Research Article

Bifurcation Analysis for a Predator-Prey Model with Time Delay and Delay-Dependent Parameters

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A class of stage-structured predator-prey model with time delay and delay-dependent parameters is considered. Its linear stability is investigated and Hopf bifurcation is demonstrated. Using normal form theory and center manifold theory, some explicit formulae for determining the stability and the direction of the Hopf bifurcation periodic solutions bifurcating from Hopf bifurcations are obtained. Finally, numerical simulations are performed to verify the analytical results.

1. Introduction

Over the past decade, a great many predator-prey models have been developed to describe the interaction between predator and prey. Their dynamical phenomena have been extensively studied because of the wide application in the field of biomathematics. In particular, the appearance of a cycle bifurcating from the equilibrium of an ordinary or a delayed predator-prey model with a single parameter, which is known as a Hopf bifurcation, has attracted much attention due to its theoretical and practical significance [1–5]. But most of the research literature on these models are only connected with parameters which are independent of time delay; thus, the corresponding characteristic equations are easy to deal with. While in most applications of delay differential equations in population dynamics, the need of incorporation of a time delay is often the result of existence of some stage structure [6–8]. Indeed, every population goes through some distinct life stages [9, 10]. Since the through-stage survival rate is often a function of time delay, it is easy to conceive that these models will inevitably involve some delay-dependent parameters. Thus, the corresponding characteristic equations dependent on the delay τ become more complicated. In view of the fact that it is

often difficult to analytically study models with delay-dependent parameters even if only a single discrete delay is present, we resort to the help of computer programs.

In 2008, Wang et al. [11] introduced and investigated the following predator-prey interaction model with time delay and delay-dependent parameters:

$$
\begin{aligned}
\dot{x}(t) &= x(t)[a - bx(t)] - \frac{cx^2(t)y(t)}{1 + \varpi x^2(t)}, \\
\dot{y}(t) &= \frac{cx^2(t - \tau)y(t - \tau)}{1 + \varpi x^2(t - \tau)}e^{-d\tau} - dy(t),
\end{aligned}
\tag{1.1}
$$

where $x(t)$ and $y(t)$ stand for prey and predator density at time t, respectively. a, b, c, ϖ are real positive parameters and the time delay τ is a positive constant. Wang et al. [11] obtained the conditions that guarantee the system asymptotically stable and permanent. For more knowledge about the model, one can see [11].

It is well known that time delays which occur in the interaction between predator-prey will affect the stability of a model by creating instability, oscillation, and chaos phenomena. Based on the discussion above, the main purpose of this paper is to investigate the stability and the properties of Hopf bifurcation of the model (1.1) which involves some delay-dependent parameters. Recently, there are few papers on the topic that involves some delay-dependent parameters, for example, Liu and Zhang [12] investigated the stability and Hopf bifurcation of the following SIS model with nonlinear birth rate:

$$
\begin{aligned}
\dot{I}(t) &= \beta(N(t) - I(t))\frac{I(t)}{N(t)} - (d + \varepsilon + \gamma)I(t), \\
\dot{N}(t) &= \frac{PN(t - \tau)}{1 + qN^3(t - \tau)}e^{-d_1\tau} - dN(t) - \varepsilon I(t).
\end{aligned}
\tag{1.2}
$$

Jiang and Wei [13] studied the stability and Hopf bifurcation of the following SIR model:

$$
\begin{aligned}
\dot{S}(t) &= \mu - \mu S(t) - \frac{\phi I(t)S(t)}{1 + I(t)} + \gamma I(t - \tau)e^{-\mu\tau}, \\
\dot{I}(t) &= \frac{\phi I(t)S(t)}{1 + I(t)} - (\mu + \gamma)I(t).
\end{aligned}
\tag{1.3}
$$

It worth pointing out that Liu and Zhang [12] investigated the Hopf bifurcation of system (1.2) by choosing p (not delay τ) as the bifurcation parameters and Jiang and Wei [13] studied the Hopf bifurcation of system (1.3) by choosing ϕ (not delay τ) as the bifurcation parameters. In this paper, we will investigate the Hopf bifurcation by regarding the delay τ as the bifurcation parameter which is different from the papers [12, 13]. To the best of our knowledge, it is the first time to deal with the stability and Hopf bifurcation of system (1.1).

This paper is organized as follows. In Section 2, the stability of the equilibrium and the existence of Hopf bifurcation at the equilibrium are studied. In Section 3, the direction of Hopf bifurcation and the stability and periodic of bifurcating periodic solutions on the center manifold are determined. In Section 4, numerical simulations are carried out to illustrate the validity of the main results. Some main conclusions are drawn in Section 5.

2. Stability of the Equilibrium and Local Hopf Bifurcations

Throughout this paper, we assume that the following condition

(H1) $ce^{-d\tau} > d\varpi$, $a^2(ce^{-d\tau} - d\varpi) > b^2d$ holds.

The hypothesis (H1) implies that system (1.1) has a unique positive equilibrium $E_*(x^*, y^*)$, where

$$x^* = \sqrt{\frac{d}{ce^{-d\tau} - d\varpi}}, \qquad y^* = \frac{(a - bx^*)(1 + \varpi x^{*2})}{cx^*}. \tag{2.1}$$

The linearized system of (1.1) around $E_*(x^*, y^*)$ takes the form

$$\frac{dx(t)}{dt} = a_1 x(t) - b_1 y(t),$$

$$\frac{dy(t)}{dt} = -dy(t) + c_1 x(t - \tau) + d_1 y(t - \tau), \tag{2.2}$$

where

$$a_1 = a - 2bx^* - \frac{2cx^* y^*}{1 + \varpi x^{*2}} + \frac{2x^{*3} y^* \varpi}{(1 + \varpi x^*)^2}, \qquad b_1 = -\frac{cx^{*2}}{1 + \varpi x^{*2}},$$

$$c_1 = e^{-d\tau}\left[\frac{2cx^* y^*}{1 + \varpi x^{*2}} - \frac{2x^{*3} y^* \varpi}{(1 + \varpi x^*)^2}\right], \qquad d_1 = e^{-d\tau}\frac{cx^{*2}}{1 + \varpi x^{*2}}. \tag{2.3}$$

The associated characteristic equation of (2.2) is

$$P(\lambda, \tau) + Q(\lambda, \tau)e^{-\lambda \tau} = 0, \tag{2.4}$$

where

$$P(\lambda, \tau) = \lambda^2 + (d - a_1)\lambda - a_1 d,$$

$$Q(\lambda, \tau) = -(d_1\lambda - a_1 d_1 - b_1 c_1). \tag{2.5}$$

When $\tau = 0$, then (2.4) becomes

$$\lambda^2 + \left(d - a_1 - d_1^0\right)\lambda + b_1 c_1^0 - a_1 d - a_1 d_1^0 = 0, \tag{2.6}$$

where

$$c_1^0 = \left[\frac{2cx^* y^*}{1 + \varpi x^{*2}} - \frac{2x^{*3} y^* \varpi}{(1 + \varpi x^*)^2}\right], \qquad d_1^0 = \frac{cx^*}{1 + \varpi x^{*2}}. \tag{2.7}$$

It is easy to obtain the following result.

