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Research Article

Existence Results for Solutions of Nonlinear Fractional Differential Equations

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This paper deals with theoretical and constructive existence results for solutions of nonlinear fractional differential equations using the method of upper and lower solutions which generate a closed set. The existence of solutions for nonlinear fractional differential equations involving Riemann-Liouville differential operator in a closed set is obtained by utilizing various types of coupled upper and lower solutions. Furthermore, these results are extended to the finite systems of nonlinear fractional differential equations leading to more general results.

1. Introduction

Fractional derivative, introduced around the 17th century, was developed almost until the 19th century. Although the introduction of the concept of fractional calculus involving fractional differentiation and integral is a few centuries old, it was realized only a few decades ago that these functional operations play an important role in various fields of science and engineering [1–8]. As a reason, since the significance of the fractional calculus has been more clearly perceived, many quality researches have been put forward on this branch of mathematical analysis in the literature (see [9–11] and the references therein), and many physical phenomena, chemical processes, biological systems, and so forth have described with fractional derivatives. In this framework, fractional differential equations have been gaining much interest and attracting the attention of many researchers. Some recent contributions on fractional differential equations can be seen in [9–20] and the references within.

On the other hand, the study for solutions of fractional ordinary and partial differential equations has received great interest by scientists. Especially, in the last decade, there are noteworthy works on the analytical and numerical solutions of fractional partial differential equations (see [21–28] and the references there in).

The attention drawn to basic theoretical concepts like the theory of existence and uniqueness of solutions to nonlinear fractional-order differential equations is obvious. Recently, there have been many paper investigating the existence and uniqueness of fractional-order differential equations [29–34].

An interesting and fruitful technique for providing existence results for nonlinear problems is the method of upper and lower solutions. This technique permits us to establish the existence results in a closed set, namely, the ordered interval, generated by upper and lower solutions. Thus, in this context, we are concerned with the existence of solutions of the following nonlinear fractional-order initial value problem (IVP):

$$D^q x(t) = F(t,x), \quad t \in J = [t_0, T], \qquad x(t)(t-t_0)^{1-q}\Big|_{t=t_0} = x^0, \tag{1.1}$$

where $F \in C[J \times \mathbb{R}, \mathbb{R}]$ and D^q is Riemann-Liouville (R-L) fractional derivative of order q, $0 < q < 1$.

The corresponding Volterra fractional integral equation of (1.1) is defined as

$$x(t) = \frac{x^0 (t-t_0)^{q-1}}{\Gamma(q)} + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} F(s, x(s)) ds. \tag{1.2}$$

In recent years, Lakshmikantham and Vatsala investigated the existence theory and established a Peanos type local existence theorem for (1.1) by using integral inequalities and perturbation techniques [35]. McRae also studied an important existence result utilizing the method of upper and lower solutions [36], by means of which, monotone iterative and quasilinearization techniques are developed to fractional differential equations [37–40].

In this paper, we utilize the technique of upper and lower solutions and establish some existence results in terms of various types of coupled upper and lower solutions. Then we will extend this idea to the finite systems of nonlinear fractional differential equations.

The organization of this paper is as follows. In Section 2, we provide necessary background. In Section 3, we focus on the existence of solutions of nonlinear fractional differential equations in a sector. Finally, in Section 4, we extend our results to the finite systems of fractional differential equations.

2. Preliminaries

Now, we present some basic definitions and theorems which are used throughout the paper.

Definition 2.1. Let $p = 1 - q$, then a function $\sigma(t)$ is said to be a C_p *function* if $\sigma \in C_p$ where

$$C_p[J, \mathbb{R}] = \{u \in C[(t_0, T], \mathbb{R}] : u(t)(t-t_0)^p \in C[J, \mathbb{R}]\}. \tag{2.1}$$

If we replace $F(t,x)$ in (1.1) by the sum of two functions such that $F = f + g$ where $f, g \in C[J \times \mathbb{R}, \mathbb{R}]$, then problem (1.1) takes the following form:

$$D^q x(t) = f(t,x) + g(t,x), \qquad x(t)(t-t_0)^{1-q}\Big|_{t=t_0} = x^0. \tag{2.2}$$

We give a variety of possible definitions of upper and lower solutions relative to (2.2).

