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Research Article

Successive Matrix Squaring Algorithm for Computing the Generalized Inverse $A^{(2)}_{T,S}$

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We investigate successive matrix squaring (SMS) algorithms for computing the generalized inverse $A^{(2)}_{T,S}$ of a given matrix $A \in C^{m\times n}$.

1. Introduction

Throughout this paper, the symbol $C^{m\times n}$ denotes a set of all $m \times n$ complex matrices. Let $A \in C^{m\times n}$, and the symbols $R(A)$, $N(A)$, $\rho(A)$, and $\|\cdot\|$ stand for the range, the null space, the spectrum of matrix A, and the matrix norm, respectively.

A matrix B is called a $\{2\}$-inverse of matrix A if $BAB = B$ holds. The symbols $A^{\dagger}$, $\mathrm{Ind}(A)$, and A^D denote, respectively, the Moore-Penrose inverse, the index, and the Drazin inverse of A, and, obviously, $\mathrm{rank}(A^{\dagger}) = \mathrm{rank}(A)$ (see [1] for details). Let $A \in C_r^{m\times n}$, $T \subset C^n$, $S \subset C^m$, and $\dim(T) = t \le r$ and $\dim(S) = m - t$, and there exists and unique matrix $B \in C^{n\times m}$ such that

$$BAB = B, \qquad R(B) = T, \qquad N(B) = S \tag{1.1}$$

then $B \in C^{n\times m}$ is called $\{2\}$-inverse of A with the prescribed range T and null space S of A, denoted by $A^{(2)}_{T,S}$.

In [1], it is well known that the generalized inverse $A^{(2)}_{T,S}$ of a given matrix $A \in C^{m\times n}$ with the prescribed range T and null space S is very important in applications of many mathematics branches such as stable approximations of ill-posed problems, linear and nonlinear problems involving rank-deficient generalized, and the applications to statistics [2]. In particular, the generalized inverse $A^{(2)}_{T,S}$ plays an important role for the iterative methods for solving nonlinear equations [1, 2].

In recent years, successive matrix squaring algorithms are investigated for computing the generalized inverse of a given matrix $A \in C^{m\times n}$ in [3–7]. In [3], the authors exhibit a deterministic iterative algorithm for linear system solution and matrix inversion based on a repeated matrix squaring scheme. Wei derives a successive matrix squaring (SMS) algorithm to approximate the Drazin inverse in [4]. Wei et al. in [5] derive a successive matrix squaring (SMS) algorithm to approximate the weighted generalized inverse $A^{\dagger}_{M,N}$, which can be expressed in the form of successive squaring of a composite matrix T. Stanimirović and Cvetković-Ilić derive a successive matrix squaring (SMS) algorithm to approximate an outer generalized inverse with prescribed range and null space of a given matrix $A \in C^{m\times n}_r$ in [6]. In [7], authors introduce a new algorithm based on the successive matrix squaring (SMS) method and this algorithm uses the strategy of ϵ-displacement rank in order to find various outer inverses with prescribed ranges and null spaces of a square Toeplitz matrix.

In this paper, based on [3–5], we investigate successive matrix squaring algorithms for computing the generalized inverse $A^{(2)}_{T,S}$ of a matrix A in Section 2 and also give a numerical example for illustrating our results in Section 3.

The following given lemma suggests that the generalized inverse $A^{(2)}_{T,S}$ is unique.

Lemma 1.1 (see [1, Theorem 2.14]). *Let $A \in C^{m\times n}$ with rank r, let T be a subspace of $\mathbb{C}^n$ of dimension $s \le r$, and let S be a subspace of $\mathbb{C}^m$ of dimension $m - s$. Then, A has a $\{2\}$-inverse X such that $\mathcal{R}(X) = T$ and $\mathcal{N}(X) = S$ if and only if*

$$AT \oplus S = \mathbb{C}^m \tag{1.2}$$

in which case X is unique.

The following nations are stated in Banach space but they are true in the finite dimension space. Throughout this paper, let H, K denote the Banach space and let $\mathcal{B}(H,K)$ stand for the set of all bounded linear operators from H to K, in particular $\mathcal{B}(H,H) = \mathcal{B}(H)$.

In the following, we state two lemmas which are given for Banach space but it can be used also for the finite dimension space.