Lemma 2.1. *If the condition*

(H2) $d - a_1 - d_1^0 > 0$, $b_1 c_1^0 - a_1 d - a_1 d_1^0 > 0$,

holds, then the positive equilibrium $E_*(x^*, y^*)$ of system (1.1) is asymptotically stable.

In the following, one investigates the existence of purely imaginary roots $\lambda = i\omega$ ($\omega > 0$) of (2.4). Equation (2.4) takes the form of a second-degree exponential polynomial in λ, which some of the coefficients of P and Q depend on τ. Beretta and Kuang [14] established a geometrical criterion which gives the existence of purely imaginary roots of a characteristic equation with delay-dependent coefficients. In order to apply the criterion due to Beretta and Kuang [14], one needs to verify the following properties for all $\tau \in [0, \tau_{max})$, where τ_{max} is the maximum value which $E_*(x^*, y^*)$ exists.

(a) $P(0, \tau) + Q(0, \tau) \neq 0$;

(b) $P(i\omega, \tau) + Q(i\omega, \tau) \neq 0$;

(c) $\lim \sup\{|Q(\lambda, \tau)/P(\lambda, \tau)| : |\lambda| \to \infty, \operatorname{Re} \lambda \geq 0\} < 1$;

(d) $F(\omega, \tau) = |P(i\omega, \tau)|^2 - |Q(i\omega, \tau)|^2$ has a finite number of zeros;

(e) Each positive root $\omega(\tau)$ of $F(\omega, \tau) = 0$ is continuous and differentiable in τ whenever it exists.

Here, $P(\lambda, \tau)$ and $Q(\lambda, \tau)$ are defined as in (2.5), respectively.

Let $\tau \in [0, \tau_{max})$. It is easy to see that

$$P(0, \tau) + Q(0, \tau) = -a_1 d + a_1 d_1 + b_1 c_1 \neq 0, \tag{2.8}$$

which implies that (a) is satisfied, and (b)

$$P(i\omega, \tau) + Q(i\omega, \tau) = -\omega^2 + i\omega(d - a_1) - a_1 d - i\omega d_1 + a_1 d_1 + b_1 c_1$$
$$= -\omega^2 - a_1 d + a_1 d_1 + b_1 c_1 + i\omega(d - a_1 - d_1) \neq 0. \tag{2.9}$$

From (2.4), one has

$$\lim_{|\lambda| \to +\infty} \left| \frac{Q(\lambda, \tau)}{P(\lambda, \tau)} \right| = \lim_{|\lambda| \to +\infty} \left| \frac{-(d_1\lambda + a_1 d_1 - b_1 c_1)}{\lambda^2 + (d - a_1)\lambda - a_1 d} \right| = 0. \tag{2.10}$$

Therefore, (c) follows. Let F be defined as in (d). From

$$|P(i\omega, \tau)|^2 = \left(\omega^2 + a_1 d\right)^2 + (d - a_1)^2 \omega^2$$
$$= \omega^4 + \left(d^2 + a_1^2\right)\omega^2 + a_1^2 d^2, \tag{2.11}$$
$$|Q(i\omega, \tau)|^2 = d_1^2 \omega^2 + (a_1 d_1 + b_1 c_1)^2,$$

one obtain

$$F(\omega, \tau) = \omega^4 + \left(d^2 + a_1^2 - d_1^2\right)\omega^2 + a_1 d^2 - (a_1 d_1 + b_1 c_1)^2. \tag{2.12}$$

Obviously, property (d) is satisfied, and by implicit function theorem, (e) is also satisfied.

Now let $\lambda = i\omega \, (\omega > 0)$ be a root of (2.4). Substituting it into (2.4) and separating the real and imaginary parts yields

$$
\begin{aligned}
(a_1 d_1 + b_1 c_1) \cos \omega \tau - d_1 \omega \sin \omega \tau &= \omega^2 + a_1 d, \\
d_1 \omega \cos \omega \tau + (a_1 d_1 + b_1 c_1) \sin \omega \tau &= (d - a_1)\omega.
\end{aligned}
\tag{2.13}
$$

From (2.13), it follows that

$$
\begin{aligned}
\sin \omega \tau &= -\frac{(\omega^2 + a_1 d) d_1 \omega - (d - a_1)\omega(a_1 d_1 + b_1 c_1)}{d_1^2 \omega^2 + (a_1 d_1 + b_1 c_1)^2}, \\
\cos \omega \tau &= \frac{(\omega^2 + a_1 d)(a_1 d_1 + b_1 c_1) + (d - a_1)\omega d_1 \omega}{d_1^2 \omega^2 + (a_1 d_1 + b_1 c_1)^2}.
\end{aligned}
\tag{2.14}
$$

By the definitions of P and Q as in (2.5), respectively, and applying the property (a), then (2.14) can be written as

$$
\begin{aligned}
\sin \omega \tau &= \ \mathrm{Im}\left[\frac{P(i\omega, \tau)}{Q(i\omega, \tau)}\right], \\
\cos \omega \tau &= -\mathrm{Re}\left[\frac{P(i\omega, \tau)}{Q(i\omega, \tau)}\right],
\end{aligned}
\tag{2.15}
$$

which yields $|P(i\omega, \tau)|^2 = |Q(i\omega, \tau)|^2$. Assume that $I \in R_{+0}$ is the set where $\omega(\tau)$ is a positive root of

$$
F(\omega, \tau) = |P(i\omega, \tau)|^2 - |Q(i\omega, \tau)|^2,
\tag{2.16}
$$

and for $\tau \bar{\in} I, \omega(\tau)$ is not definite. Then for all τ in $I, \omega(\tau)$ satisfied $F(\omega, \tau) = 0$. The polynomial function F can be written as

$$
F(\omega, \tau) = h\left(\omega^2, \tau\right),
\tag{2.17}
$$

where h is a second degree polynomial, defined by

$$
h(z, \tau) = z^2 + \left(d^2 + a_1^2 - d_1^2\right) z + a_1 d^2 - (a_1 d_1 + b_1 c_1)^2.
\tag{2.18}
$$

It is easy to see that

$$
h(z, \tau) = z^2 + \left(d^2 + a_1^2 - d_1^2\right) z + a_1 d^2 - (a_1 d_1 - b_1 c_1)^2 = 0
\tag{2.19}
$$

has only one positive real root if the following condition (H3) holds:

(H3) $a_1 d^2 < (a_1 d_1 + b_1 c_1)^2$.

One denotes this positive real root by z_+. Hence, (2.17) has only one positive real root $\omega = \sqrt{z_+}$. Since the critical value of τ and $\omega(\tau)$ are impossible to solve explicitly, so one will use the procedure described in Beretta and Kuang [14]. According to this procedure, one defines $\theta(\tau) \in [0, 2\pi)$ such that $\sin \theta(\tau)$ and $\cos \theta(\tau)$ are given by the righthand sides of (2.14), respectively, with $\theta(\tau)$ given by (2.19). This define $\theta(\tau)$ in a form suitable for numerical evaluation using standard software. And the relation between the argument θ and $\omega\tau$ in (2.18) for $\tau > 0$ must be $\omega\tau = \theta + 2n\pi$, $n = 1, 2, \ldots$.

