + a^{mn}\left\{u^0_{m\|\alpha}(t)\eta_{n\|\beta} + u^0_{n\|\beta}(t)\eta_{m\|\alpha}\right\}\right) \in L^\infty\left(0,T;L^2(\Omega)\right), \end{aligned} \tag{3.28}$$

which are thus also independent of x_3, may be likewise identified with functions (denoted for convenience by the same symbols)

$$\begin{aligned} E^0_{\alpha\|\beta}(t) &:= \frac{1}{2}\left(\zeta^0_{\alpha\|\beta}(t) + \zeta^0_{\beta\|\alpha}(t) + a^{mn}\zeta^0_{m\|\alpha}(t)\zeta^0_{n\|\beta}(t)\right) \in L^\infty\left(0,T;L^2(\omega)\right), \\ F^0_{\alpha\|\beta}(t,\boldsymbol{\eta}) &:= \frac{1}{2}\left(\eta_{\alpha\|\beta} + \eta_{\beta\|\alpha} + a^{mn}\left\{\zeta^0_{m\|\alpha}(t)\eta_{n\|\beta} + \zeta^0_{n\|\beta}(t)\eta_{m\|\alpha}\right\}\right) \in L^\infty\left(0,T;L^2(\omega)\right), \end{aligned} \tag{3.29}$$

where

$$\eta_{\alpha\|\beta} = \partial_\beta\eta_\alpha - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - b_{\alpha\beta}\eta_3, \qquad \eta_{3\|\eta} := \partial_3\eta_3 + b^\sigma_\beta\eta_\sigma, \tag{3.30}$$

for all $\boldsymbol{\eta} \in \mathbf{W}(\omega)$. The last variational problem is thus indeed two-dimensional.

The definition of $a(t)$ implies

$$\widehat{a}(s)\left(\widehat{\lambda}(s)+2\widehat{\mu}(s)\right)=2\widehat{\lambda}(s)\widehat{\mu}(s). \tag{3.31}$$

Applying the inverse Laplace transformation to (3.31), we get

$$\int_0^t a(t-\tau)\left(\lambda(\tau)+2\mu(\tau)\right)d\tau=\int_0^t 2\lambda(t-\tau)\mu(\tau)d\tau. \tag{3.32}$$

Therefore

$$a(0)\left(\lambda(t)+2\mu(t)\right)+\int_0^t a'(t-\tau)\left(\lambda(\tau)+2\mu(\tau)\right)d\tau=2\lambda(0)\mu(t)+\int_0^t 2\lambda'(t-\tau)\mu(\tau)d\tau. \tag{3.33}$$

Letting $t=0$ in (3.33), we obtain immediately that

$$a(0)=\frac{2\lambda(0)\mu(0)}{\lambda(0)+2\mu(0)}. \tag{3.34}$$

□

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