Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 250538, 20 pages
doi:10.1155/2012/250538

Research Article

Hybrid Method with Perturbation for Lipschitzian Pseudocontractions

Lu-Chuan Ceng[1] and Ching-Feng Wen[2]

[1] *Department of Mathematics, Shanghai Normal University and Scientific Computing Key Laboratory of Shanghai Universities, Shanghai 200234, China*
[2] *Center for General Education, Kaohsiung Medical University, Kaohsiung 807, Taiwan*

Correspondence should be addressed to Ching-Feng Wen, cfwen@kmu.edu.tw

Received 21 May 2012; Accepted 5 June 2012

Academic Editor: Jen-Chih Yao

Assume that F is a nonlinear operator which is Lipschitzian and strongly monotone on a nonempty closed convex subset C of a real Hilbert space H. Assume also that Ω is the intersection of the fixed point sets of a finite number of Lipschitzian pseudocontractive self-mappings on C. By combining hybrid steepest-descent method, Mann's iteration method and projection method, we devise a hybrid iterative algorithm with perturbation F, which generates two sequences from an arbitrary initial point $x_0 \in H$. These two sequences are shown to converge in norm to the same point $P_\Omega x_0$ under very mild assumptions.

1. Introduction and Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$ and C a nonempty closed convex subset of H. Let $T : C \to C$ be a self-mapping of C. Recall that T is said to be a pseudocontractive mapping if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C, \tag{1.1}$$

and T is said to be a strictly pseudo-contractive mapping if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{1.2}$$

For such cases, we also say that T is a k-strict pseudo-contractive mapping. We use $F(T)$ to denote the set of fixed points of T.

It is well known that the class of strictly pseudo-contractive mappings strictly includes the class of nonexpansive mappings which are the mappings T on C such that

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.3}$$

Iterative methods for nonexpansive mappings have been extensively investigated; see [1–16] and the references therein.

However, iterative methods for strictly pseudo-contractive mappings are far less developed than those for nonexpansive mappings though Browder and Petryshyn initiated their work in 1967; the reason is probably that the second term appearing on the right-hand side of (1.2) impedes the convergence analysis for iterative algorithms used to find a fixed point of the strictly pseudo-contractive mapping T. However, on the other hand, strictly pseudo-contractive mappings have more powerful applications than nonexpansive mappings do in solving inverse problems; see Scherzer [17]. Therefore, it is interesting to develop iterative methods for strictly pseudo-contractive mappings. As a matter of fact, Browder and Petryshyn [18] showed that if a k-strict pseudo-contractive mapping T has a fixed point in C, then starting with an initial $x_0 \in C$, the sequence $\{x_n\}$ generated by the recursive formula:

$$x_{n+1} = \alpha x_n + (1-\alpha)Tx_n, \quad \forall n \geq 0, \tag{1.4}$$

where α is a constant such that $k < \alpha < 1$ converges weakly to a fixed point of T.

Recently, Marino and Xu [19] have extended Browder and Petryshyn's result by proving that the sequence $\{x_n\}$ generated by the following Mann's algorithm:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad \forall n \geq 0 \tag{1.5}$$

converges weakly to a fixed point of T, provided that the control sequence $\{\alpha_n\}$ satisfies the condition that $k < \alpha_n < 1$ for all n and $\sum_{n=0}^{\infty}(\alpha_n - k)(1-\alpha_n) = \infty$. However, this convergence is in general not strong. It is well known that if C is a bounded and closed convex subset of H, and $T : C \to C$ is a demicontinuous pseudocontraction, then T has a fixed point in C (Theorem 2.3 in [20]). However, all efforts to approximate such a fixed point by virtue of the normal Mann's iteration algorithm failed.

In 1974, Ishikawa [21] introduced a new iteration algorithm and proved the following convergence theorem.

Theorem I (see [21]). *If C is a compact convex subset of a Hilbert space $H, T : C \to C$ is a Lipschitzian pseudocontraction and $x_0 \in C$ is chosen arbitrarily, then the sequence $\{x_n\}_{n\geq 0}$ converges strongly to a fixed point of T, where $\{x_n\}$ is defined iteratively for each positive integer $n \geq 0$ by*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, \end{aligned} \tag{1.6}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers satisfying the conditions (i) $0 \leq \alpha_n \leq \beta_n < 1$; (ii) $\beta_n \to 0$ *as* $n \to \infty$; (iii) $\sum_{n=0}^{\infty} \alpha_n \beta_n = \infty$.

Since its publication in 1974, it remains an open question whether or not Mann's iteration algorithm converges under the setting of Theorem I to a fixed point of T if the mapping T is Lipschitzian pseudo-contractive. In [22], Chidume and Mutangadura gave an example of a Lipschitzian pseudocontraction with a unique fixed point for which Mann's iteration algorithm fails to converge.

In an infinite-dimensional Hilbert space, Mann and Ishikawa's iteration algorithms have only weak convergence, in general, even for nonexpansive mapping. So, in order to get strong convergence for strictly pseudo-contractive mappings, several attempts have been made based on the CQ method (see, e.g., [19, 23, 24]). The last scheme, in such a direction, seems for us to be the following due to Zhou [25]:

$$\begin{gathered}
x_0 \in C \quad \text{chosen arbitrarily}, \\
y_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\
z_n = (1-\beta_n)x_n + \beta_n T y_n, \\
C_n = \left\{ z \in C : \|z_n - z\|^2 \le \|x_n - z\|^2 - \alpha_n\beta_n\left(1 - 2\alpha_n - L^2\alpha_n^2\right)\|x_n - Tx_n\|^2 \right\}, \\
Q_n = \{ z \in C : \langle x_n - z, x_0 - x_n \rangle \ge 0 \}, \\
x_{n+1} = P_{C_n \cap Q_n} x_0, \quad \forall n \ge 0.
\end{gathered} \tag{1.7}$$

He proved, under suitable choice of the parameters α_n and β_n, that the sequence $\{x_n\}$ generated by (1.7) strongly converges to $P_{F(T)}x_0$.

Among classes of nonlinear mappings, the class of pseudocontractions is one of the most important. This is due to the relation between the class of pseudocontractions and the class of monotone mappings (we recall that a mapping A is monotone if $\langle Ax - Ay, x - y \rangle \ge 0$ for all $x, y \in H$). A mapping A is monotone if and only if $(I - A)$ is pseudo-contractive. It is well known (see, e.g., [26]) that if S is monotone, then the solutions of the equation $Sx = 0$ correspond to the equilibrium points of some evolution systems. Consequently, considerable research efforts, especially within the past 30 years or so, have been devoted to iterative methods for approximating fixed points of a pseudo-contractive mapping T (see e.g., [27–32] and the references therein).

Very recently, motivated by the work in [19, 25, 33] and the related work in the literature, Yao et al. [34] suggested and analyzed a hybrid algorithm for pseudo-contractive mappings in Hilbert spaces. Further, they proved the strong convergence of the proposed iterative algorithm for Lipschitzian pseudo-contractive mappings.