Hence, one can define the maps: $\tau_n : I \to R_{+0}$ given by

$$\tau_n(\tau) := \frac{\theta(\tau) + 2n\pi}{\omega(\tau)}, \quad \tau_n > 0, \ n = 0, 1, 2, \ldots, \tag{2.20}$$

where a positive root $\omega(\tau)$ of $F(\omega, \tau) = 0$ exists in I. Let us introduce the functions $S_n(\tau)$: $I \to R$,

$$S_n(\tau) = \tau - \frac{\theta(\tau) + 2n\pi}{\omega(\tau)}, \quad n = 0, 1, 2, \ldots, \tag{2.21}$$

which are continuous and differentiable in τ. Thus, one gives the following theorem which is due to Beretta and Kuang [14].

Theorem 2.2. *Assume that $\omega(\tau)$ is a positive root of (2.4) defined for $\tau \in I, I \subseteq R_{+0}$, and at some $\tau_0 \in I, S_n(\tau_0) = 0$ for some $n \in N_0$. Then, a pair of simple conjugate pure imaginary roots $\lambda = \pm i\omega$ exists at $\tau = \tau_0$ which crosses the imaginary axis from left to right if $\delta(\tau_0) > 0$ and crosses the imaginary axis from right to left if $\delta(\tau_0) < 0$, where $\delta(\tau_0) = \text{sign}[F'_\omega(\omega\tau_0, \tau_0)] \, \text{sign}[(dS_n(\tau))/d\tau|_{\tau=\tau_0}]$.*

Applying Lemma 2.1 and the Hopf bifurcation theorem for functional differential equation [5], we can conclude the existence of a Hopf bifurcation as stated in the following theorem.

Theorem 2.3. *For system (1.1), if (H1)–(H3) hold, then there exists s $\tau_0 \in I$ such that the positive equilibrium $E_*(x^*, y^*)$ is asymptotically stable for $0 \leq \tau < \tau_0$ and becomes unstable for τ staying in some right neighborhood of τ_0, with a Hopf bifurcation occurring when $\tau = \tau_0$.*

3. Direction and Stability of the Hopf Bifurcation

In the previous section, we obtained some conditions which guarantee that the stage-structured predator-prey model with time delay undergoes the Hopf bifurcation at some values of $\tau = \tau_0$. In this section, we will derive the explicit formulae determining the direction, stability, and period of these periodic solutions bifurcating from the positive equilibrium $E_*(x^*, y^*)$ at these critical value of τ, by using techniques from normal form and center manifold theory [15]. Throughout this section, we always assume that system (1.1) undergoes Hopf bifurcation at the positive equilibrium $E_*(x^*, y^*)$ for $\tau = \tau_0$, and then $\pm i\omega_0$ is corresponding purely imaginary roots of the characteristic equation at the equilibrium $E_*(x^*, y^*)$.

For convenience, let $\tau = \tau_0 + \mu, \mu \in R$. Then $\mu = 0$ is the Hopf bifurcation value of (1.1). Thus, one will study Hopf bifurcation of small amplitude periodic solutions of (1.1) from the positive equilibrium point $E_*(x^*, y^*)$ for μ close to 0.

Let $u_1(t) = x(t) - x^*, u_2(t) = y(t) - y^*, x_i(t) = u_i(\tau t), (i = 1, 2), \tau = \tau_0 + \mu$, then system (1.1) can be transformed into an functional differential equation (FDE) in $(C = C(-1, 0], R^2)$ as

$$\frac{du}{dt} = L_\mu(u_t) + f(\mu, u_t), \tag{3.1}$$

where $u(t) = (x_1(t), x_2(t))^T \in R^2$ and $L_\mu : C \to R, f : R \times C \to R$ are given, respectively, by

$$L_\mu \phi = (\tau_0 + \mu) B \phi(0) + (\tau_0 + \mu) G \phi(-1), \tag{3.2}$$

where

$$B = \begin{pmatrix} a_1 & -b_1 \\ 0 & -d \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 \\ c_1 & d_1 \end{pmatrix},$$

$$a_1 = a - 2bx^* - \frac{2cx^*y^*}{1 + \varpi x^{*2}} + \frac{2x^{*3}y^*\varpi}{(1 + \varpi x^*)^2}, \qquad b_1 = -\frac{cx^*}{1 + \varpi x^{*2}},$$

$$c_1 = e^{-d\tau}\left[\frac{2cx^*y^*}{1 + \varpi x^{*2}} - \frac{2x^{*3}y^*\varpi}{(1 + \varpi x^*)^2}\right], \qquad d_1 = e^{-d\tau}\frac{cx^*}{1 + \varpi x^{*2}} \tag{3.3}$$

$$f(\mu, \phi) = \begin{pmatrix} f_1(\mu, \phi) \\ f_2(\mu, \phi) \end{pmatrix},$$

where

$$f_1(\mu, \phi) = (\tau_0 + \mu)\left[m_1\phi_1^2(0) + m_2\phi_1(0)\phi_2(0) + m_3\phi_1^3(0) + m_4\phi_1^2(0)\phi_2(0) + \text{ h.o.t. }\right],$$

$$f_2(\mu, \phi) = (\tau_0 + \mu)\left[n_1\phi_1^2(-1) + n_2\phi_1(-1)\phi_2(-1) + n_3\phi_1^3(-1) + m_4\phi_1^2(-1)\phi_2(-1) + \text{ h.o.t.}\right], \tag{3.4}$$

where

$$m_1 = \frac{4x^{*2}y^*\varpi}{(1 + \varpi x^*)^2} - \frac{cy^*}{1 + \varpi x^{*2}} - \frac{cx^{*2}y^*(4\varpi^2 x^* - \varpi x^{*2} - \varpi)}{(1 + \varpi x^*)^4},$$

$$m_2 = \frac{2c\varpi x^{*3}}{(1 + \varpi x^{*2})^2} - \frac{2cx^*}{1 + \varpi x^{*2}},$$

$$m_3 = \frac{2c\varpi x^* y^*}{(1 + \varpi x^{*2})^2} - \frac{cx^{*2} y^* g}{6} - \frac{2cx^* y^* (4\varpi^2 x^* - \varpi^2 x^{*2} - \varpi)}{(1 + \varpi x^{*2})^4},$$

$$m_4 = \frac{4c\varpi x^{*2}}{(1 + \varpi x^{*2})^2} - \frac{c}{1 + \varpi x^*} - \frac{cx^{*2}(4\varpi^2 x^* - \varpi^2 x^{*2} - \varpi)}{(1 + \varpi x^{*2})^4},$$

$$n_1 = e^{-d\tau_0}\left[\frac{cy^*}{1 + \varpi x^{*2}} + \frac{cx^{*2} y^* (4\varpi^2 x^* - \varpi x^{*2} - \varpi)}{(1 + \varpi x^*)^4} - \frac{4x^{*2} y^* \varpi}{(1 + \varpi x^*)^2}\right],$$