Definition 2.2. Let $v,\ w \in C_p[J,\mathbb{R}]$, $p = 1-q$, $0 < q < 1$ be locally Hölder continuous with exponent $\lambda > q, D^q v,\ D^q w$ exist, and $f, g \in C[J \times \mathbb{R}, \mathbb{R}]$, then v and w are said to be as follows:

(i) *natural upper and lower solutions* of (2.2), respectively, if

$$\begin{aligned} &D^q v \le f(t,v) + g(t,v), \quad v^0 \le x^0, \\ &D^q w \ge f(t,w) + g(t,w), \quad w^0 \ge x^0,\ t \in J, \end{aligned} \tag{2.3}$$

(ii) *coupled upper and lower solutions of type I* of (2.2), respectively, if

$$\begin{aligned} &D^q v \le f(t,v) + g(t,w), \quad v^0 \le x^0, \\ &D^q w \ge f(t,w) + g(t,v), \quad w^0 \ge x^0,\ t \in J, \end{aligned} \tag{2.4}$$

(iii) *coupled upper and lower solutions of type II of* (2.2), respectively, if

$$\begin{aligned} &D^q v \le f(t,w) + g(t,v), \quad v^0 \le x^0, \\ &D^q w \ge f(t,v) + g(t,w), \quad w^0 \ge x^0,\ t \in J, \end{aligned} \tag{2.5}$$

(iv) *coupled upper and lower solutions of type III of* (2.2), respectively, if

$$\begin{aligned} &D^q v \le f(t,w) + g(t,w), \quad v^0 \le x^0, \\ &D^q w \ge f(t,v) + g(t,v), \quad w^0 \ge x^0,\ t \in J, \end{aligned} \tag{2.6}$$

where $v^0 = v(t)(t-t_0)^{1-q}|_{t=t_0}$ and $w^0 = w(t)(t-t_0)^{1-q}|_{t=t_0}$.

Lemma 2.3. *Let $m \in C_p([t_0,T],R)$ be locally Hölder continuous with exponent $\lambda > q$ and for any $t_1 \in (t_0,T]$, and one has*

$$m(t_1) = 0, \quad m(t) \le 0 \quad \text{for } t_0 \le t \le t_1, \tag{2.7}$$

then it follows that

$$D^q m(t_1) \ge 0. \tag{2.8}$$

Proof. For the proof, see [16]. □

Remark 2.4. A dual result for Lemma 2.3 is valid.

The explicit solution of the following nonhomogeneous linear fractional differential equation

$$D^q x = \alpha x + f(t), \qquad x^0 = x(t)(t-t_0)^{1-q}\Big|_{t=t_0}, \tag{2.9}$$

involving R-L fractional differential operator of order q $(0 < q < 1)$, is necessary for further development of our main results. In (2.9), α is a real number and $f \in C_p([t_0, T], \mathbb{R})$.

When we apply the method of successive approximations [16] to find the solution $x(t) = x(t, t_0, x^0)$ explicitly for the given nonhomogeneous IVP (2.9), we obtain

$$x(t) = x^0(t-t_0)^{q-1}E_{q,q}\left(\alpha(t-t_0)^q\right) + \int_{t_0}^{t}(t-s)^{q-1}E_{q,q}\left(\alpha(t-s)^q\right)f(s)ds, \quad t \in [t_0, T], \tag{2.10}$$

where $E_{q,q}$ denotes the two-parameter Mittag-Leffler function and $E_{q,q}(t^q) = \sum_{k=0}^{\infty} t^{qk}/\Gamma(q(k+1))$, $q > 0$.

If $f(t) \equiv 0$, we get

$$x(t) = x^0(t-t_0)^{q-1}E_{q,q}\left(\alpha(t-t_0)^q\right), \quad t \in [t_0, T], \tag{2.11}$$

as the solution of the corresponding homogeneous IVP for (2.9).

We next give a Peano's type local existence result and then an existence result in a special closed set generated by upper and lower solutions.