Theorem YLM (see [34]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a L-Lipschitzian pseudo-contractive mapping such that $F(T) \neq \emptyset$. Assume that the sequence $\alpha_n \in [a, b]$ for some $a, b \in (0,\ 1/(L+1))$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, let $\{x_n\}$ be the sequence in C generated iteratively by*

$$\begin{gathered}
y_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\
C_{n+1} = \left\{ z \in C_n : \left\|\alpha_n (I - T) y_n\right\|^2 \le 2\alpha_n \left\langle x_n - z, (I - T) y_n \right\rangle \right\}, \\
x_{n+1} = P_{C_{n+1}} x_0, \quad n \ge 1.
\end{gathered} \tag{1.8}$$

Then $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

Inspired by the above research work of Yao et al. [34], in this paper we will continue this direction of research. Let C be a nonempty closed convex subset of a real Hilbert space H. We will propose a new hybrid iterative scheme with perturbed mapping for approximating fixed points of a Lipschitzian pseudo-contractive self-mapping on C. We will establish a strong convergence theorem for this hybrid iterative scheme. To be more specific, let $T : C \to C$ be a L-Lipschitzian pseudo-contractive mapping and $F : C \to H$ a mapping such that for some constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strong monotone. Let $\{\alpha_n\} \subset (0,1)$, $\{\lambda_n\} \subset [0,1)$ and take a fixed number $\mu \in (0, 2\eta/\kappa^2)$. We introduce the following hybrid iterative process with perturbed mapping F. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, two sequences $\{x_n\}, \{y_n\}$ are generated as follows:

$$
\begin{aligned}
y_n &= (1-\alpha_n)x_n + \alpha_n P_C\left[Tx_n - \lambda_n\mu F(Tx_n)\right], \\
C_{n+1} &= \Big\{ z \in C_n : \left\|\alpha_n\left(I - P_C(I-\lambda_n\mu F)T\right)y_n\right\|^2 \le 2\alpha_n\left[\langle x_n - z, \left(I - P_C(I-\lambda_n\mu F)T\right)y_n\rangle \right. \\
&\qquad \left. - \langle Ty_n - P_C(I-\lambda_n\mu F)Ty_n, y_n - z\rangle\right] \Big\} \\
x_{n+1} &= P_{C_{n+1}}x_0, \quad n \ge 1.
\end{aligned}
\tag{1.9}
$$

It is clear that if $\lambda_n = 0$, for all $n \ge 1$, then the hybrid iterative scheme (1.9) reduces to the hybrid iterative process (1.8). Under very mild assumptions, we obtain a strong convergence theorem for the sequences $\{x_n\}$ and $\{y_n\}$ generated by the introduced method. Our proposed hybrid method with perturbation is quite general and flexible and includes the hybrid method considered in [34] and several other iterative methods as special cases. Our results represent the modification, supplement, extension, and improvement of [34, Algorithm 3.1 and Theorem 3.1]. Further, we consider the more general case, where $\{T_i\}_{i=1}^N$ are N L-Lipschitzian pseudo-contractive self-mappings on C with $N \ge 1$ an integer. In this case, we propose another hybrid iterative process with perturbed mapping F for approximating a common fixed point of $\{T_i\}_{i=1}^N$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, two sequences $\{x_n\}$ and $\{y_n\}$ are generated as follows:

$$
\begin{aligned}
y_n &= (1-\alpha_n)x_n + \alpha_n P_C\left[T_nx_n - \lambda_n\mu F(T_nx_n)\right], \\
C_{n+1} &= \Big\{ z \in C_n : \left\|\alpha_n\left(I - P_C(I-\lambda_n\mu F)T_n\right)y_n\right\|^2 \le 2\alpha_n\left[\langle x_n - z, \left(I - P_C(I-\lambda_n\mu F)T_n\right)y_n\rangle \right. \\
&\qquad \left. - \langle T_ny_n - P_C(I-\lambda_n\mu F)T_ny_n, y_n - z\rangle\right] \Big\} \\
x_{n+1} &= P_{C_{n+1}}x_0, \quad n \ge 1,
\end{aligned}
\tag{1.10}
$$

where $T_n := T_{n \bmod N}$, for integer $n \ge 1$, with the mod function taking values in the set $\{1, 2, \ldots, N\}$ (i.e., if $n = jN + q$ for some integers $j \ge 0$ and $0 \le q < N$, then $T_n = T_N$ if $q = 0$ and $T_n = T_q$ if $1 < q < N$). It is clear that if $N = 1$, then the hybrid iterative scheme (1.10) reduces to the hybrid iterative process (1.9). Under quite appropriate conditions, we derive a strong convergence theorem for the sequences $\{x_n\}$ and $\{y_n\}$ generated by the proposed method.

We now give some preliminaries and results which will be used in the rest of this paper. A Banach space X is said to satisfy Opial's condition if whenever $\{x_n\}$ is a sequence in X which converges weakly to x, then

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \quad \forall y \in X,\ y \neq x. \tag{1.11}$$

It is well known that every Hilbert space H satisfies Opial's condition (see, e.g., [35]). Throughout this paper, we shall use the notations: "$\rightharpoonup$" and "$\to$" standing for the weak convergence and strong convergence, respectively. Moreover, we shall use the following notation: for a given sequence $\{x_n\} \subset X, \omega_w(x_n)$ denotes the weak ω-limit set of $\{x_n\}$, that is,

$$\omega_w(x_n) := \left\{x \in X : x_{n_j} \rightharpoonup x \text{ for some subsequence } \{n_j\} \text{ of } \{n\}\right\}. \tag{1.12}$$

In addition, for each point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C, \tag{1.13}$$

where P_C is called the metric projection of H onto C. It is known that P_C is a nonexpansive mapping.

Now we collect some lemmas which will be used in the proof of the main result in the next section. We note that Lemmas 1.1 and 1.2 are well known.

Lemma 1.1. *Let H be a real Hilbert space. There holds the following identity:*

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle, \quad \forall x, y \in H. \tag{1.14}$$

Lemma 1.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Given $x \in H$ and $z \in C$. Then $z = P_C x$ if and only if there holds the relation:*

$$\langle x - z, y - z\rangle \leq 0, \quad \forall y \in C. \tag{1.15}$$

Lemma 1.3 (see [23]). *Let C be a nonempty closed convex subset of H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_C u$. If $\{x_n\}$ is such that $\omega_w(x_n) \subset C$ and satisfies the condition:*

$$\|x_n - u\| \leq \|u - q\|, \quad \forall n \geq 0. \tag{1.16}$$

Then $x_n \to q$.

Lemma 1.4 (see [27]). *Let X be a real reflexive Banach space which satisfies Opial's condition. Let C be a nonempty closed convex subset of X, and $T : C \to C$ be a continuous pseudo-contractive mapping. Then, $I - T$ is demiclosed at zero.*

Let $T : C \to C$ be a nonexpansive mapping and $F : C \to H$ be a mapping such that for some constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone, that is, F satisfies the following conditions:

$$\|Fx - Fy\| \leq \kappa\|x - y\|, \quad \forall x, y \in C,$$
$$\langle Fx - Fy, x - y\rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in C, \tag{1.17}$$

respectively. For any given numbers $\lambda \in [0,1)$ and $\mu \in (0,\ 2\eta/\kappa^2)$, we define the mapping $T^\lambda : C \to H$:

$$T^\lambda x := Tx - \lambda\mu F(Tx), \quad \forall x \in C. \tag{1.18}$$

Lemma 1.5 (see [36]). *If $0 \leq \lambda < 1$ and $0 < \mu < 2\eta/\kappa^2$, then there holds for $T^\lambda : C \to H$:*

$$\left\|T^\lambda x - T^\lambda y\right\| \leq (1 - \lambda\tau)\|x - y\|, \quad \forall x, y \in C, \tag{1.19}$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0,1)$.

In particular, whenever $T = I$ the identity operator of H, we have

$$\|(I - \lambda\mu F)x - (I - \lambda\mu F)y\| \leq (1 - \lambda\tau)\|x - y\|, \quad \forall x, y \in C. \tag{1.20}$$

2. Main Result

In this section, we introduce a hybrid iterative algorithm with perturbed mapping for pseudo-contractive mappings in a real Hilbert space H.