$$n_2 = e^{-d\tau_0}\left[\frac{2cx^*}{1 + \varpi x^{*2}} - \frac{2c\varpi x^{*3}}{(1 + \varpi x^{*2})^2}\right],$$

$$n_3 = e^{-d\tau_0}\left[\frac{cx^{*2} y^* g_1}{6} + \frac{2cx^* y^* (4\varpi^2 x^* - \varpi^2 x^{*2} - \varpi)}{(1 + \varpi x^{*2})^4} - \frac{2c\varpi x^* y^*}{(1 + \varpi x^{*2})^2}\right],$$

$$n_4 = e^{-d\tau_0}\left[\frac{c}{1 + \varpi x^*} + \frac{cx^{*2}(4\varpi^2 x^* - \varpi^2 x^{*2} - \varpi)}{(1 + \varpi x^{*2})^4} - \frac{4c\varpi x^{*2}}{(1 + \varpi x^{*2})^2}\right],$$

$$(3.5)$$

where

$$g = \frac{18\varpi^2 x^* (1 + \varpi^2 x^{*2})^3 (\varpi^2 x^{*4} - 4\varpi^2 x^{*3} + 2\varpi x^{*2} - 4\varpi x^* + 1)}{(1 + \varpi x^{*2})^8}$$
$$- \frac{8\varpi^2 (x^* + 4x^{*3} - 3\varpi x^{*2} - 4\varpi)}{(1 + \varpi x^{*2})^4}.$$

$$(3.6)$$

Clearly, L_μ is a linear continuous operator from C to R^2. By the Riesz representation theorem, there exists a matrix function with bounded variation components $\eta(\theta, \mu), \theta \in [-1, 0]$ such that

$$L_\mu \phi = \int_{-1}^{0} d\eta(\theta, \mu)\phi(\theta), \quad \text{for } \phi \in C. \tag{3.7}$$

In fact, we can choose

$$\eta(\theta, \mu) = (\tau_0 + \mu)\begin{pmatrix} a_1 & -b_1 \\ 0 & -d \end{pmatrix}\delta(\theta) - (\tau_0 + \mu)\begin{pmatrix} 0 & 0 \\ c_1 & d_1 \end{pmatrix}\delta(\theta + 1), \tag{3.8}$$

where δ is the Dirac delta function.

For $\phi \in C([-1,0], R^2)$, define

$$A(\mu)\phi = \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, & -1 \le \theta < 0, \\ \displaystyle\int_{-1}^{0} d\eta(s,\mu)\phi(s), & \theta = 0, \end{cases} \tag{3.9}$$

$$R(\mu)\phi = \begin{cases} 0, & -1 \le \theta < 0, \\ f(\mu,\phi), & \theta = 0. \end{cases}$$

Then (1.1) is equivalent to the abstract differential equation

$$\dot{u}_t = A(\mu)u_t + R(\mu)u_t, \tag{3.10}$$

where $u = (x_1, x_2)^T$, $u_t(\theta) = u(t+\theta)$, $\theta \in [-1,0]$.

For $\psi \in C([0,1], (R^2)^*)$, define

$$A^*\psi(s) = \begin{cases} -\dfrac{d\psi(s)}{ds}, & s \in (0,1], \\ \displaystyle\int_{-1}^{0} d\eta^T(t,0)\psi(-t), & s = 0. \end{cases} \tag{3.11}$$

For $\phi \in C([-1,0], R^2)$ and $\psi \in C([0,1], (R^2)^*)$, define the bilinear form

$$\langle \psi, \phi \rangle = \overline{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \overline{\psi}(\xi - \theta)\, d\eta(\theta)\phi(\xi)\, d\xi, \tag{3.12}$$

where $\eta(\theta) = \eta(\theta, 0)$. We have the following result on the relation between the operators $A = A(0)$ and A^*.

Lemma 3.1. $A = A(0)$ *and* A^* *are adjoint operators.*

The proof is easy from (3.12), so we omit it.

By the discussions in Section 2, we know that $\pm i\omega_0\tau_0$ are eigenvalues of $A(0)$, and they are also eigenvalues of A^* corresponding to $i\omega_0\tau_0$ and $-i\omega_0\tau_0$, respectively. We have the following result.

Lemma 3.2. *The vector*

$$q(\theta) = (1, \gamma)^T e^{i\omega_0\tau_0\theta}, \quad \theta \in [-1, 0], \tag{3.13}$$

where

$$\gamma = \frac{i\omega_0 - a_1}{b_1} \tag{3.14}$$

is the eigenvector of $A(0)$ corresponding to the eigenvalue $i\omega_0\tau_0$, and

$$q^*(s) = D(1, \gamma^*)e^{i\omega_0\tau_0 s}, \quad s \in [0,1], \tag{3.15}$$

where

$$\gamma^* = -\frac{i\omega_0 a_1}{c_1 e^{-i\omega\tau_0}} \tag{3.16}$$

is the eigenvector of A^* corresponding to the eigenvalue $-i\omega_0\tau_0$; moreover, $\langle q^*(s), q(\theta) \rangle = 1$, where

$$D = 1 + \overline{\gamma}\overline{\gamma^*} + c_1\gamma^* e^{i\omega_0\tau_0} + d_1\overline{\gamma}\gamma^* e^{i\omega_0\tau_0}. \tag{3.17}$$

Proof. Let $q(\theta)$ be the eigenvector of $A(0)$ corresponding to the eigenvalue $i\omega_0\tau_0$ and $q^*(s)$ be the eigenvector of A^* corresponding to the eigenvalue $-i\omega_0\tau_0$, namely, $A(0)q(\theta) = i\omega_0\tau_0 q(\theta)$ and $A^*q^{*T}(s) = -i\omega_0\tau_0 q^{*T}(s)$. From the definitions of $A(0)$ and A^*, we have $A(0)q(\theta) = dq(\theta)/d\theta$ and $A^*q^{*T}(s) = -dq^{*T}(s)/ds$. Thus, $q(\theta) = q(0)e^{i\omega_0\tau_0\theta}$ and $q^*(s) = q^*(0)e^{i\omega_0\tau_0 s}$. In addition,

$$\int_{-1}^{0} d\eta(\theta)q(\theta) = \tau_0 Bq(0) + \tau_0 Gq(-1) = \tau_0 A(0)q(0) = i\omega_0\tau_0 q(0). \tag{3.18}$$

That is,

$$\begin{pmatrix} i\omega_0 - a_1 & -b_1 \\ -c_1 e^{-i\omega_0\tau_0} & i\omega_0 + d - d_1 e^{-i\omega_0\tau_0} \end{pmatrix} q(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.19}$$

Therefore, we can easily obtain

$$\gamma = \frac{i\omega_0 - a_1}{b_1}. \tag{3.20}$$

And so

$$q(0) = \left(1, \frac{i\omega_0 - a_1}{b_1}\right)^T, \tag{3.21}$$

and hence

$$q(\theta) = \left(1, \frac{i\omega_0 - a_1}{b_1}\right)^T e^{i\omega_0\tau_0\theta}. \tag{3.22}$$