Lemma 1.2 (see [8, Section 4]). *Let $A \in \mathcal{B}(H,K)$ and T and S, respectively, closed subspaces of H and K. Then the following statements are equivalent:*

(i) *A has a $\{2\}$-inverse $B \in K, H$ such that $R(B) = T$ and $N(B) = S$,*

(ii) *T is a complemented subspace of H, $A|_T : T \longrightarrow A(T)$ is invertible and $A(T) \oplus S = K$.*

Lemma 1.3 (see [9, Section 3]). *Suppose that the conditions of Lemma 1.2 are satisfied. If we take $T_1 = N(A^{(2)}_{T,S}A)$, then $H = T \oplus T_1$ holds and A has the following matrix form:*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} : \begin{bmatrix} T \\ T_1 \end{bmatrix} \longrightarrow \begin{bmatrix} A(T) \\ S \end{bmatrix}, \tag{1.3}$$

where A_1 is invertible. Moreover, $A^{(2)}_{T,S}$ has the matrix following form:

$$A^{(2)}_{T,S} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} A(T) \\ S \end{bmatrix} \longrightarrow \begin{bmatrix} T \\ T_1 \end{bmatrix}. \tag{1.4}$$

From (1.5), we obtain the following projections (see [9]):

$$\begin{aligned} P_{A(T),S} = AA^{(2)}_{T,S} &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} A(T) \\ S \end{bmatrix} \longrightarrow \begin{bmatrix} A(T) \\ S \end{bmatrix}, \\ P_{T,T_1} = A^{(2)}_{T,S}A &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} T \\ T_1 \end{bmatrix} \longrightarrow \begin{bmatrix} T \\ T_1 \end{bmatrix}. \end{aligned} \tag{1.5}$$

2. Main Result

In this section, we consider successive matrix squaring (SMS) algorithms for computing the generalized inverse $A^{(2)}_{T,S}$.

Let $A \in C^{m\times n}$ and the sequence $\{X_n\}$ in $C^{n\times m}$, and we can define the iterative form as follows ([10, Theorem 2.2] for computing the generalized inverse $A^{(2)}_{T,S}$ in the infinite space case):

$$\begin{aligned} R_k &= P_{A(T),S} - P_{A(T),S}AX_k, \\ X_{k+1} &= X_0R_k + X_k, \quad k = 0,1,2,\ldots. \end{aligned} \tag{2.1}$$

From [10], the authors have proved that the iteration (2.1) converges to the generalized inverse $A^{(2)}_{T,S}$ if and only if $R(X_0) \subset T$, $\rho(R_0) < 1$, where $T \subset C^n$ and $P_{A(T),S} = AA^{(2)}_{T,S}$ (for the proof see [11] and [10, Theorem 2.1] when $p = 2$).

In the following, we give the algorithm for computing the generalized inverse $A^{(2)}_{T,S}$ of a matrix $A \in C^{m\times n}$.

Let $P = R_0 = P_{A(T),S} - P_{A(T),S}AX_0$ and $Q = X_0$. It is not difficult to see that the above fact can be written as follows:

$$M = \begin{bmatrix} R_0 & 0 \\ X_0 & I \end{bmatrix} = \begin{bmatrix} P & 0 \\ Q & I \end{bmatrix}. \tag{2.2}$$

From (2.2) and letting $X_k = Q\sum_{i=0}^{k} P^i$, we have

$$M^k = \begin{bmatrix} P^k & 0 \\ Q\sum\limits_{i=0}^{k-1} P^i & I \end{bmatrix} = \begin{bmatrix} P^k & 0 \\ X_{k-1} & I \end{bmatrix}. \tag{2.3}$$

By (2.3), we prove that the iterative (2.1) X_k is equal to the right upper block in the matrix M^k. Note that we defined the new iterative form $\{M_k\}$ as follows:

$$M_0 = M, \qquad M_{k+1} = M_k^2, \quad k = 0,1,2,\ldots. \tag{2.4}$$

Input: Input the initial value matrices $A, X_0, P_{A(T),S}$ and the accurate value ϵ;
Output: The algorithm export the matrix, that is $X \approx A^{(2)}_{T,S}$;
Begin: Assignment the matrix Q by the initial value matrix X_0, that is $Q \Leftarrow X_0$;
Assigned the matrix P by $P_{A(T),S} - P_{A(T),S}AQ$, that is $P \Leftarrow P_{A(T),S} - P_{A(T),S}AQ$;
Computed matrix X_1, that is $X_1 \Leftarrow I + P$;
Computed the error between X_1 and X_0, that is $e \Leftarrow \|X_1 - X_0\|$;
Judged that whether e is lower than ϵ or not,
that is while $e < \epsilon$, do $P \Leftarrow P \cdot P$;
Defined the loop function: $X_{k+1} \Leftarrow X_k + P$;
Computed the error between X_k and X_{k+1}, that is $e \Leftarrow \|X_{k+1} - X_k\|$;
Finished the loop function
The $k+1$ matrix X_{k+1} multiplied by Q and assigned to X, that is $X \Leftarrow QX_{k+1}$;
End the algorithm.