Algorithm 2.1. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a pseudo-contractive mapping and $F : C \to H$ be a mapping such that for some constants $\kappa, \eta > 0, F$ is κ-Lipschitzian and η-strong monotone. Let $\{\alpha_n\} \subset (0,1)$, $\{\lambda_n\} \subset [0,1)$ and take a fixed number $\mu \in (0, 2\eta/\kappa^2)$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, define two sequences: $\{x_n\}$ and $\{y_n\}$ of C as follows:

$$\begin{aligned} y_n &= (1 - \alpha_n)x_n + \alpha_n P_C\left[Tx_n - \lambda_n\mu F(Tx_n)\right], \\ C_{n+1} &= \Big\{z \in C_n : \|\alpha_n(I - P_C(I - \lambda_n\mu F)T)y_n\|^2 \leq 2\alpha_n\left[\langle x_n - z, (I - P_C(I - \lambda_n\mu F)T)y_n\rangle\right. \\ &\qquad \left. - \langle Ty_n - P_C(I - \lambda_n\mu F)Ty_n, y_n - z\rangle\right]\Big\} \\ x_{n+1} &= P_{C_{n+1}}x_0, \quad n \geq 1. \end{aligned} \tag{2.1}$$

Now we prove the strong convergence of the above iterative algorithm for Lipschitzian pseudo-contractive mappings.

Theorem 2.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a L-Lipschitzian pseudo-contractive mapping such that $F(T) \neq \emptyset$, and let $F : C \to H$ be a mapping such that for some constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strong monotone. Assume that $\{\alpha_n\} \subset [a,b]$ for some $a, b \in (0,\ 1/(L+1))$ and $\{\lambda_n\} \subset [0,1)$ such that $\lim_{n\to\infty}\lambda_n = 0$. Take a fixed number $\mu \in (0,\ 2\eta/\kappa^2)$. Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by* (2.1) *converge strongly to the same point $P_{F(T)}x_0$.*

Proof. Firstly, we observe that $P_{F(T)}$ and $\{x_n\}$ are well defined. From [19, 27], we note that $F(T)$ is closed and convex. Indeed, by [27], we can define a mapping $g : C \to C$ by $g(x) = (2I - T)^{-1}$ for every $x \in C$. It is clear that g is a nonexpansive self-mapping such that $F(T) = F(g)$. Hence, by [23, Proposition 2.1 (iii)], we conclude that $F(g) = F(T)$ is a closed convex set. This implies that the projection $P_{F(T)}$ is well defined. It is obvious that $\{C_n\}$ is closed and convex. Thus, $\{x_n\}$ is also well defined.

Now, we show that $F(T) \subset C_n$ for all $n \geq 0$. Indeed, taking $p \in F(T)$, we note that $(I - T)p = 0$, and (1.1) is equivalent to

$$\langle (I-T)x - (I-T)y, x-y\rangle \geq 0, \quad \forall x, y \in C. \tag{2.2}$$

Using Lemma 1.1 and (2.2), we obtain

$$\begin{aligned}
&\|x_n - p - \alpha_n(I - P_C(I-\lambda_n\mu F)T)y_n\|^2 \\
&\quad = \|x_n - p\|^2 - \|\alpha_n(I - P_C(I-\lambda_n\mu F)T)y_n\|^2 \\
&\qquad - 2\alpha_n\langle (I - P_C(I-\lambda_n\mu F)T)y_n, x_n - p - \alpha_n(I - P_C(I-\lambda_n\mu F)T)y_n\rangle \\
&\quad = \|x_n - p\|^2 - \|\alpha_n(I - P_C(I-\lambda_n\mu F)T)y_n\|^2 - 2\alpha_n\langle (I-T)y_n - (I-T)p, y_n - p\rangle \\
&\qquad - 2\alpha_n\langle Ty_n - P_C(I-\lambda_n\mu F)Ty_n, y_n - p\rangle \\
&\qquad - 2\alpha_n\langle (I - P_C(I-\lambda_n\mu F)T)y_n, x_n - y_n - \alpha_n(I - P_C(I-\lambda_n\mu F)T)y_n\rangle \\
&\quad \leq \|x_n - p\|^2 - \|\alpha_n(I - P_C(I-\lambda_n\mu F)T)y_n\|^2 - 2\alpha_n\langle Ty_n - P_C(I-\lambda_n\mu F)Ty_n, y_n - p\rangle \\
&\qquad - 2\alpha_n\langle (I - P_C(I-\lambda_n\mu F)T)y_n, x_n - y_n - \alpha_n(I - P_C(I-\lambda_n\mu F)T)y_n\rangle \\
&\quad = \|x_n - p\|^2 - \|x_n - y_n + y_n - x_n + \alpha_n(I - P_C(I-\lambda_n\mu F)T)y_n\|^2 \\
&\qquad - 2\alpha_n\langle Ty_n - P_C(I-\lambda_n\mu F)Ty_n, y_n - p\rangle \\
&\qquad - 2\alpha_n\langle (I - P_C(I-\lambda_n\mu F)T)y_n, x_n - y_n - \alpha_n(I - P_C(I-\lambda_n\mu F)T)y_n\rangle \\
&\quad = \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - x_n + \alpha_n(I - P_C(I-\lambda_n\mu F)T)y_n\|^2 \\
&\qquad - 2\langle x_n - y_n, y_n - x_n + \alpha_n(I - P_C(I-\lambda_n\mu F)T)y_n\rangle \\
&\qquad + 2\alpha_n\langle (I - P_C(I-\lambda_n\mu F)T)y_n, y_n - x_n + \alpha_n(I - P_C(I-\lambda_n\mu F)T)y_n\rangle \\
&\qquad - 2\alpha_n\langle Ty_n - P_C(I-\lambda_n\mu F)Ty_n, y_n - p\rangle \\
&\quad = \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - x_n + \alpha_n(I - P_C(I-\lambda_n\mu F)T)y_n\|^2 \\
&\qquad - 2\langle x_n - y_n - \alpha_n(I - P_C(I-\lambda_n\mu F)T)y_n, y_n - x_n + \alpha_n(I - P_C(I-\lambda_n\mu F)T)y_n\rangle \\
&\qquad - 2\alpha_n\langle Ty_n - P_C(I-\lambda_n\mu F)Ty_n, y_n - p\rangle
\end{aligned}$$

$$\begin{aligned}
&\le \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - x_n + \alpha_n (I - P_C(I - \lambda_n \mu F)T) y_n\|^2 \\
&\quad + 2\left|\langle x_n - y_n - \alpha_n (I - P_C(I - \lambda_n \mu F)T) y_n, y_n - x_n + \alpha_n (I - P_C(I - \lambda_n \mu F)T) y_n \rangle\right| \\
&\quad - 2\alpha_n \langle T y_n - P_C(I - \lambda_n \mu F) T y_n, y_n - p \rangle.
\end{aligned} \tag{2.3}$$

Since T is L-Lipschitzian, utilizing Lemma 1.5 we derive

$$\begin{aligned}
&\|(I - P_C(I - \lambda_n \mu F)T) x_n - (I - P_C(I - \lambda_n \mu F)T) y_n\| \\
&\quad \le \|x_n - y_n\| + \|P_C(I - \lambda_n \mu F) T x_n - P_C(I - \lambda_n \mu F) T y_n\| \\
&\quad \le \|x_n - y_n\| + \|(I - \lambda_n \mu F) T x_n - (I - \lambda_n \mu F) T y_n\| \\
&\quad \le \|x_n - y_n\| + (1 - \lambda_n \tau)\|T x_n - T y_n\| \\
&\quad \le \|x_n - y_n\| + \|T x_n - T y_n\| \\
&\quad \le (L+1)\|x_n - y_n\|.
\end{aligned} \tag{2.4}$$