Theorem 2.5. *Assume that $F \in C(\mathbb{R}_0, \mathbb{R}^n)$ and $|F(t,x)| \le M$ on R_0 where $R_0 = \{(t,x) : t_0 \le t \le t_0 + a$ and $|x - x^0(t)| \le b\}$ and $x^0(t) = x^0(t-t_0)^{q-1}/\Gamma(q)$, then IVP* (1.1) *possesses at least one solution $x(t)$ on $[t_0, t_0+\alpha]$ where $\alpha = \min\{a, ((b/M)\Gamma(q+1))^{1/q}\}$.*

For the proof of the theorem, see [35].

If the existence of upper and lower solutions w, v such that $v(t) \le w(t)$, $t \in J$ for IVP (1.1) is known, the existence of solutions can be proved in the closed set

$$\Omega = [(t,x) : v(t) \le x(t) \le w(t),\ t \in [t_0, T]]. \tag{2.12}$$

Theorem 2.6. *Let v and $w \in C_p[[t_0, T], \mathbb{R}]$ be natural upper and lower solutions of IVP* (1.1)*, which are locally Hölder continuous with exponent $\lambda > q$ such that $v(t) \le w(t)$ on $J = [t_0, T]$ and $f \in C(\Omega, \mathbb{R})$, then there exists a solution $x(t)$ of IVP* (1.1) *satisfying $v(t) \le x(t) \le w(t)$, $t \in [t_0, T]$.*

For the detailed proof of the above theorem, see [36].

3. Existence Theorems

We are in position to give existence of solutions in the closed set Ω.

Theorem 3.1. *Let v and $w \in C_p[[t_0,T],\mathbb{R}]$ be coupled upper and lower solutions of type I of* (2.2) *such that $f, g \in C(\Omega,\mathbb{R})$ and $v(t) \le w(t)$, $t \in [t_0,T]$. Moreover, assume that $g(t,x)$ is nonincreasing in x for each t, then there exists a solution $x(t)$ of* (2.2) *satisfying $v(t) \le x(t) \le w(t)$ on $[t_0,T]$.*

Proof. Let $p : [t_0,T] \times \mathbb{R} \to \mathbb{R}$ be defined by

$$p(t,x) = \min[w(t), \max(x(t), v(t))]. \tag{3.1}$$

Then $f(t,p(t,x)) + g(t,p(t,x))$ defines a continuous extension of $f+g$ to $[t_0,T]\times\mathbb{R}$ which is also bounded since $f+g$ is bounded on Ω. Employing Theorem 2.5, we get the following equation:

$$D^q x(t) = f\left(t,p(t,x)\right) + g\left(t,p(t,x)\right), \qquad x(t)(t-t_0)^{1-q}\Big|_{t=t_0} = x^0, \tag{3.2}$$

having a solution $x(t)$ on $[t_0,T]$. We wish to prove that $v(t) \le x(t) \le w(t)$ on $[t_0,T]$. For this purpose, consider the following equations:

$$w_\epsilon(t) = w(t) + \epsilon\gamma(t), \qquad v_\epsilon(t) = v(t) - \epsilon\gamma(t), \tag{3.3}$$

where $\gamma(t) = (t-t_0)^{q-1}E_{q,q}((t-t_0)^q)$ and $\varepsilon > 0$. This implies that

$$\begin{aligned} w_\epsilon(t)(t-t_0)^{1-q}\Big|_{t=t_0} &= w_\epsilon^0 = w(t)(t-t_0)^{1-q}\Big|_{t=t_0} + \epsilon\gamma(t)(t-t_0)^{1-q}\Big|_{t=t_0}, \\ v_\epsilon(t)(t-t_0)^{1-q}\Big|_{t=t_0} &= v_\epsilon^0 = v(t)(t-t_0)^{1-q}\Big|_{t=t_0} - \epsilon\gamma(t)(t-t_0)^{1-q}\Big|_{t=t_0}, \end{aligned} \tag{3.4}$$

which gives $w_\epsilon^0 = w^0 + \epsilon\gamma^0$, $v_\epsilon^0 = v^0 - \epsilon\gamma^0$ where $\gamma^0 > 0$. It follows that $v_\epsilon^0 < x^0 < w_\epsilon^0$ in view of the upper and lower definitions of $w(t)$ and $v(t)$. It is enough to show that

$$v_\epsilon(t) < x(t) < w_\epsilon(t) \quad \text{on } [t_0,T], \tag{3.5}$$

which proves the claim of the theorem as $\epsilon \to 0$. First, suppose that the inequality $x(t) < w_\epsilon(t)$ on $[t_0,T]$ is not true, then there would exist a $t_1 \in (t_0,T]$ such that

$$x(t_1) = w_\epsilon(t_1), \quad x(t) < w_\epsilon(t) \quad \text{on } [t_0,t_1). \tag{3.6}$$