Algorithm 1: SMS algorithm for computing the generalized inverse $A^{(2)}_{T,S}$.

From the new iterative form (2.4), we arrive at

$$M_k = M^{2^k} = \begin{bmatrix} P^{2^k} & 0 \\ Q\sum_{i=0}^{2^k-1} P^i & I \end{bmatrix} = \begin{bmatrix} P^{2^k} & 0 \\ X_{2^k-1} & I \end{bmatrix}. \tag{2.5}$$

Assume that $X_{2^k-1} = \widehat{X}_k$, and by (2.5), we have

$$M_k = \begin{bmatrix} P^{2^k} & 0 \\ \widehat{X}_k & I \end{bmatrix} = \begin{bmatrix} P^{2^k} & 0 \\ X_{2^k-1} & I \end{bmatrix} = \begin{bmatrix} P^{2^k} & 0 \\ Q\sum_{i=0}^{2^k-1} P^i & I \end{bmatrix}. \tag{2.6}$$

By (2.4)–(2.6), we have Algorithm 1.

From (2.4)–(2.6) and Algorithm 1, we obtain the following result.

Theorem 2.1. *Let $A \in C^{m\times n}$, and the sequence $\{\widehat{X}_k\}$ converges to the generalized inverse $A^{(2)}_{T,S}$ if and only if $R(X_0) \subset T,\ \rho(R_0) < 1$. In this case*

$$\left\| A^{(2)}_{T,S} - \widehat{X}_k \right\| \le q^{2^{k+1}} (1-q)^{-1} \|X_0\|, \tag{2.7}$$

where $q = \|R_0\|$ and

$$T \subset C^n, \quad P_{A(T),S} = AA^{(2)}_{T,S}. \tag{2.8}$$

Proof. From the proof in [11] and [10, Theorem 2.1] when $p = 2$ and according to (2.4), (2.5) and (2.6), we easily finish the proof of the former of the theorem. In the following, we only prove the last section, that is, prove that the inequality (2.7) holds.

By applying (2.5) and (2.6), we obtain

$$\widehat{X}_k = X_{2^k-1} = \sum_{i=0}^{2^k-1} P^i Q. \tag{2.9}$$

By the iteration (2.4) and (2.9), we arrive at

$$\begin{aligned}\left\|A_{T,S}^{(2)}-\widehat{X}_k\right\| &= \left\|X_0(I-R_0)^{-1}-X_0\sum_{i=0}^{2^k-1}R_0^i\right\| = \left\|X_0\sum_{i=0}^{\infty}R_0^i-X_0\sum_{i=0}^{2^k-1}R_0^i\right\| \\ &= \left\|X_0\sum_{i=2^k}^{\infty}R_0^i\right\| = \left\|X_0R_0^{2^k}\sum_{i=0}^{\infty}R_0^i\right\| \le \left\|R_0^{2^k}\right\|\left\|\sum_{i=0}^{\infty}\right\|R_0^i\left\|\right\|\|X_0\| \\ &\le q^{2^{k+1}}(1-q)^{-1}\|X_0\|.\end{aligned} \tag{2.10}$$

□

The following corollary given the result is the same as theorem in [6, Theorem 2.3]. It also presents an explicit representation of the the generalized inverse $A_{T,S}^{(2)}$ and the sequence (2.4) converges to a $\{2\}$-inverse of a given matrix A by its full-rank decomposition.