On the other hand,

$$\int_{-1}^{0} q^*(-t)\, d\eta(t) = \tau_0 B^T q^{*T}(0) + \tau_0 G^T q^{*T}(-1) = \tau_0 A^* q^{*T}(0) = -i\omega_0\tau_0 q^{*T}(0). \tag{3.23}$$

Namely,

$$\begin{pmatrix} -i\omega_0 - a_1 & -c_1 e^{-i\omega_0 \tau_0} \\ b_1 & -i\omega_0 + d - d_1 e^{-i\omega_0 \tau_0} \end{pmatrix} q^{*T}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.24}$$

Therefore, we can easily obtain

$$\gamma^* = -\frac{i\omega_0 a_1}{c_1 e^{-i\omega_0 \tau_0}}, \tag{3.25}$$

and so

$$q^*(0) = \left(1, -\frac{i\omega_0 a_1}{c_1 e^{-i\omega \tau_0}}\right), \tag{3.26}$$

and hence

$$q^*(s) = \left(1, -\frac{i\omega_0 a_1}{c_1 e^{-i\omega \tau_0}}\right) e^{i\omega_0 \tau_0 s}. \tag{3.27}$$

In the sequel, one will verify that $\langle q^*(s), q(\theta) \rangle = 1$. In fact, from (3.12), we have

$$\begin{aligned}
\langle q^*(s), q(\theta) \rangle &= \overline{D}\left(1, \overline{\gamma^*}\right)(1, \gamma)^T \\
&\quad - \int_{-1}^{0} \int_{\xi=0}^{0} \overline{D}\left(1, \overline{\gamma^*}\right) e^{-i\omega_0(\xi-\theta)} \, d\eta(\theta)(1,\gamma)^T e^{i\omega_0\xi} \, d\xi \\
&= \overline{D}\left[1 + \gamma\overline{\gamma^*} - \int_{-1}^{0}\left(1, \overline{\gamma^*}\right)\theta e^{i\omega_0\theta} \, d\eta(\theta)(1,\gamma)^T\right] \\
&= \overline{D}\left\{1 + \gamma\overline{\gamma^*} - \left(1, \overline{\gamma^*}\right)\left[-\tau_0 C e^{-i\omega_0\tau_0}\right](1,\gamma)^T\right\} \\
&= \overline{D}\left[1 + \gamma\overline{\gamma^*} + c_1\overline{\gamma^*}e^{-i\omega_0\tau_0} + d_1\gamma\overline{\gamma^*}e^{-i\omega_0\tau_0}\right] = 1.
\end{aligned} \tag{3.28}$$

Next, we use the same notations as those in Hassard et al. [15], and we first compute the coordinates to describe the center manifold C_0 at $\mu = 0$. Let u_t be the solution of (1.1) when $\mu = 0$.

Define

$$z(t) = \langle q^*, u_t \rangle, \qquad W(t,\theta) = u_t(\theta) - 2\operatorname{Re}\{z(t)q(\theta)\}, \tag{3.29}$$

on the center manifold C_0, and we have

$$W(t,\theta) = W(z(t), \overline{z}(t), \theta), \tag{3.30}$$

where

$$W(z(t), \overline{z}(t), \theta) = W(z, \overline{z}) = W_{20}\frac{z^2}{2} + W_{11}z\overline{z} + W_{02}\frac{\overline{z}^2}{2} + \cdots, \tag{3.31}$$

and z and $\overline{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\overline{q}^*$. Noting that W is also real if u_t is real, we consider only real solutions. For solutions $u_t \in C_0$ of (1.1),

$$
\begin{aligned}
\dot{z}(t) &= \langle q^*(s), \dot{u}_t \rangle = \langle q^*(s), A(0)u_t + R(0)u_t \rangle \\
&= \langle q^*(s), A(0)u_t \rangle + \langle q^*(s), R(0)u_t \rangle \\
&= \langle A^*q^*(s), u_t \rangle + \overline{q}^*(0)R(0)u_t - \int_{-1}^{0}\int_{\xi=0}^{\theta} \overline{q^*}(\xi - \theta)\, d\eta(\theta)A(0)R(0)u_t(\xi)\, d\xi \\
&= \langle i\omega_0 q^*(s), u_t \rangle + \overline{q}^*(0)f(0, u_t(\theta)) \\
&\overset{\text{def}}{=} i\omega_0 z(t) + \overline{q}^*(0)f_0(z(t), \overline{z}(t)).
\end{aligned}
\tag{3.32}
$$

That is,

$$\dot{z}(t) = i\omega_0 z + g(z, \overline{z}), \tag{3.33}$$

where

$$g(z, \overline{z}) = g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots. \tag{3.34}$$

Hence, we have

$$g(z, \overline{z}) = \overline{q}^*(0)f_0(z, \overline{z}) = f(0, u_t) = \overline{D}(1, \gamma^*)\left(f_1(0, u_t), f_2(0, u_t)\right)^T, \tag{3.35}$$

where

$$
\begin{aligned}
f_1(0, u_t) &= \tau_0\left[m_1 x_t^2(0) + m_2 x_t(0)y_t(0) + m_3 x_t^3(0) + m_4 x_t^2(0)y_t(0) + \text{ h.o.t. }\right], \\
f_2(0, u_t) &= \tau_0\left[n_1 x_t^2(-1) + n_2 x_t(-1)y_t(-1) + n_3 x_t^3(-1) + m_4 x_t^2(-1)y_t(-1) + \text{ h.o.t. }\right].
\end{aligned}
\tag{3.36}
$$

Noticing $u_t(\theta) = (x_t(\theta), y_t(\theta))^T = W(t,\theta) + zq(\theta) + \overline{z}\,\overline{q}(\overline{\theta})$ and $q(\theta) = (1, \gamma)^T e^{i\omega_0\tau_0\theta}$, we have

$$x_t(0) = z + \overline{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\overline{z} + W_{02}^{(1)}(0)\frac{\overline{z}^2}{2} + \cdots,$$

$$y_t(0) = \gamma z + \overline{\gamma}\,\overline{z} + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\overline{z} + W_{02}^{(2)}(0)\frac{\overline{z}^2}{2} + \cdots,$$

$$x_t(-1) = e^{-i\omega_0\tau_0}z + e^{i\omega_0\tau_0}\overline{z} + W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\overline{z} + W_{02}^{(1)}(-1)\frac{\overline{z}^2}{2} + \cdots,$$

$$y_t(-1) = \gamma e^{-i\omega_0\tau_0}\beta z + \overline{\gamma}e^{i\omega_0\tau_0}\overline{z} + W_{20}^{(2)}(-1)\frac{z^2}{2} + W_{11}^{(2)}(-1)z\overline{z} + W_{02}^{(2)}(-1)\frac{\overline{z}^2}{2} + \cdots.$$

(3.37)