Hence, $x(t_1) > w(t_1) \geq v(t_1)$, therefore $p(t_1, x(t_1)) = w(t_1)$ and $v(t_1) \leq p(t_1, x(t_1)) \leq w(t_1)$. If we construct $m(t) = x(t) - w_\epsilon(t)$, we get $m(t_1) = 0$ and $m(t) \leq 0,\ t_0 \leq t \leq t_1$. Employing Lemma 2.3,we obtain $D^q m(t_1) \geq 0$ which gives a contradiction

$$\begin{aligned} f(t_1, w(t_1)) + g(t_1, w(t_1)) &= f\left(t_1, p(t_1, x(t_1))\right) + g\left(t_1, p(t_1, x(t_1))\right) \\ &= D^q x(t_1) \\ &\geq D^q w_\epsilon(t_1) \\ &= D^q w(t_1) + \epsilon\gamma(t_1) \\ &> D^q w(t_1) \\ &\geq f(t_1, w(t_1)) + g(t_1, v(t_1)) \\ &\geq f(t_1, w(t_1)) + g(t_1, w(t_1)). \end{aligned} \tag{3.7}$$

Here, we have used the nonincreasing property of $g(t, x)$ in x for each t and the fact that $\gamma(t_1) > 0$.

Similarly, it can be proved that the other side of the inequality (3.5) is valid for $t_0 \leq t \leq T$. To do this, we must show that $v_\epsilon(t) < x(t)$ on $[t_0, T]$. Suppose that it is not true and so there exists a t_1 such that $v_\epsilon(t_1) = x(t_1)$ and $v_\epsilon(t) < x(t)$ for $t_0 \leq t < t_1$, then $x(t_1) < v(t_1) \leq w(t_1)$ and $p(t_1, x(t_1)) = v(t_1)$. Hence, $v(t_1) \leq p(t_1, x(t_1)) \leq w(t_1)$. If we put $m(t) = v_\epsilon(t) - x(t)$, we get $m(t_1) = 0$ and $m(t) \leq 0,\ t_0 \leq t \leq t_1$. Employing Lemma 2.3, we find $D^q m(t_1) \geq 0$. Since the nonincreasing property of $g(t, x)$ in x for each t and the fact that $\gamma(t_1) > 0$, we arrive at the contradiction

$$\begin{aligned} f(t_1, v(t_1)) + g(t_1, v(t_1)) &= f\left(t_1, p(t_1, x(t_1))\right) + g\left(t_1, p(t_1, x(t_1))\right) \\ &= D^q x(t_1) \\ &\leq D^q v_\epsilon(t_1) \\ &= D^q v(t_1) - \epsilon\gamma(t_1) \\ &< D^q v(t_1) \\ &\leq f(t_1, v(t_1)) + g(t_1, w(t_1)) \\ &\leq f(t_1, v(t_1)) + g(t_1, v(t_1)). \end{aligned} \tag{3.8}$$

Consequently, we have $v_\epsilon(t) < x(t) < w_\epsilon(t)$ on $[t_0, T]$, and letting $\epsilon \to 0$, we get $v(t) \leq x(t) \leq w(t)$ on $[t_0, T]$ proving the theorem. □

Theorem 3.2. *Let v and $w \in C_p[[t_0, T], \mathbb{R}]$ be coupled upper and lower solutions of type II of* (2.2) *such that $f, g \in C(\Omega, \mathbb{R})$ and $v(t) \leq w(t),\ t \in [t_0, T]$. Moreover, assume that $f(t, x)$ is nonincreasing in x for each t, then there exists a solution $x(t)$ of* (2.2) *satisfying $v(t) \leq x(t) \leq w(t)$ on $[t_0, T]$.*