Corollary 2.2. *Let $A \in C_r^{m\times n}$, $A = FG$ be full rank decomposition, and the sequence $\{\widehat{X}_k\}$ converges to the $\{2\}$-inverse $X = F(GAF)^{-1}G$ if and only if $\rho(R_0) < 1$. In this case*

$$\left\|X-\widehat{X}_k\right\| \le q^{2^{k+1}}(1-q)^{-1}\|X_0\|, \tag{2.11}$$

where $q = \|R_0\|$ and

$$F \in C_s^{m\times s}, \quad G \in C_s^{s\times n}, \quad P_{R(AX),N(AX)} = AX. \tag{2.12}$$

Proof. From Theorem 2.5 and by [6, Theorem 2.3], we have the result. □

In the following, we consider the improvement of the iterative form (2.1) (see [11] for computing the Moore-Penrose inverse and the Drazin inverse of the matrix case and [10, Theorem 2.2] for computing the generalized inverse $A_{T,S}^{(2)}$ in the infinite space case):

$$\begin{aligned}R_k &= P_{A(T),S}-P_{A(T),S}AX_k, \\ X_{k+1} &= X_k\left(I+R_k+\cdots+R_k^{p-1}\right), \quad p\ge 2,\ k=0,1,2,\ldots.\end{aligned} \tag{2.13}$$

Let M be a $m\times m$ block matrix and

$$M = \begin{bmatrix} P^{m-1} & 0 & \cdots & 0 \\ P^{m-2} & 0 & \cdots & 0 \\ * & & & \mathbf{0} \\ P & 0 & \cdots & 0 \\ Q & Q & \cdots & I \end{bmatrix}, \tag{2.14}$$

then

$$M^2 = \begin{bmatrix} P^{2m-1} & 0 & \cdots & 0 \\ P^{2m-2} & 0 & \cdots & 0 \\ * & & & \mathbf{0} \\ P^m & 0 & \cdots & 0 \\ Q\sum_{i=0}^{m-1} P^i & Q & \cdots & I \end{bmatrix}. \tag{2.15}$$

By induction if M^{k-1} has the following form:

$$M^{k-1} = \begin{bmatrix} P^{(k-1)m-1} & 0 & \cdots & 0 \\ P^{(k-1)m-2} & 0 & \cdots & 0 \\ * & & & \mathbf{0} \\ P^{(k-2)m} & 0 & \cdots & 0 \\ Q\sum_{i=0}^{(k-2)m-1} P^i & Q & \cdots & I \end{bmatrix}, \tag{2.16}$$

then

$$M^k = \begin{bmatrix} P^{km-1} & 0 & \cdots & 0 \\ P^{km-2} & 0 & \cdots & 0 \\ * & & & \mathbf{0} \\ P^{(k-1)m} & 0 & \cdots & 0 \\ Q\sum_{i=0}^{(k-1)m-1} P^i & Q & \cdots & I \end{bmatrix}. \tag{2.17}$$

Similarly to the iterative form (2.4), we also define the new iterative scheme $\{M_k\}$

$$M_0 = M, \qquad M_{k+1} = M_k^p, \quad k = 0, 1, 2, \ldots. \tag{2.18}$$

Note that from (2.18)

$$M_k = M^{p^k} = \begin{bmatrix} P^{p^k m-1} & 0 & \cdots & 0 \\ P^{p^k m-2} & 0 & \cdots & 0 \\ * & & & \mathbf{0} \\ P^{(p^k-1)m} & 0 & \cdots & 0 \\ Q\sum_{i=0}^{(p^k-1)m-1} P^i & Q & \cdots & I \end{bmatrix} = \begin{bmatrix} P^{p^k m-1} & 0 & \cdots & 0 \\ P^{p^k m-2} & 0 & \cdots & 0 \\ * & & & \mathbf{0} \\ P^{(p^k-1)m} & 0 & \cdots & 0 \\ X_{(p^k-1)m-1} & Q & \cdots & I \end{bmatrix}. \tag{2.19}$$

Let $X_{(p^k-1)m-1} = \widehat{X}_k$, and by (2.18), and (2.19), we arrive at

$$M_k = \begin{bmatrix} * & 0 \\ X_{(p^k-1)m-1} & * \end{bmatrix} = \begin{bmatrix} * & 0 \\ \widehat{X}_k & * \end{bmatrix}. \tag{2.20}$$