From (3.34) and (3.35), we have

$$
\begin{aligned}
g(z,\overline{z}) &= \overline{q}^*(0)f_0(z,\overline{z}) = \overline{D}\Big[f_1(0,x_t) + \overline{\gamma^*}\,f_2(0,x_t)\Big] \\
&= \overline{D}\tau_0\Big[(m_1 + m_2\gamma) + \overline{\gamma^*}(n_1 + n_2\gamma)\Big]z^2 \\
&\quad + \overline{D}\tau_0\Big[2m_1 + m_2(\gamma + \overline{\gamma^*})(2n_1 + n_2(\gamma + \overline{\gamma^*}))\Big]z\overline{z} \\
&\quad + \overline{D}\tau_0\Big[m_1 + m_2\overline{\gamma} + \overline{\gamma^*}(n_1 + n_2\overline{\gamma})\Big]\overline{z}^2 \\
&\quad + \overline{D}\tau_0\Bigg[m_1\Big(W_{20}^{(1)}(0) + 2W_{11}^{(1)}(0)\Big) \\
&\qquad\quad + m_2\left(\frac{\overline{\gamma}W_{20}^{(1)}(0)}{2} + W_{20}^{(2)}(0) + \gamma W_{11}^{(1)}(0) + W_{11}^{(2)}(0) + 3m_3 + m_4(\overline{\gamma} + 2\gamma)\right) \\
&\qquad\quad + n_1\Big(W_{20}^{(1)}(-1) + 2W_{11}^{(1)}(-1)\Big) \\
&\qquad\quad + n_2\Bigg(\frac{\overline{\gamma}W_{20}^{(1)}(-1)}{2} + W_{20}^{(2)}(-1) + \gamma W_{11}^{(1)}(-1) \\
&\qquad\qquad\quad + W_{11}^{(2)}(-1) + 3n_3 + n_4(\overline{\gamma} + 2\gamma)\Bigg)\Bigg]z^2\overline{z} + \text{h.o.t,}
\end{aligned}
$$

(3.38)

and we obtain

$$g_{20} = 2\overline{D}\tau_0\Big[(m_1 + m_2\gamma) + \overline{\gamma^*}(n_1 + n_2\gamma)\Big],$$

$$g_{11} = \overline{D}\tau_0\Big[2m_1 + m_2(\gamma + \overline{\gamma^*}) + \overline{\gamma^*}(2n_1 + n_2(\gamma + \overline{\gamma^*}))\Big],$$

$$g_{02} = 2\overline{D}\tau_0\Big[m_1 + m_2\overline{\gamma} + \overline{\gamma^*}(n_1 + n_2\overline{\gamma})\Big],$$

$$g_{21} = 2\overline{D}\tau_0 \left[m_1 \left(W_{20}^{(1)}(0) + 2W_{11}^{(1)}(0) \right) \right.$$

$$+ m_2 \left(\frac{\overline{\gamma} W_{20}^{(1)}(0)}{2} + W_{20}^{(2)}(0) + \gamma W_{11}^{(1)}(0) + W_{11}^{(2)}(0) + 3m_3 + m_4 (\overline{\gamma} + 2\gamma) \right)$$

$$+ n_1 \left(W_{20}^{(1)}(-1) + 2W_{11}^{(1)}(-1) \right)$$

$$\left. + n_2 \left(\frac{\overline{\gamma} W_{20}^{(1)}(-1)}{2} + W_{20}^{(2)}(-1) + \gamma W_{11}^{(1)}(-1) + W_{11}^{(2)}(-1) + 3n_3 + n_4 (\overline{\gamma} + 2\gamma) \right) \right].$$

$$(3.39)$$

For unknown

$$W_{20}^{(1)}(0), W_{20}^{(1)}(-1), W_{11}^{(1)}(0), W_{11}^{(2)}(-1), W_{11}^{(2)}(0), W_{11}^{(2)}(-1), \tag{3.40}$$

in g_{21}, we still need to compute them.

From (3.10) and (3.29), we have

$$W' = \begin{cases} W - 2\operatorname{Re}\left\{ \overline{q}^*(0)\overline{f}q(\theta) \right\}, & -1 \leq \theta < 0, \\ W - 2\operatorname{Re}\left\{ \overline{q}^*(0)\overline{f}q(\theta) \right\} + \overline{f}, & \theta = 0. \end{cases}$$

$$\stackrel{\text{def}}{=} AW + H(z, \overline{z}, \theta),$$

$$(3.41)$$

where

$$H(z, \overline{z}, \theta) = H_{20}(\theta) \frac{z^2}{2} + H_{11}(\theta) z\overline{z} + H_{02}(\theta) \frac{\overline{z}^2}{2} + \cdots. \tag{3.42}$$

Comparing the coefficients, we obtain

$$(A - 2i\omega_0\tau_0)W_{20} = -H_{20}(\theta), \tag{3.43}$$

$$AW_{11}(\theta) = -H_{11}(\theta), \tag{3.44}$$

and we know that for $\theta \in [-1, 0)$,

$$H(z, \overline{z}, \theta) = -\overline{q}^*(0)f_0 q(\theta) - q^*(0)\overline{f}_0\overline{q}(\theta) = -g(z, \overline{z})q(\theta) - \overline{g}(z, \overline{z})\overline{q}(\theta). \tag{3.45}$$

Comparing the coefficients of (3.42) with (3.45) gives that

$$H_{20}(\theta) = -g_{20}q(\theta) - \overline{g}_{02}\overline{q}(\theta), \tag{3.46}$$

$$H_{11}(\theta) = -g_{11}q(\theta) - \overline{g}_{11}\overline{q}(\theta). \tag{3.47}$$

From (3.43) and (3.46) and the definition of A, we get

$$\dot{W}_{20}(\theta) = 2i\omega_0\tau_0 W_{20}(\theta) + g_{20}q(\theta) + \overline{g_{02}}\,\overline{q}(\theta). \tag{3.48}$$

Noting that $q(\theta) = q(0)e^{i\omega_0\tau_0\theta}$, we have

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_0\tau_0}q(0)e^{i\omega_0\tau_0\theta} + \frac{i\overline{g}_{02}}{3\omega_0\tau_0}\overline{q}(0)e^{-i\omega_0\tau_0\theta} + E_1 e^{2i\omega_0\tau_0\theta}, \tag{3.49}$$

where $E_1 = (E_1^{(1)}, E_1^{(2)})^T$ is a constant vector.

Similarly, from (3.44) and (3.47) and the definition of A, we have

$$\dot{W}_{11}(\theta) = g_{11}q(\theta) + \overline{g_{11}}\,\overline{q}(\theta), \tag{3.50}$$

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega_0\tau_0}q(0)e^{i\omega_0\tau_0\theta} + \frac{i\overline{g}_{11}}{\omega_0\tau_0}\overline{q}(0)e^{-i\omega_0\theta} + E_2, \tag{3.51}$$

where $E_2 = (E_2^{(1)}, E_2^{(2)})^T$ is a constant vector.