Proof. Let $p : [t_0, T] \times \mathbb{R} \to \mathbb{R}$ be defined by

$$p(t, x) = \min[w(t), \max(x(t), v(t))]. \tag{3.9}$$

Then $f(t,p(t,x))+g(t,p(t,x))$ defines a continuous extension of $f+g$ to $[t_0,T]\times\mathbb{R}$ which is also bounded since $f+g$ is bounded on Ω. Therefore, by Theorem 2.5,

$$D^q x(t) = f(t,p(t,x)) + g(t,p(t,x)), \qquad x(t)(t-t_0)^{1-q}\Big|_{t=t_0} = x^0 \tag{3.10}$$

has a solution $x(t)$ on $[t_0,T]$. We intend to show that $v(t)\le x(t)\le w(t)$ on $[t_0,T]$. For this purpose, consider the following equations:

$$w_\epsilon(t) = w(t)+\epsilon\gamma(t), \qquad v_\epsilon(t) = v(t)-\epsilon\gamma(t), \tag{3.11}$$

where $\gamma(t)$ and ε are defined as before. It follows that $v_\epsilon^0 < x^0 < w_\epsilon^0$. It is enough to show that

$$v_\epsilon(t) < x(t) < w_\epsilon(t) \quad \text{on } [t_0,T]. \tag{3.12}$$

Suppose that it is not true. Thus, there would exist a $t_1\in(t_0,T]$ such that

$$x(t_1) = w_\epsilon(t_1), \quad v_\epsilon(t) < x(t) < w_\epsilon(t) \quad \text{on } [t_0,t_1). \tag{3.13}$$

Hence, $x(t_1) > w(t_1) \ge v(t_1)$; therefore, we get $p(t_1,x(t_1)) = w(t_1)$ and $v(t_1)\le p(t_1,x(t_1))\le w(t_1)$. Setting $m(t)=x(t)-w_\epsilon(t)$, we have $m(t_1)=0$ and $m(t)\le 0$, $t_0\le t\le t_1$. Employing Lemma 2.3, we obtain $D^q m(t_1)\ge 0$ which yields a contradiction

$$\begin{aligned} f(t_1,w(t_1))+g(t_1,w(t_1)) &= f(t_1,p(t_1,x(t_1)))+g(t_1,p(t_1,x(t_1)))\\ &= D^q x(t_1)\\ &\ge D^q w_\epsilon(t_1)\\ &= D^q w(t_1)+\epsilon\gamma(t_1)\\ &> D^q w(t_1)\\ &\ge f(t_1,v(t_1))+g(t_1,w(t_1))\\ &\ge f(t_1,w(t_1))+g(t_1,w(t_1)). \end{aligned} \tag{3.14}$$

Here, we have used the nonincreasing property of $f(t,x)$ in x for each t and the fact that $\gamma(t_1)>0$. Thus, we get $x(t)<w_\epsilon(t)$ on $[t_0,T]$.

After utilizing the similar procedure, the other case can be proved easily. Consequently, we have $v_\epsilon(t)<x(t)<w_\epsilon(t)$ on $[t_0,T]$, and letting $\epsilon\to 0$, we get $v(t)\le x(t)\le w(t)$ on $[t_0,T]$ which proves the theorem. □

Theorem 3.3. *Let v and $w\in C_p[[t_0,T],\mathbb{R}]$ be coupled upper and lower solutions of type III of* (2.2) *such that $f,g\in C(\Omega,\mathbb{R})$ and $v(t)\le w(t)$, $t\in[t_0,T]$. Moreover, assume that both $f(t,x)$ and $g(t,x)$ are nonincreasing in x for each t, then there exists a solution $x(t)$ of* (2.2) *satisfying $v(t)\le x(t)\le w(t)$ on $[t_0,T]$.*

Proof. In a similar manner in previous theorems, the existence of the solution can be proved. Thus, we omit the details. □

4. Extensions to the Systems of Differential Equations

We can generalize the result of Theorem 2.6 to finite systems of fractional differential equations. Consider the following fractional differential system:

$$D^q x(t) = F(t,x), \qquad x(t)(t-t_0)^{1-q}\Big|_{t=t_0} = x^0, \tag{4.1}$$

where $F \in C[[t_0,T]\times\mathbb{R}^n,\mathbb{R}^n]$, and $D^q x$ is the fractional derivative of $x \in \mathbb{R}^n$ and $0 < q < 1$, $p = 1-q$.