Input: Input the matrices $A, X_0, P_{A(T),S}$ and the accurate value ϵ;
Output: The algorithm export the matrix: $X \approx A^{(2)}_{T,S}$;
Begin: Assignment the matrix Q by the initial value matrix X_0, that is $Q \Leftarrow X_0$;
Assigned the matrix P_1 by $P_{A(T),S} - P_{A(T),S}AQ$, that is $P_1 \Leftarrow P_{A(T),S} - P_{A(T),S}AQ$;
Computed the product of P_1 and P_1, and assigned its value to P_2. that is $P_2 \Leftarrow P_1 \cdot P_1$;
Similarly, we repeatedly do the computation for the product P_i and P_1 as well as above the computation, where $i = 2, \ldots, m-2$.
Computed the product of the matrix P_{m-1} and P_1, and assigned its value to P_m as well as above computations, that is $P_m \Leftarrow P_{m-1} \cdot P_1$;
Assigned the matrix X_1 by the sum of the matrices P_i, where $i = 0, 1, 2, \ldots, m$ and $P_0 = I$. that is $X_1 \Leftarrow I + P_1 + \cdots + P_{m-1} + P_m$;
Take the norm of $\|X_1 - X_0\|$ and assigned its value to e. that is $e \Leftarrow \|X_1 - X_0\|$;
while $e < \epsilon$ do;
We need the iteration not to exceed 500 times. that is $n = 500$; (In fact $(p^k - 1)m - 1 = 500$)
Do 500 step repeatedly computations in the following.
that is For $i = 1 : n$
Computed the product of the given matrix $P_m = P^m$ and the iteration matrix P_i, and assigned its value to the new matrix P_i. that is $P_i \Leftarrow P_m \cdot P_i$;
From the iteration $P_i \Leftarrow P_m \cdot P_i$, we obtain the new matrix P_i and add its value to X_i, and assigned the sum of P_i and X_i to the matrix X_{i+1}..that is $X_{i+1} \Leftarrow X_i + P_i$; After these, return the the step $P_i = P_m \cdot P_i$.
Finished the *For* loop function that is end
Computed the error between X_k and X_{k+1}, that is $e \Leftarrow \|X_{k+1} - X_k\|$;
Finished the *While* loop function. that is end
The $k+1$ matrix X_{k+1} multiplied by Q and assigned to X, that is $X \Leftarrow QX_{k+1}$;
End the algorithm.

Algorithm 2: SMS algorithm for computing the generalized inverse $A^{(2)}_{T,S}$.

From (2.14) to (2.20), we find that if one wants to compute the generalized inverse $A^{(2)}_{T,S}$ then we only compute the element $(m, 1)$ of the matrix M^{2^k}. Similarly to Algorithm 1, we also obtain Algorithm 2.

Analogous to Theorem 2.5 by Algorithm 2 and sequence (2.18), we also have the following theorem.

Theorem 2.3. *Let $A \in C^{m\times n}$, and the sequence $\{\widehat{X}_k\}$ converges to the generalized inverse $A^{(2)}_{T,S}$ if and only if $R(X_0) \subset T$, $\rho(R_0) < 1$. In this case*

$$\left\| A^{(2)}_{T,S} - \widehat{X}_k \right\| \le q^{(p^k-1)m+1} (1-q)^{-1} \|X_0\|, \tag{2.21}$$

where $q = \|X_0\|$ and

$$T \subset C^n, \quad P_{A(T),S} = AA^{(2)}_{T,S}. \tag{2.22}$$

Proof. Similarly the proof in [10, Theorem 2.1], we can prove the former of this theorem. Analogous to the proof of Theorem 2.5, we finish the proof of the theorem. □

In the following, we extend the sequence (2.4) to

$$M_0 = M, \qquad M_{k+1} = M_k^t, \quad k = 0, 1, 2, \ldots, \text{ for any } t \ge 2. \tag{2.23}$$

By (2.26) and by induction, we have

$$M_k = M^{t^k} = \begin{bmatrix} P^{t^k} & 0 \\ Q\sum_{i=0}^{t^k-1} P^i & I \end{bmatrix}. \tag{2.24}$$

Assume that $X_{t^k-1} = \widehat{X}_k$, we easily have

$$M_k = \begin{bmatrix} P^{t^k} & 0 \\ \widehat{X}_k & I \end{bmatrix} = \begin{bmatrix} P^{t^k} & 0 \\ X_{t^k-1} & I \end{bmatrix} = \begin{bmatrix} P^{t^k} & 0 \\ Q\sum_{i=0}^{t^k-1} P^i & I \end{bmatrix}. \tag{2.25}$$

Similarly, from (2.23) and (2.25), we obtain the following result.