From (2.1), we observe that $x_n - y_n = \alpha_n (I - P_C(I - \lambda_n \mu F)T) x_n$. Hence, utilizing Lemma 1.5 and (2.4) we obtain

$$\begin{aligned}
&\left|\langle x_n - y_n - \alpha_n (I - P_C(I - \lambda_n \mu F)T) y_n, y_n - x_n + \alpha_n (I - P_C(I - \lambda_n \mu F)T) y_n \rangle\right| \\
&\quad = \alpha_n \left|\langle (I - P_C(I - \lambda_n \mu F)T) x_n - (I - P_C(I - \lambda_n \mu F)T) y_n,\right. \\
&\qquad\qquad \left. y_n - x_n + \alpha_n (I - P_C(I - \lambda_n \mu F)T) y_n \rangle\right| \\
&\quad \le \alpha_n \|(I - P_C(I - \lambda_n \mu F)T) x_n - (I - P_C(I - \lambda_n \mu F)T) y_n\| \\
&\qquad \times \|y_n - x_n + \alpha_n (I - P_C(I - \lambda_n \mu F)T) y_n\| \\
&\quad \le \alpha_n (L+1)\|x_n - y_n\| \|y_n - x_n + \alpha_n (I - P_C(I - \lambda_n \mu F)T) y_n\| \\
&\quad \le \frac{\alpha_n (L+1)}{2}\left(\|x_n - y_n\|^2 + \|y_n - x_n + \alpha_n (I - P_C(I - \lambda_n \mu F)T) y_n\|^2\right).
\end{aligned} \tag{2.5}$$

Combining (2.3) and (2.5), we get

$$\begin{aligned}
&\|x_n - p - \alpha_n (I - P_C(I - \lambda_n \mu F)T) y_n\|^2 \\
&\quad \le \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - x_n + \alpha_n (I - P_C(I - \lambda_n \mu F)T) y_n\|^2 \\
&\qquad + \alpha_n (L+1)\left(\|x_n - y_n\|^2 + \|y_n - x_n + \alpha_n (I - P_C(I - \lambda_n \mu F)T) y_n\|^2\right) \\
&\qquad - 2\alpha_n \langle T y_n - P_C(I - \lambda_n \mu F) T y_n, y_n - p \rangle \\
&\quad = \|x_n - p\|^2 + [\alpha_n (L+1) - 1]\left(\|x_n - y_n\|^2 + \|y_n - x_n + \alpha_n (I - P_C(I - \lambda_n \mu F)T) y_n\|^2\right) \\
&\qquad - 2\alpha_n \langle T y_n - P_C(I - \lambda_n \mu F) T y_n, y_n - p \rangle \\
&\quad \le \|x_n - p\|^2 - 2\alpha_n \langle T y_n - P_C(I - \lambda_n \mu F) T y_n, y_n - p \rangle.
\end{aligned} \tag{2.6}$$

At the same time, we observe that

$$\left\|x_n - p - \alpha_n(I - P_C(I - \lambda_n\mu F)T)y_n\right\|^2 = \left\|x_n - p\right\|^2 - 2\alpha_n\langle x_n - p, (I - P_C(I - \lambda_n\mu F)T)y_n\rangle + \left\|\alpha_n(I - P_C(I - \lambda_n\mu F)T)y_n\right\|^2. \tag{2.7}$$

Therefore, from (2.6) and (2.7) we have

$$\left\|\alpha_n(I - P_C(I - \lambda_n\mu F)T)y_n\right\|^2 \le 2\alpha_n\left[\langle x_n - p, (I - P_C(I - \lambda_n\mu F)T)y_n\rangle - \langle Ty_n - P_C(I - \lambda_n\mu F)Ty_n, y_n - p\rangle\right], \tag{2.8}$$

which implies that

$$p \in C_n, \tag{2.9}$$

that is,

$$F(T) \subset C_n, \quad \forall n \ge 0. \tag{2.10}$$

From $x_n = P_{C_n}x_0$, we have

$$\left\langle x_0 - x_n, x_n - y\right\rangle \ge 0, \quad \forall y \in C_n. \tag{2.11}$$

Utilizing $F(T) \subset C_n$, we also have

$$\langle x_0 - x_n, x_n - u\rangle \ge 0, \quad \forall u \in F(T). \tag{2.12}$$

So, for all $u \in F(T)$ we have

$$\begin{aligned} 0 &\le \langle x_0 - x_n, x_n - u\rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - u\rangle \\ &= -\|x_0 - x_n\|^2 + \langle x_0 - x_n, x_0 - u\rangle \\ &\le -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - u\|, \end{aligned} \tag{2.13}$$

which hence implies that

$$\|x_0 - x_n\| \le \|x_0 - u\|, \quad \forall u \in F(T). \tag{2.14}$$

Thus, $\{x_n\}$ is bounded and so are $\{y_n\}$ and $\{Ty_n\}$.

From $x_n = P_{C_n} x_0$ and $x_{n+1} = P_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we have

$$\langle x_0 - x_n, x_n - x_{n+1} \rangle \geq 0. \tag{2.15}$$

Hence,

$$\begin{aligned} 0 &\leq \langle x_0 - x_n, x_n - x_{n+1} \rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1} \rangle \\ &= -\|x_0 - x_n\|^2 + \langle x_0 - x_n, x_0 - x_{n+1} \rangle \\ &\leq -\|x_0 - x_n\|^2 + \|x_0 - x_n\| \|x_0 - x_{n+1}\|, \end{aligned} \tag{2.16}$$

and therefore

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\|. \tag{2.17}$$

This implies that $\lim_{n \to \infty} \|x_n - x_0\|$ exists.

From Lemma 1.1 and (2.15), we obtain

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\ &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0 \rangle \\ &\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 \\ &\longrightarrow 0. \end{aligned} \tag{2.18}$$

Since $x_{n+1} \in C_{n+1} \subset C_n$, from $\|x_n - x_{n+1}\| \to 0$ and $\lambda_n \to 0$ it follows that

$$\begin{aligned} &\|\alpha_n (I - P_C (I - \lambda_n \mu F) T) y_n\|^2 \\ &\quad \leq 2\alpha_n [\langle x_n - x_{n+1}, (I - P_C (I - \lambda_n \mu F) T) y_n \rangle - \langle T y_n - P_C (I - \lambda_n \mu F) T y_n, y_n - x_{n+1} \rangle] \\ &\quad \leq 2\alpha_n [\|x_n - x_{n+1}\| \|y_n - P_C (I - \lambda_n \mu F) T y_n\| + \|T y_n - P_C (I - \lambda_n \mu F) T y_n\| \|y_n - x_{n+1}\|] \\ &\quad \leq 2\alpha_n [\|x_n - x_{n+1}\| \|y_n - P_C (I - \lambda_n \mu F) T y_n\| + \|T y_n - (I - \lambda_n \mu F) T y_n\| \|y_n - x_{n+1}\|] \\ &\quad = 2\alpha_n [\|x_n - x_{n+1}\| \|y_n - P_C (I - \lambda_n \mu F) T y_n\| + \lambda_n \mu \|F(T y_n)\| \|y_n - x_{n+1}\|] \\ &\quad \longrightarrow 0. \end{aligned} \tag{2.19}$$