In what follows, one will seek appropriate E_1, E_2 in (3.49) and (3.51), respectively. It follows from the definition of A and (3.46) and (3.47) that

$$\int_{-1}^{0} d\eta(\theta)W_{20}(\theta) = 2i\omega_0\tau_0 W_{20}(0) - H_{20}(0), \tag{3.52}$$

$$\int_{-1}^{0} d\eta(\theta)W_{11}(\theta) = -H_{11}(0), \tag{3.53}$$

where $\eta(\theta) = \eta(0, \theta)$.

From (3.43), we have

$$H_{20}(0) = -g_{20}q(0) - \overline{g_{02}}\,\overline{q}(0) + \tau_0(M_1, M_2)^T, \tag{3.54}$$

where

$$\begin{aligned} M_1 &= m_1 + m_2\gamma, \\ M_2 &= n_1 + n_2\gamma. \end{aligned} \tag{3.55}$$

From (3.44), we have

$$H_{11}(0) = -g_{11}q(0) - \overline{g_{11}}(0)\overline{q}(0) + \tau_0(N_1, N_2)^T, \tag{3.56}$$

where

$$\begin{aligned} N_1 &= 2m_1 + m_2(\gamma + \overline{\gamma}), \\ N_2 &= 2n_1 + n_2(\gamma + \overline{\gamma}). \end{aligned} \tag{3.57}$$

Noting that

$$\left(i\omega_0\tau_0 I - \int_{-1}^0 e^{i\omega_0\tau_0\theta}d\eta(\theta) \right)q(0) = 0,$$

$$\left(-i\omega_0\tau_0 I - \int_{-1}^0 e^{-i\omega_0\tau_0\theta}d\eta(\theta) \right)\overline{q}(0) = 0,$$

(3.58)

and substituting (3.49) and (3.54) into (3.52), we have

$$\left(2i\omega_0\tau_0 I - \int_{-1}^0 e^{2i\omega_0\tau_0\theta}d\eta(\theta) \right)E_1 = \tau_0(M_1, M_2)^T.$$

(3.59)

That is,

$$\left(2i\omega_0\tau_0 I - \tau_0 B - \tau_0 G e^{-2i\omega_0\tau_0} \right)E_1 = \tau_0(M_1, M_2)^T,$$

(3.60)

then

$$\begin{pmatrix} 2i\omega_0 - a_1 & b_1 \\ c_1 e^{-2i\omega_0\tau_0} & 2i\omega_0 + d - d_1 e^{-2i\omega_0\tau_0} \end{pmatrix} \begin{pmatrix} E_1^{(1)} \\ E_1^{(2)} \end{pmatrix} = \begin{pmatrix} m_1 + m_2\gamma \\ n_1 + n_2\gamma \end{pmatrix}.$$

(3.61)

Hence,

$$E_1^{(1)} = \frac{\Delta_{11}}{\Delta_1}, \qquad E_1^{(2)} = \frac{\Delta_{12}}{\Delta_1},$$

(3.62)

where

$$\Delta_1 = \det\begin{pmatrix} 2i\omega_0 - a_1 & b_1 \\ c_1 e^{-2i\omega_0\tau_0} & 2i\omega_0 + d - d_1 e^{-2i\omega_0\tau_0} \end{pmatrix},$$

$$\Delta_{11} = \det\begin{pmatrix} m_1 + m_2\gamma & b_1 \\ n_1 + n_2\gamma & 2i\omega_0 + d - d_1 e^{-2i\omega_0\tau_0} \end{pmatrix},$$

$$\Delta_{12} = \det\begin{pmatrix} 2i\omega_0 - a_1 & m_1 + m_2\gamma \\ c_1 e^{-2i\omega_0\tau_0} & n_1 + n_2\gamma \end{pmatrix}.$$

(3.63)

Similarly, substituting (3.51) and (3.56) into (3.53), we have

$$\left(\int_{-1}^0 d\eta(\theta) \right)E_2 = \tau_0(N_1, N_2)^T.$$

(3.64)

Then,

$$(B + G)E_2 = (-N_1, -N_2)^T. \tag{3.65}$$

That is,

$$\begin{pmatrix} a_1 & -b_1 \\ c_1 & D_1 - d \end{pmatrix} \begin{pmatrix} E_2^{(1)} \\ E_2^{(2)} \end{pmatrix} = \begin{pmatrix} -N_1 \\ -N_2 \end{pmatrix}. \tag{3.66}$$

Hence,

$$E_2^{(1)} = \frac{\Delta_{21}}{\Delta_2}, \qquad E_2^{(2)} = \frac{\Delta_{22}}{\Delta_2}, \tag{3.67}$$

where

$$
\begin{aligned}
\Delta_2 &= \det \begin{pmatrix} a_1 & -b_1 \\ c_1 & d_1 - d \end{pmatrix}, \\
\Delta_{21} &= \det \begin{pmatrix} -2m_1 - m_2(\gamma + \overline{\gamma}) & -b_1 \\ -2n_1 - n_2(\gamma + \overline{\gamma}) & d_1 - d \end{pmatrix}, \\
\Delta_{22} &= \det \begin{pmatrix} a_1 & -2m_1 - m_2(\gamma + \overline{\gamma}) \\ c_1 & -2n_1 - n_2(\gamma + \overline{\gamma}) \end{pmatrix}.
\end{aligned}
\tag{3.68}
$$

From (3.49) and (3.51), we can calculate g_{21} and derive the following values:

$$
\begin{aligned}
c_1(0) &= \frac{i}{2\omega_0 \tau_0} \left(g_{20} g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3} \right) + \frac{g_{21}}{2}, \\
\mu_2 &= -\frac{\operatorname{Re}\{c_1(0)\}}{\operatorname{Re}\{\lambda'(\tau_0)\}}, \\
\beta_2 &= 2\operatorname{Re}(c_1(0)), \\
T_2 &= -\frac{\operatorname{Im}\{c_1(0)\} + \mu_2 \operatorname{Im}\{\lambda'(\tau_0)\}}{\omega_0 \tau_0}.
\end{aligned}
\tag{3.69}
$$

These formulae give a description of the Hopf bifurcation periodic solutions of (1.1) at $\tau = \tau_0$ on the center manifold. From the discussion above, we have the following result. $\square$