Theorem 2.4. *Let $A \in C^{m\times n}$, and the sequence $\{\widehat{X}_k\}$ converges to the generalized inverse $A^{(2)}_{T,S}$ if and only if $R(X_0) \subset T,\ \rho(R_0) < 1$. In this case*

$$\left\| A^{(2)}_{T,S} - \widehat{X}_k \right\| \le q^{t^{k+1}} (1-q)^{-1} \|X_0\|, \tag{2.26}$$

where $q = \|R_0\|$ and

$$T \subset C^n, \quad P_{A(T),S} = AA^{(2)}_{T,S}. \tag{2.27}$$

Proof. From (2.25) and only using t instead of 2 in Theorem 2.1, we easily have that $\{\widehat{X}_k\}$ converges to the generalized inverse $A^{(2)}_{T,S}$ if and only if $R(X_0) \subset T,\ \rho(R_0) < 1$. Similarly to the formula (2.29), we obtain that

$$\left\| A^{(2)}_{T,S} - \widehat{X}_k \right\| \le q^{t^{k+1}} (1-q)^{-1} \|X_0\|, \tag{2.28}$$

where q, T, and $P_{A(T),S}$ are the same as Theorem 2.5. □

In the following, we consider the dually iterative form.

Let $A \in C^{m\times n}$ and the sequence $\{X_n\}$ in $C^{n\times m}$, and we can define the iterative form as follows (see [11] and [10, Theorem 2.3]):

$$\begin{aligned} R_k &= P_{T,T_1} - AX_k P_{T,T_1}, \\ X_{k+1} &= R_k X_0 + X_k, \quad k = 0, 1, 2, \ldots. \end{aligned} \tag{2.29}$$

Let $P = R_0 = P_{T,T_1} - X_0 A P_{T,T_1}$ and $Q = X_0$. It is not difficult to see that the above fact can be written as follows:

$$M = \begin{bmatrix} R_0 & 0 \\ X_0 & I \end{bmatrix} = \begin{bmatrix} P & 0 \\ Q & I \end{bmatrix}. \tag{2.30}$$

From iterative forms (2.26) and (2.29), we have the following theorem.

Theorem 2.5. *Let $A \in C^{m\times n}$, and the sequence $\{\widehat{X}_k\}$ converges to the generalized inverse $A^{(2)}_{T,S}$ if and only if $R(X_0) \subset T,\ \rho(R_0) < 1$. In this case*

$$\left\| A^{(2)}_{T,S} - \widehat{X}_k \right\| \leq q^{2^{k+1}} (1-q)^{-1} \|X_0\|, \tag{2.31}$$

where $q = \|R_0\|$ and

$$T \subset C^n, \quad P_{A(T),S} = AA^{(2)}_{T,S}. \tag{2.32}$$

Similarly to Corollary 2.2, we have the result as follows.

Corollary 2.6. *Let $A \in C_r^{m\times n}$, $A = FG$ full rank decomposition, and the sequence $\{\widehat{X}_k\}$ converges to the $\{2\}$-inverse $X = F(GAF)^{-1}G$ if and only if $\rho(R_0) < 1$. In this case*

$$\left\| X - \widehat{X}_k \right\| \leq q^{2^{k+1}} (1-q)^{-1} \|X_0\|, \tag{2.33}$$

where $q = \|R_0\|$ and

$$F \in C_s^{m\times s}, \qquad G \in C_s^{s\times n}, \qquad P_{R(XA),N(XA)} = XA. \tag{2.34}$$

In the following, we consider the improvement of the iterative form (2.29) (see [11] for computing the Moore-Penrose inverse and the Drazin inverse of the matrix case and [10, Theorem 2.3] for computing the generalized inverse $A^{(2)}_{T,S}$ in the infinite space case):

$$\begin{gathered} R_k = P_{T,T_1} - AX_kP_{T,T_1}, \\ X_{k+1} = \left(I + R_k + \cdots + R_k^{p-1}\right)X_k, \quad p \geq 2,\ k = 0,1,2,\ldots. \end{gathered} \tag{2.35}$$

It is similar to (2.14), and we have

$$M = \begin{bmatrix} P^m & P^{m-1} & \cdots & P & Q \\ 0 & 0 & \cdots & 0 & Q \\ * & & & & \mathbf{0} \\ 0 & 0 & \cdots & 0 & Q \\ 0 & 0 & \cdots & 0 & I \end{bmatrix}. \tag{2.36}$$

Analogous to Theorem 2.5 by Algorithm 2 and from (2.36), we obtain the theorem in the following.