Noticing that $\alpha_n \in [a, b]$ for some $a, b \in (0, \ 1/(L+1))$, thus, we obtain

$$\|y_n - P_C (I - \lambda_n \mu F) T y_n\| \longrightarrow 0. \tag{2.20}$$

Also, we note that $\|Ty_n - P_C(I - \lambda_n \mu F)Ty_n\| \le \lambda_n \mu \|F(Ty_n)\| \to 0$. Therefore, we get

$$\|y_n - Ty_n\| \le \|y_n - P_C(I - \lambda_n \mu F)Ty_n\| + \|Ty_n - P_C(I - \lambda_n \mu F)Ty_n\| \longrightarrow 0. \tag{2.21}$$

On the other hand, utilizing Lemma 1.5 we deduce that

$$\begin{aligned}
&\|x_n - P_C(I - \lambda_n \mu F)Tx_n\| \\
&\quad \le \|x_n - y_n\| + \|y_n - P_C(I - \lambda_n \mu F)Ty_n\| + \|P_C(I - \lambda_n \mu F)Ty_n - P_C(I - \lambda_n \mu F)Tx_n\| \\
&\quad \le \|x_n - y_n\| + \|y_n - P_C(I - \lambda_n \mu F)Ty_n\| + \|(I - \lambda_n \mu F)Ty_n - (I - \lambda_n \mu F)Tx_n\| \\
&\quad \le \|x_n - y_n\| + \|y_n - P_C(I - \lambda_n \mu F)Ty_n\| + (1 - \lambda_n \tau)\|Ty_n - Tx_n\| \\
&\quad \le \|x_n - y_n\| + \|y_n - P_C(I - \lambda_n \mu F)Ty_n\| + L\|y_n - x_n\| \\
&\quad = (L+1)\|x_n - y_n\| + \|y_n - P_C(I - \lambda_n \mu F)Ty_n\| \\
&\quad = \alpha_n (L+1)\|x_n - P_C(I - \lambda_n \mu F)Tx_n\| + \|y_n - P_C(I - \lambda_n \mu F)Ty_n\|,
\end{aligned} \tag{2.22}$$

that is,

$$\|x_n - P_C(I - \lambda_n \mu F)Tx_n\| \le \frac{1}{1 - \alpha_n (L+1)} \|y_n - P_C(I - \lambda_n \mu F)Ty_n\| \longrightarrow 0. \tag{2.23}$$

Meantime, it is clear that

$$\|Tx_n - P_C(I - \lambda_n \mu F)Tx_n\| \le \lambda_n \mu \|F(Tx_n)\| \longrightarrow 0. \tag{2.24}$$

Consequently,

$$\|x_n - Tx_n\| \le \|x_n - P_C(I - \lambda_n \mu F)Tx_n\| + \|Tx_n - P_C(I - \lambda_n \mu F)Tx_n\| \longrightarrow 0. \tag{2.25}$$

Now (2.25) and Lemma 1.4 guarantee that every weak limit point of $\{x_n\}$ is a fixed point of T, that is, $\omega_w(x_n) \subset F(T)$. In fact, the inequality (2.14) and Lemma 1.3 ensure the strong convergence of $\{x_n\}$ to $P_{F(T)}x_0$. Since $\|x_n - y_n\| = \|\alpha_n(I - P_C(I - \lambda_n \mu F)T)x_n\| \to 0$, it is immediately known that $\{y_n\}$ converges strongly to $P_{F(T)}x_0$. This completes the proof. □

Corollary 2.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$, and let $F : C \to H$ be a mapping such that for some constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strong monotone. Assume that $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1/2)$ and $\{\lambda_n\} \subset [0, 1)$ such that $\lim_{n\to\infty}\lambda_n = 0$. Take a fixed number $\mu \in (0, 2\eta/\kappa^2)$. Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by* (2.1) *converge strongly to the same point $P_{F(T)}x_0$.*

Corollary 2.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a L-Lipschitzian pseudo-contractive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\} \subset [a, b]$ for some*

$a, b \in (0,\ 1/(L+1))$ and $\{\lambda_n\} \subset [0,1)$ such that $\lim_{n\to\infty}\lambda_n = 0$. Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by the scheme

$$\begin{aligned} y_n &= (1-\alpha_n)x_n + \alpha_n P_C((1-\lambda_n)Tx_n), \\ C_{n+1} &= \Big\{z \in C_n : \left\|\alpha_n\left(y_n - P_C\left((1-\lambda_n)Ty_n\right)\right)\right\|^2 \leq 2\alpha_n\left[\left\langle x_n - z, y_n - P_C\left((1-\lambda_n)Ty_n\right)\right\rangle \right. \\ &\qquad \left. -\left\langle Ty_n - P_C\left((1-\lambda_n)Ty_n\right), y_n - z\right\rangle\right]\Big\} \\ x_{n+1} &= P_{C_{n+1}}x_0 \end{aligned} \tag{2.26}$$

converge strongly to the same point $P_{F(T)}x_0$.

Proof. Put $\mu = 2$ and $F = (1/2)I$ in Theorem 2.2. Then, in this case we have $\kappa = \eta = 1/2$, and hence

$$\left(0, \frac{2\eta}{\kappa^2}\right) = (0,4). \tag{2.27}$$

This implies that $\mu = 2 \in (0,\ 2\eta/\kappa^2) = (0,4)$. Meantime, it is easy to see that the scheme (2.1) reduces to (2.26). Therefore, by Theorem 2.2, we obtain the desired result. □

Corollary 2.5 ([34, Corollary 3.2]). *Let $A : H \to H$ be a L-Lipschitzian monotone mapping for which $A^{-1}(0) \neq \emptyset$. Assume that the sequence $\{\alpha_n\} \subset [a,b]$ for some $a, b \in (0,\ 1/(L+2))$. Then the sequence $\{x_n\}$ generated by the scheme*

$$\begin{aligned} y_n &= x_n - \alpha_n A x_n, \\ C_{n+1} &= \left\{z \in C_n : \left\|\alpha_n A y_n\right\|^2 \leq 2\alpha_n\left\langle x_n - z, Ay_n\right\rangle\right\}, \\ x_{n+1} &= P_{C_{n+1}}x_0 \end{aligned} \tag{2.28}$$

strongly converges to $P_{A^{-1}(0)}x_0$.

Proof. Put $\lambda_n = 0$ and $T = I - A$ in Corollary 2.4. Then, it is easy to see that the scheme (2.26) reduces to (2.28). Therefore, by Corollary 2.4, we derive the desired result. □

Next, consider the more general case where Ω is expressed as the intersection of the fixed-point sets of N pseudo-contractive mappings $T_i : C \to C$ with $N \geq 1$ an integer, that is,

$$\Omega = \bigcap_{i=1}^{N} F(T_i). \tag{2.29}$$

In this section, we propose another hybrid iterative algorithm with perturbed mapping for a finite family of pseudo-contractive mappings in a real Hilbert space H.