Theorem 3.3. *The periodic solution is supercritical (subcritical) if $\mu_2 > 0$ ($\mu_2 < 0$). The bifurcating periodic solutions are orbitally asymptotically stable with asymptotical phase (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$). The periods of the bifurcating periodic solutions increase (decrease) if $T_2 > 0$ ($T_2 < 0$).*
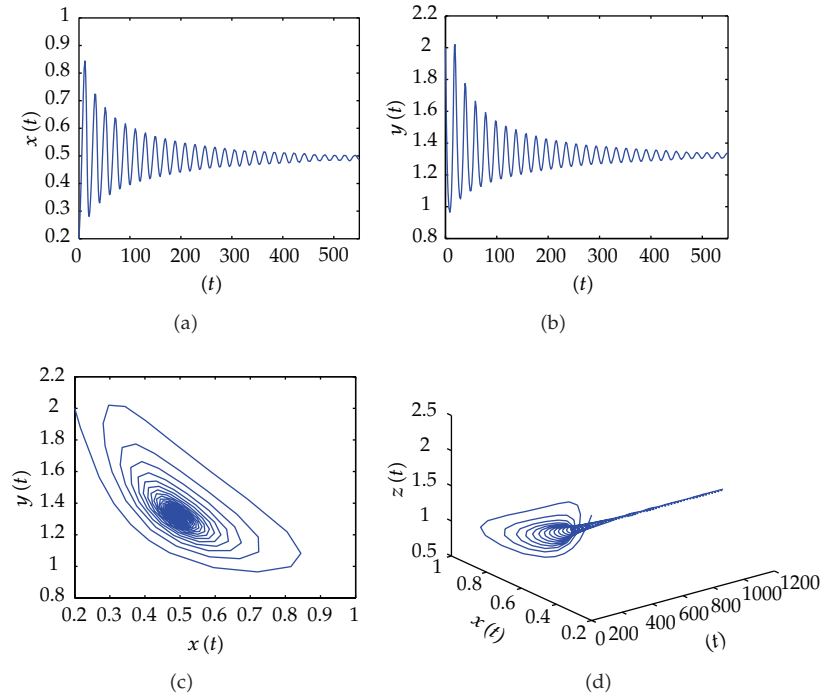
(a)

(b)

(c)

(d)

Figure 1: The time histories and phase portrait of system (4.1) with the following parameters: $a = 0.5, b = 0.5, c = 2, \varpi = 2, d = 0.2$, and $\tau = 1.7 < \tau_0 \approx 1.7355$. The positive equilibrium $E_*(x^*, y^*) = (0.4928, 1.3217)$ is asymptotically stable. The initial value is $(0.2, 2)$.

4. Numerical Examples

In this section, we present some numerical results to verify the analytical predictions obtained in the previous section. As an example, we consider the following special case of system (1.1) with the parameters $a = 0.5, b = 0.5, c = 2, \varpi = 2$, and $d = 0.2$. Then system (1.1) becomes

$$\dot{x}(t) = x(t)[1 - 0.5x(t)] - \frac{2x^2(t)y(t)}{1 + 3x^2(t)},$$

$$\dot{y}(t) = \frac{2x^2(t - \tau)y(t - \tau)}{1 + 3x^2(t - \tau)} e^{-0.2\tau} - 0.2y(t),$$

(4.1)

which has a positive equilibrium $E_*(x^*, y^*) = (1.4928, 1.3217)$. By some complicated computation by means of Matlab 7.0, we get only one critical values of the delay $\tau_0 \approx 1.7355, \lambda'(\tau_0) \approx 0.2035 - 0.5423i$. Thus, we derive $c_1(0) \approx -1.3122 - 5.0131i, \mu_2 \approx 0.6177, \beta_2 \approx -3.3326, T_2 \approx 9.3042$. We obtain that the conditions indicated in Theorem 2.3 are satisfied. Furthermore, it follows that $\mu_2 > 0$ and $\beta_2 < 0$. Thus, the positive equilibrium $E_*(x^*, y^*)$ is stable when $\tau < \tau_0$ which is illustrated by the computer simulations (see Figures 1(a)–1(d)). When τ passes through the critical value τ_0, the positive equilibrium $E_*(x^*, y^*)$ loses its stability and a Hopf bifurcation occurs, that is, a family of periodic solutions bifurcations
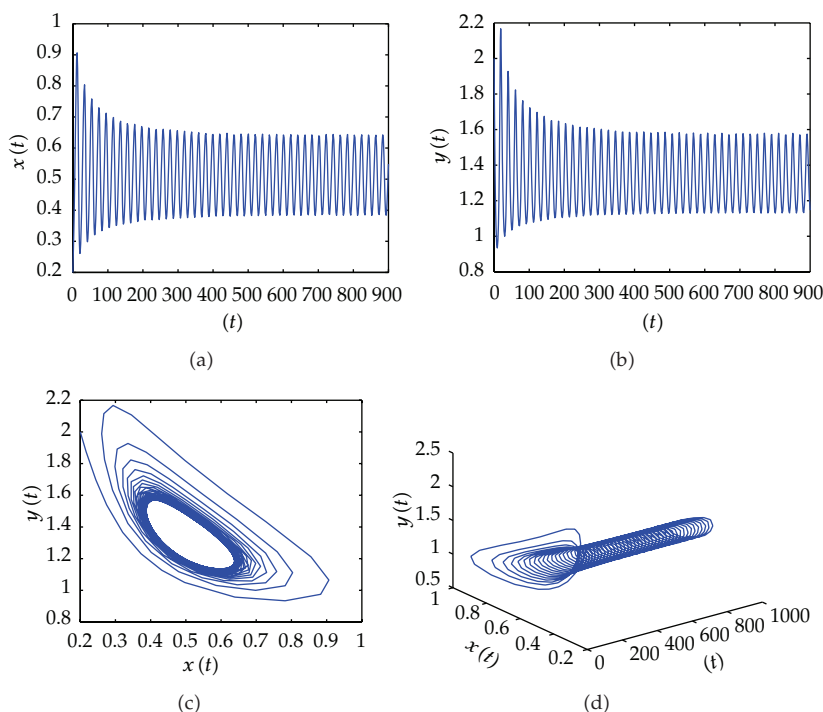
(a)

(b)

(c)

(d)

Figure 2: The time histories and phase portrait of system (4.1) with the following parameters: $a = 0.5, b = 0.5, c = 2, \varpi = 2, d = 0.2$, and $\tau = 1.8 > \tau_0 \approx 1.7355$. Hopf bifurcation occurs from the positive equilibrium $E_*(x^*, y^*) = (0.5014, 1.3108)$. The initial value is $(0.2, 2)$.

from the positive equilibrium $E_*(x^*, y^*)$. Since $\mu_2 > 0$ and $\beta_2 < 0$, the direction of the Hopf bifurcation is $\tau > \tau_0$, and these bifurcating periodic solutions from $E_*(x^*, y^*)$ at τ_0 are stable, which are depicted in Figures 2(a)–2(d).

5. Conclusions

In this paper, the main object is to investigate the local stability and Hopf bifurcation and also to study the stability of bifurcating periodic solutions and some formulae for determining the direction of Hopf bifurcation for a stage-structured predator-prey model with time delay and delay. By choosing the delay as a bifurcation parameter, It is shown that under certain condition, the positive equilibrium $E_*(x^*, y^*)$ of system (1.1) is asymptotically stable for all $\tau \in [0, \tau_0)$ and unstable for $\tau > \tau_0$ and under another condition; when the delay τ increases, the equilibrium loses its stability and a sequence of Hopf bifurcations occur at the positive equilibrium $E_*(x^*, y^*)$, that is, a family of periodic orbits bifurcate from the positive equilibrium $E_*(x^*, y^*)$. At the same time, using the normal form theory and the center manifold theorem, the direction of Hopf bifurcation and the stability of the bifurcating periodic orbits are discussed. Finally, numerical simulations are carried out to validate the theorems obtained.

Acknowledgments

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