At this point, we shall need an important property, known as quasimonotone nondecreasing relative to systems of inequalities.

Definition 4.1. A function $F \in C[J\times\mathbb{R}^n,\mathbb{R}^n]$ is said to possess quasimonotone nondecreasing property if $u \le v$, $u_i = v_i$ for some i, $1 \le i \le n$, then $F_i(t,u) \le F_i(t,v)$.

Here, we shall be using vectorial inequalities, which are understood to mean the same inequalities hold between their corresponding components.

Next, we give the following existence result for systems of differential equations.

Theorem 4.2. *Let v and $w \in C_p[[t_0,T],\mathbb{R}^n]$ be upper and lower solutions of* (4.1)*, respectively, such that $F \in C(\Omega,\mathbb{R}^n)$ and $v(t) \le w(t), t \in [t_0,T]$ where $\Omega = [(t,x) : v(t) \le x(t) \le w(t),\ t \in [t_0,T]]$. Moreover, assume that $F(t,x)$ is quasimonotone nondecreasing in x for each t, then there exists a solution $x(t)$ of* (4.1) *satisfying $v(t) \le x(t) \le w(t)$ on $[t_0,T]$.*

The proof of this theorem is a special case of the following theorem in which we choose F not to be quasimonotone nondecreasing in x provided that we strengthen the notion of upper and lower solutions of (4.1) as follows:

For each i, $1 \le i \le n$,

$$\begin{aligned} D^q v_i(t) &\le F_i(t,\rho) \quad \forall\rho \text{ such that } v_i(t) = \rho_i(t), \qquad v(t) \le x(t) \le w(t) \text{ on } [t_0,T], \\ D^q w_i(t) &\le F_i(t,\rho) \quad \forall\rho \text{ such that } w_i(t) = \rho_i(t), \qquad v(t) \le x(t) \le w(t) \text{ on } [t_0,T]. \end{aligned} \tag{4.2}$$

We state and prove the following existence result relative to the definition of upper and lower solutions in (4.2).

Theorem 4.3. *Let v and $w \in C_p[[t_0,T],\mathbb{R}^n]$ be upper and lower solutions of* (4.1)*, respectively, satisfying the relations given in* (4.2)*, which are also locally Hölder continuous with exponent $\lambda > q$ such that $v(t) \le w(t)$ and $F \in C(\Omega,\mathbb{R}^n)$, then there exists a solution $x(t)$ of* (4.1) *satisfying $v(t) \le x(t) \le w(t)$ on $[t_0,T]$.*

Proof. Let $p : [t_0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ given by

$$p(t,x) = \min[w(t),\max(x(t),v(t))] \quad \text{for each } i, \tag{4.3}$$

then $F(t,p(t,x))$ defines a continuous extension of F to $[t_0,T]\times\mathbb{R}^n$ which is also bounded since $f+g$ is bounded on Ω. Therefore, by Theorem 2.5,

$$D^q x(t) = F(t,p(t,x)), \qquad x(t)(t-t_0)^{1-q}\Big|_{t=t_0} = x^0 \tag{4.4}$$

has a solution $x(t)$ on $[t_0,T]$. For $\epsilon>0$ and $e=(1,1,\ldots,1)$, consider $w_\epsilon(t)=w(t)+\epsilon\gamma(t)e$ and $v_\epsilon(t)=v(t)-\epsilon\gamma(t)e$ where $\gamma(t)=(t-t_0)^{q-1}E_{q,q}((t-t_0)^q)$. It is clear that $v_\epsilon^0<x^0<w_\epsilon^0$. We wish to show that $v_\epsilon(t)<x(t)<w_\epsilon(t)$ on $[t_0,T]$. Suppose that it is not true, then there exists an index $j, 1\le j\le n$ and a $t_1\in(t_0,T]$ such that

$$x_j(t_1)=w_{\epsilon j}(t_1), \quad x(t)\le w_\epsilon(t), \quad t_0\le t\le t_1, \qquad x_i(t_1)\le w_{\epsilon i}(t_1) \quad \text{for } i\ne j. \tag{4.5}$$