Algorithm 2.6. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^{N}$ be N pseudo-contractive self-mappings on C with $N \geq 1$ an integer, and let $F : C \to H$ be a mapping such that for some constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strong monotone. Let $\{\alpha_n\} \subset (0,1), \{\lambda_n\} \subset [0,1)$, and take a fixed number $\mu \in (0, 2\eta/\kappa^2)$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, define two sequences $\{x_n\}$, $\{y_n\}$ of C as follows:

$$\begin{aligned} y_n &= (1-\alpha_n)x_n + \alpha_n P_C\left[T_n x_n - \lambda_n \mu F(T_n x_n)\right], \\ C_{n+1} &= \Big\{z \in C_n : \left\|\alpha_n(I - P_C(I-\lambda_n\mu F)T_n)y_n\right\|^2 \\ &\qquad \leq 2\alpha_n\left[\langle x_n - z, (I - P_C(I-\lambda_n\mu F)T_n)y_n\rangle \right. \\ &\qquad \left. -\langle T_n y_n - P_C(I-\lambda_n\mu F)T_n y_n, y_n - z\rangle\right]\Big\}, \\ x_{n+1} &= P_{C_{n+1}}x_0, \quad n \geq 1, \end{aligned} \tag{2.30}$$

where

$$T_n := T_{n \bmod N}, \tag{2.31}$$

for integer $n \geq 1$, with the mod function taking values in the set $\{1, 2, \ldots, N\}$ (i.e., if $n = jN + q$ for some integers $j \geq 0$ and $0 \leq q < N$, then $T_n = T_N$ if $q = 0$ and $T_n = T_q$ if $1 < q < N$).

Theorem 2.7. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^{N}$ be N L-Lipschitzian pseudo-contractive self-mappings on C such that $\Omega = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$, and let $F : C \to H$ be a mapping such that for some constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strong monotone. Assume that $\{\alpha_n\} \subset [a,b]$ for some $a, b \in (0,\ 1/(L+1))$ and $\{\lambda_n\} \subset [0,1)$ such that $\lim_{n\to\infty}\lambda_n = 0$. Take a fixed number $\mu \in (0,\ 2\eta/\kappa^2)$. Then the sequences $\{x_n\}$, $\{y_n\}$ generated by* (2.30) *converge strongly to the same point $P_\Omega x_0$.*

Proof. Firstly, as stated in the proof of Theorem 2.2, we can readily see that each $F(T_i)$ is closed and convex for $i = 1, 2, \ldots, N$. Hence, Ω is closed and convex. This implies that the projection P_Ω is well defined. It is clear that the sequence $\{C_n\}$ is closed and convex. Thus, $\{x_n\}$ is also well defined.

Now let us show that $\Omega \subset C_n$ for all $n \geq 0$. Indeed, taking $p \in \Omega$, we note that $(I - T_n)p = 0$ and

$$\langle (I-T_n)x - (I-T_n)y, x - y\rangle \geq 0, \quad \forall x, y \in C. \tag{2.32}$$

Using Lemma 1.1 and (2.32), we obtain

$$\begin{aligned} &\left\|x_n - p - \alpha_n(I - P_C(I-\lambda_n\mu F)T_n)y_n\right\|^2 \\ &\quad = \left\|x_n - p\right\|^2 - \left\|\alpha_n(I - P_C(I-\lambda_n\mu F)T_n)y_n\right\|^2 \\ &\qquad - 2\alpha_n\langle (I - P_C(I-\lambda_n\mu F)T_n)y_n, x_n - p - \alpha_n(I - P_C(I-\lambda_n\mu F)T_n)y_n\rangle \end{aligned}$$

$$
\begin{aligned}
&= \|x_n - p\|^2 - \|\alpha_n (I - P_C(I - \lambda_n \mu F)T_n) y_n\|^2 - 2\alpha_n \langle (I - T_n) y_n - (I - T_n) p, y_n - p \rangle \\
&\quad - 2\alpha_n \langle T_n y_n - P_C(I - \lambda_n \mu F) T_n y_n, y_n - p \rangle \\
&\quad - 2\alpha_n \langle (I - P_C(I - \lambda_n \mu F)T_n) y_n, x_n - y_n - \alpha_n (I - P_C(I - \lambda_n \mu F)T_n) y_n \rangle \\
&\le \|x_n - p\|^2 - \|\alpha_n (I - P_C(I - \lambda_n \mu F)T_n) y_n\|^2 - 2\alpha_n \langle T_n y_n - P_C(I - \lambda_n \mu F) T_n y_n, y_n - p \rangle \\
&\quad - 2\alpha_n \langle (I - P_C(I - \lambda_n \mu F)T_n) y_n, x_n - y_n - \alpha_n (I - P_C(I - \lambda_n \mu F)T_n) y_n \rangle \\
&= \|x_n - p\|^2 - \|x_n - y_n + y_n - x_n + \alpha_n (I - P_C(I - \lambda_n \mu F)T_n) y_n\|^2 \\
&\quad - 2\alpha_n \langle T_n y_n - P_C(I - \lambda_n \mu F) T_n y_n, y_n - p \rangle \\
&\quad - 2\alpha_n \langle (I - P_C(I - \lambda_n \mu F)T_n) y_n, x_n - y_n - \alpha_n (I - P_C(I - \lambda_n \mu F)T_n) y_n \rangle \\
&= \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - x_n + \alpha_n (I - P_C(I - \lambda_n \mu F)T_n) y_n\|^2 \\
&\quad - 2\langle x_n - y_n, y_n - x_n + \alpha_n (I - P_C(I - \lambda_n \mu F)T_n) y_n \rangle \\
&\quad + 2\alpha_n \langle (I - P_C(I - \lambda_n \mu F)T_n) y_n, y_n - x_n + \alpha_n (I - P_C(I - \lambda_n \mu F)T_n) y_n \rangle \\
&\quad - 2\alpha_n \langle T_n y_n - P_C(I - \lambda_n \mu F) T_n y_n, y_n - p \rangle \\
&= \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - x_n + \alpha_n (I - P_C(I - \lambda_n \mu F)T_n) y_n\|^2 \\
&\quad - 2\langle x_n - y_n - \alpha_n (I - P_C(I - \lambda_n \mu F)T_n) y_n, y_n - x_n + \alpha_n (I - P_C(I - \lambda_n \mu F)T_n) y_n \rangle \\
&\quad - 2\alpha_n \langle T_n y_n - P_C(I - \lambda_n \mu F) T_n y_n, y_n - p \rangle \\
&\le \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - x_n + \alpha_n (I - P_C(I - \lambda_n \mu F)T_n) y_n\|^2 \\
&\quad + 2\left| \langle x_n - y_n - \alpha_n (I - P_C(I - \lambda_n \mu F)T_n) y_n, y_n - x_n + \alpha_n (I - P_C(I - \lambda_n \mu F)T_n) y_n \rangle \right| \\
&\quad - 2\alpha_n \langle T_n y_n - P_C(I - \lambda_n \mu F) T_n y_n, y_n - p \rangle.
\end{aligned}
\tag{2.33}
$$

Since each T_i is L-Lipschitzian for $i = 1, 2, \ldots, N$, utilizing Lemma 1.5 we derive

$$
\begin{aligned}
&\|(I - P_C(I - \lambda_n \mu F)T_n) x_n - (I - P_C(I - \lambda_n \mu F)T_n) y_n\| \\
&\quad \le \|x_n - y_n\| + \|P_C(I - \lambda_n \mu F) T_n x_n - P_C(I - \lambda_n \mu F) T_n y_n\| \\
&\quad \le \|x_n - y_n\| + \|(I - \lambda_n \mu F) T_n x_n - (I - \lambda_n \mu F) T_n y_n\| \\
&\quad \le \|x_n - y_n\| + (1 - \lambda_n \tau) \|T_n x_n - T_n y_n\| \\
&\quad \le \|x_n - y_n\| + \|T_n x_n - T_n y_n\| \\
&\quad \le (L + 1) \|x_n - y_n\|.
\end{aligned}
\tag{2.34}
$$