Theorem 2.7. *Let $A \in C^{m\times n}$, and the sequence $\{\widehat{X}_k\}$ converges to the generalized inverse $A^{(2)}_{T,S}$ if and only if $R(X_0) \subset T,\ \rho(R_0) < 1$. In this case*

$$\left\| A^{(2)}_{T,S} - \widehat{X}_k \right\| \leq q^{(p^k-1)m+1} (1-q)^{-1} \|X_0\|, \tag{2.37}$$

where $q = \|X_0\|$ *and*

$$T \subset C^n, \quad P_{T,T_1} = A_{T,S}^{(2)} A. \tag{2.38}$$

Dually, we give the SMS algorithm for computing the generalized inverse $A_{T,S}^{(2)}$ which are analogous to the iterative form (2.23) as follows and omit their proofs:

$$M_k = \begin{bmatrix} P^{t^k} & \widehat{X}_k \\ 0 & I \end{bmatrix} = \begin{bmatrix} P^{t^k} & X_{t^k-1} \\ 0 & I \end{bmatrix} = \begin{bmatrix} P^{t^k} & Q\displaystyle\sum_{i=0}^{t^k-1} P^i \\ 0 & I \end{bmatrix}. \tag{2.39}$$

Similarly Theorem 2.4, from (2.35) and (2.39), we obtain the following result.

Theorem 2.8. *Let* $A \in C^{m\times n}$*, and the sequence* $\{\widehat{X}_k\}$ *converges to the generalized inverse* $A_{T,S}^{(2)}$ *if and only if* $R(X_0) \subset T,\ \rho(R_0) < 1$*. In this case*

$$\left\| A_{T,S}^{(2)} - \widehat{X}_k \right\| \le q^{t^{k+1}} (1-q)^{-1} \|X_0\|, \tag{2.40}$$

where $q = \|R_0\|$ *and*

$$T \subset C^n, \quad P_{T,T_1} = A_{T,S}^{(2)} A. \tag{2.41}$$

3. Example

Here is an example to verify the effectiveness of the SMS method.

Example 3.1. Let

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}. \tag{3.1}$$

Let $T \in C^2$; $e = (0;0;1)^T \in C^3,\ S = \operatorname{span}\{e\}$.

Take

$$X_0 = \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix}. \tag{3.2}$$

By (2.2), we have

$$R_0 = \begin{bmatrix} 0.2 & -0.4 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.3}$$

Table 1

Method	Iteration (2.1)	Algorithm 1
Steps	5	2

From [10, 12], we easily have the generalized inverse $A_{T,S}^{(2)}$ in

$$A_{T,S}^{(2)} = \begin{bmatrix} 0.5 & -0.25 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}. \tag{3.4}$$

Then, from Algorithm 1, we obtain

$$X_1 = \begin{bmatrix} 0.4800 & -0.1600 & 0 \\ 0 & 0.4800 & 0 \end{bmatrix}, \qquad X_2 = \begin{bmatrix} 0.5600 & -0.3200 & 0 \\ 0 & 0.5600 & 0 \end{bmatrix}. \tag{3.5}$$

But by the iteration (2.1), we get

$$\begin{gathered} X_1 = \begin{bmatrix} 0.4800 & -0.1600 & 0 \\ 0 & 0.4800 & 0 \end{bmatrix}, \qquad X_2 = \begin{bmatrix} 0.4960 & -0.2240 & 0 \\ 0 & 0.4960 & 0 \end{bmatrix}, \\ X_3 = \begin{bmatrix} 0.4992 & -0.2432 & 0 \\ 0 & 0.4992 & 0 \end{bmatrix}, \qquad X_4 = \begin{bmatrix} 0.4998 & -0.2483 & 0 \\ 0 & 0.4998 & 0 \end{bmatrix}, \\ X_5 = \begin{bmatrix} 0.5000 & -0.2496 & 0 \\ 0 & 0.5000 & 0 \end{bmatrix}, \qquad X_6 = \begin{bmatrix} 0.5000 & -0.2499 & 0 \\ 0 & 0.5000 & 0 \end{bmatrix}. \end{gathered} \tag{3.6}$$

From the data in (3.5) and (3.6), we obtain Table 1.

From the above in (3.5), (3.6), and Table 1, we know that we only need two steps by Algorithm 1, but five steps by using iterative form (2.1).

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