Thus, we have $v(t_1)\le p(t_1,x(t_1))\le w(t_1)$ and $p_j(t_1,x(t_1))=w_j(t_1)$. Setting $m_j(t)=v_j(t)-w_j(t)$, it follows that

$$m_j(t_1)=0, \quad m_j(t)\le 0, \quad t\in[t_0,t_1]; \quad m_i(t_1)\le 0, \quad i\ne j. \tag{4.6}$$

Applying Lemma 2.3 to the component $m_j(t)$, we get $D^q m_j(t_1)\ge 0$ or $D^q x_j(t_1)\ge D^q w_{\epsilon j}(t_1)$ which yields a contradiction

$$\begin{aligned} F_j(t_1,w(t_1)) &= F_j(t_1,p(t_1,x(t_1))) = D^q x_j(t_1) \\ &\ge D^q w_{\epsilon j}(t_1) \\ &= D^q w_j(t_1)+\epsilon\gamma(t_1) \\ &> D^q w_j(t_1) \\ &\ge F_j(t_1,w(t_1)). \end{aligned} \tag{4.7}$$

Now, letting $\epsilon\to 0$, we arrive at $v(t)\le x(t)\le w(t)$ on $[t_0,T]$ which proves the conclusion of the theorem.

Sometimes, we can have arbitrary coupling relative to upper and lower solutions. Let p_i and q_i be nonnegative integers for each i, $1\le i\le n$, so that we can split the vector x into $(x_i,[x]_{p_i},[x]_{q_i})$. Then the system (4.1) can be written as

$$D^q x_i(t) = F_i\Big(t,x_i,[x]_{p_i},[x]_{q_i}\Big), \qquad x(t)(t-t_0)^{1-q}\Big|_{t=t_0} = x^0, \tag{4.8}$$

where $F\in C[[t_0,T]\times\mathbb{R}^n,\mathbb{R}^n]$. □

Definition 4.4. A function $F\in C[[t_0,T]\times\mathbb{R}^n,\mathbb{R}^n]$ is said to possess a mixed quasimonotone property if for each i, $F_i(t,x_i,[x]_{p_i},[x]_{q_i})$ is monotone nondecreasing in $[x]_{p_i}$ and monotone nonincreasing in $[x]_{q_i}$.

Definition 4.5. The functions v and $w \in C_p[[t_0, T], \mathbb{R}^n]$ are said to be coupled upper and lower quasisolutions of (4.8) if they satisfy

$$
\begin{aligned}
D^q v_i &\le F_i\left(t, v_i, [v]_{p_i}, [v]_{q_i}\right), \quad v^0 \le x^0, \\
D^q w_i &\ge F_i\left(t, w_i, [w]_{p_i}, [w]_{q_i}\right), \quad w^0 \ge x^0,
\end{aligned}
\tag{4.9}
$$

for each i, $1 \le i \le n$.

Next, we give an existence result which is also a special case of Theorem 4.3.

Theorem 4.6. *Let $v, w \in C_p[[t_0, T], \mathbb{R}^n]$ be coupled upper and lower quasisolutions of* (4.8) *and $F \in C[[t_0, T] \times \mathbb{R}^n, \mathbb{R}^n]$. If $F(t, x)$ possesses a mixed quasimonotone property, then there exists a solution $x(t)$ of* (4.8) *such that $v(t) \le x(t) \le w(t)$ on $[t_0, T]$.*

It should be noted that if F satisfies a mixed quasimonotone property, then (4.2) holds for coupled upper and lower quasisolutions given by (4.9). Therefore, Theorem 4.3 includes Theorem 4.6 as a special case.

5. Conclusion

In this work, some existence theorems have been established for nonlinear fractional-order differential equations relative to coupled upper and lower solutions. The differential operator is taken in the Riemann-Liouville sense. For the further developments in applications of dynamical systems, we have generalized these results to the finite systems of nonlinear fractional differential equations. Being defined by a suitable differential operator, the process of finding a solution between upper and lower solutions generating a closed set could be applied to various types of linear and nonlinear fractional partial differential equations as a future work.

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