From (2.30), we observe that $x_n - y_n = \alpha_n (I - P_C(I - \lambda_n \mu F)T_n) x_n$. Hence, utilizing Lemma 1.5

and (2.34) we obtain

$$\begin{aligned}
&\left|\left\langle x_n - y_n - \alpha_n (I - P_C(I - \lambda_n \mu F)T_n) y_n, y_n - x_n + \alpha_n (I - P_C(I - \lambda_n \mu F)T_n) y_n \right\rangle\right| \\
&\quad = \alpha_n \left|\left\langle (I - P_C(I - \lambda_n \mu F)T_n) x_n - (I - P_C(I - \lambda_n \mu F)T_n) y_n, \right.\right. \\
&\qquad \left.\left. y_n - x_n + \alpha_n (I - P_C(I - \lambda_n \mu F)T_n) y_n \right\rangle\right| \\
&\quad \le \alpha_n \left\| (I - P_C(I - \lambda_n \mu F)T_n) x_n - (I - P_C(I - \lambda_n \mu F)T_n) y_n \right\| \\
&\qquad \times \left\| y_n - x_n + \alpha_n (I - P_C(I - \lambda_n \mu F)T_n) y_n \right\| \\
&\quad \le \alpha_n (L+1) \|x_n - y_n\| \left\| y_n - x_n + \alpha_n (I - P_C(I - \lambda_n \mu F)T_n) y_n \right\| \\
&\quad \le \frac{\alpha_n (L+1)}{2} \left(\|x_n - y_n\|^2 + \left\| y_n - x_n + \alpha_n (I - P_C(I - \lambda_n \mu F)T_n) y_n \right\|^2 \right).
\end{aligned} \tag{2.35}$$

Combining (2.33) and (2.35), we get

$$\begin{aligned}
&\left\| x_n - p - \alpha_n (I - P_C(I - \lambda_n \mu F)T_n) y_n \right\|^2 \\
&\quad \le \|x_n - p\|^2 - \|x_n - y_n\|^2 - \left\| y_n - x_n + \alpha_n (I - P_C(I - \lambda_n \mu F)T_n) y_n \right\|^2 \\
&\qquad + \alpha_n (L+1) \left(\|x_n - y_n\|^2 + \left\| y_n - x_n + \alpha_n (I - P_C(I - \lambda_n \mu F)T_n) y_n \right\|^2 \right) \\
&\qquad - 2\alpha_n \langle T_n y_n - P_C(I - \lambda_n \mu F)T_n y_n, y_n - p \rangle \\
&\quad = \|x_n - p\|^2 + [\alpha_n (L+1) - 1] \left(\|x_n - y_n\|^2 + \left\| y_n - x_n + \alpha_n (I - P_C(I - \lambda_n \mu F)T_n) y_n \right\|^2 \right) \\
&\qquad - 2\alpha_n \langle T_n y_n - P_C(I - \lambda_n \mu F)T_n y_n, y_n - p \rangle \\
&\quad \le \|x_n - p\|^2 - 2\alpha_n \langle T_n y_n - P_C(I - \lambda_n \mu F)T_n y_n, y_n - p \rangle.
\end{aligned} \tag{2.36}$$

Meantime, we observe that

$$\begin{aligned}
&\left\| x_n - p - \alpha_n (I - P_C(I - \lambda_n \mu F)T_n) y_n \right\|^2 \\
&\quad = \|x_n - p\|^2 - 2\alpha_n \langle x_n - p, (I - P_C(I - \lambda_n \mu F)T_n) y_n \rangle \\
&\qquad + \left\| \alpha_n (I - P_C(I - \lambda_n \mu F)T_n) y_n \right\|^2.
\end{aligned} \tag{2.37}$$

Therefore, from (2.36) and (2.37) we have

$$\begin{aligned}
\left\| \alpha_n (I - P_C(I - \lambda_n \mu F)T_n) y_n \right\|^2 \le 2\alpha_n [&\langle x_n - p, (I - P_C(I - \lambda_n \mu F)T_n) y_n \rangle \\
&- \langle T_n y_n - P_C(I - \lambda_n \mu F)T_n y_n, y_n - p \rangle],
\end{aligned} \tag{2.38}$$

which implies that

$$p \in C_n, \tag{2.39}$$

that is,

$$\Omega \subset C_n, \quad \forall n \geq 0. \tag{2.40}$$

From $x_n = P_{C_n} x_0$, we have

$$\langle x_0 - x_n, x_n - y \rangle \geq 0, \quad \forall y \in C_n. \tag{2.41}$$

Utilizing $\Omega \subset C_n$, we also have

$$\langle x_0 - x_n, x_n - u \rangle \geq 0, \quad \forall u \in \Omega. \tag{2.42}$$

So, for all $u \in \Omega$ we have

$$\begin{aligned} 0 &\leq \langle x_0 - x_n, x_n - u \rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - u \rangle \\ &= -\|x_0 - x_n\|^2 + \langle x_0 - x_n, x_0 - u \rangle \\ &\leq -\|x_0 - x_n\|^2 + \|x_0 - x_n\| \|x_0 - u\|, \end{aligned} \tag{2.43}$$

which hence implies that

$$\|x_0 - x_n\| \leq \|x_0 - u\|, \quad \forall u \in \Omega. \tag{2.44}$$

Thus $\{x_n\}$ is bounded and so are $\{y_n\}$ and $\{T_n y_n\}$.

From $x_n = P_{C_n} x_0$ and $x_{n+1} = P_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we have

$$\langle x_0 - x_n, x_n - x_{n+1} \rangle \geq 0. \tag{2.45}$$

Hence,

$$\begin{aligned} 0 &\leq \langle x_0 - x_n, x_n - x_{n+1} \rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1} \rangle \\ &= -\|x_0 - x_n\|^2 + \langle x_0 - x_n, x_0 - x_{n+1} \rangle \\ &\leq -\|x_0 - x_n\|^2 + \|x_0 - x_n\| \|x_0 - x_{n+1}\|, \end{aligned} \tag{2.46}$$

and therefore

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\|. \tag{2.47}$$

This implies that $\lim_{n\to\infty}\|x_n - x_0\|$ exists.

From Lemma 1.1 and (2.45), we obtain

$$\begin{aligned}\|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\ &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0\rangle \\ &\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{2.48}$$

Thus,

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{2.49}$$

Obviously, it is easy to see that $\lim_{n\to\infty}\|x_n - x_{n+i}\| = 0$ for each $i = 1, 2, \ldots, N$. Since $x_{n+1} \in C_{n+1} \subset C_n$, from $\|x_n - x_{n+1}\| \to 0$ and $\lambda_n \to 0$ it follows that

$$\begin{aligned}&\left\|\alpha_n(I - P_C(I - \lambda_n\mu F)T_n)y_n\right\|^2 \\ &\quad\leq 2\alpha_n\left[\langle x_n - x_{n+1}, (I - P_C(I - \lambda_n\mu F)T_n)y_n\rangle - \langle T_n y_n - P_C(I - \lambda_n\mu F)T_n y_n, y_n - x_{n+1}\rangle\right] \\ &\quad\leq 2\alpha_n\left[\|x_n - x_{n+1}\|\left\|y_n - P_C(I - \lambda_n\mu F)T_n y_n\right\| + \left\|T_n y_n - P_C(I - \lambda_n\mu F)T_n y_n\right\|\left\|y_n - x_{n+1}\right\|\right] \\ &\quad\leq 2\alpha_n\left[\|x_n - x_{n+1}\|\left\|y_n - P_C(I - \lambda_n\mu F)T_n y_n\right\| + \left\|T_n y_n - (I - \lambda_n\mu F)T_n y_n\right\|\left\|y_n - x_{n+1}\right\|\right] \\ &\quad= 2\alpha_n\left[\|x_n - x_{n+1}\|\left\|y_n - P_C(I - \lambda_n\mu F)T_n y_n\right\| + \lambda_n\mu\left\|F(T_n y_n)\right\|\left\|y_n - x_{n+1}\right\|\right] \longrightarrow 0.\end{aligned} \tag{2.50}$$

Noticing that $\alpha_n \in [a, b]$ for some $a, b \in (0,\ 1/(L+1))$, thus, we obtain

$$\left\|y_n - P_C(I - \lambda_n\mu F)T_n y_n\right\| \longrightarrow 0. \tag{2.51}$$

Also, we note that $\|T_n y_n - P_C(I - \lambda_n\mu F)T_n y_n\| \leq \lambda_n\mu\|F(T_n y_n)\| \to 0$. Therefore, we get

$$\left\|y_n - T_n y_n\right\| \leq \left\|y_n - P_C(I - \lambda_n\mu F)T_n y_n\right\| + \left\|T_n y_n - P_C(I - \lambda_n\mu F)T_n y_n\right\| \longrightarrow 0. \tag{2.52}$$

On the other hand, utilizing Lemma 1.5 we deduce that

$$\begin{aligned}
&\|x_n - P_C(I-\lambda_n\mu F)T_nx_n\| \\
&\quad\le \|x_n-y_n\| + \|y_n - P_C(I-\lambda_n\mu F)T_ny_n\| + \|P_C(I-\lambda_n\mu F)T_ny_n - P_C(I-\lambda_n\mu F)T_nx_n\| \\
&\quad\le \|x_n-y_n\| + \|y_n - P_C(I-\lambda_n\mu F)T_ny_n\| + \|(I-\lambda_n\mu F)T_ny_n - (I-\lambda_n\mu F)T_nx_n\| \\
&\quad\le \|x_n-y_n\| + \|y_n - P_C(I-\lambda_n\mu F)T_ny_n\| + (1-\lambda_n\tau)\|T_ny_n - T_nx_n\| \\
&\quad\le \|x_n-y_n\| + \|y_n - P_C(I-\lambda_n\mu F)T_ny_n\| + L\|y_n - x_n\| \\
&\quad= (L+1)\|x_n-y_n\| + \|y_n - P_C(I-\lambda_n\mu F)T_ny_n\| \\
&\quad= \alpha_n(L+1)\|x_n - P_C(I-\lambda_n\mu F)T_nx_n\| + \|y_n - P_C(I-\lambda_n\mu F)T_ny_n\|,
\end{aligned} \tag{2.53}$$

that is,

$$\|x_n - P_C(I-\lambda_n\mu F)T_nx_n\| \le \frac{1}{1-\alpha_n(L+1)}\|y_n - P_C(I-\lambda_n\mu F)T_ny_n\| \longrightarrow 0. \tag{2.54}$$

Furthermore, it is clear that

$$\|T_nx_n - P_C(I-\lambda_n\mu F)T_nx_n\| \le \lambda_n\mu\|F(T_nx_n)\| \longrightarrow 0 \quad \text{as } n\longrightarrow\infty. \tag{2.55}$$

Consequently,

$$\|x_n - T_nx_n\| \le \|x_n - P_C(I-\lambda_n\mu F)T_nx_n\| + \|T_nx_n - P_C(I-\lambda_n\mu F)T_nx_n\| \longrightarrow 0, \tag{2.56}$$

and hence for each $i = 1,2,\dots,N$:

$$\begin{aligned}
\|x_n - T_{n+i}x_n\| &\le \|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}x_{n+i}\| + \|T_{n+i}x_{n+i} - T_{n+i}x_n\| \\
&\le (L+1)\|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}x_{n+i}\| \longrightarrow 0 \quad \text{as } n\longrightarrow\infty.
\end{aligned} \tag{2.57}$$

So, we obtain $\lim_{n\to\infty}\|x_n - T_{n+i}x_n\| = 0$ for each $i = 1,2,\dots,N$. This implies that

$$\lim_{n\to\infty}\|x_n - T_lx_n\| = 0 \quad \text{for each } l = 1,2,\dots,N. \tag{2.58}$$

Now (2.58) and Lemma 1.4 guarantee that every weak limit point of $\{x_n\}$ is a fixed point of T_l. Since l is an arbitrary element in the finite set $\{1,2,\dots,N\}$, it is known that every weak limit point of $\{x_n\}$ lies in Ω, that is, $\omega_w(x_n)\subset\Omega$. This fact, the inequality (2.44) and Lemma 1.3 ensure the strong convergence of $\{x_n\}$ to $P_\Omega x_0$. Since $\|x_n - y_n\| = \|\alpha_n(I-P_C(I-\lambda_n\mu F)T_n)x_n\| \to 0$, it follows immediately that $\{y_n\}$ converges strongly to $P_\Omega x_0$. This completes the proof. □

Remark 2.8. Algorithm 3.1 in [34] for a Lipschitzian pseudocontraction is extended to develop our hybrid iterative algorithm with perturbation for N-Lipschitzian pseudocontractions; that is, Algorithm 2.6. Theorem 2.7 is more general and more flexible than Theorem 3.1 in [34]. Also, the proof of Theorem 2.7 is very different from that of Theorem 3.1 in [34] because our technique of argument depends on Lemma 1.5. Finally, we observe that several recent results for pseudocontractive and related mappings can be found in [37–42].

References

[1] H. H. Bauschke, "The approximation of fixed points of compositions of nonexpansive mappings in Hilbert space," *Journal of Mathematical Analysis and Applications*, vol. 202, no. 1, pp. 150–159, 1996.

[2] D. Wu, S.-s. Chang, and G. X. Yuan, "Approximation of common fixed points for a family of finite nonexpansive mappings in Banach space," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 63, no. 5–7, pp. 987–999, 2005.

[3] T.-H. Kim and H.-K. Xu, "Strong convergence of modified Mann iterations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 61, no. 1-2, pp. 51–60, 2005.

[4] S. Reich, "Weak convergence theorems for nonexpansive mappings in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 67, no. 2, pp. 274–276, 1979.

[5] H.-K. Xu, "Strong convergence of an iterative method for nonexpansive and accretive operators," *Journal of Mathematical Analysis and Applications*, vol. 314, no. 2, pp. 631–643, 2006.

[6] K. Nakajo and W. Takahashi, "Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups," *Journal of Mathematical Analysis and Applications*, vol. 279, no. 2, pp. 372–379, 2003.

[7] J. G. O'Hara, P. Pillay, and H.-K. Xu, "Iterative approaches to finding nearest common fixed points of nonexpansive mappings in Hilbert spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 54, no. 8, pp. 1417–1426, 2003.

[8] N. Shioji and W. Takahashi, "Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 125, no. 12, pp. 3641–3645, 1997.

[9] Y. Yao and J.-C. Yao, "On modified iterative method for nonexpansive mappings and monotone mappings," *Applied Mathematics and Computation*, vol. 186, no. 2, pp. 1551–1558, 2007.

[10] S.-y. Matsushita and W. Takahashi, "A strong convergence theorem for relatively nonexpansive mappings in a Banach space," *Journal of Approximation Theory*, vol. 134, no. 2, pp. 257–266, 2005.

[11] L.-C. Zeng and J.-C. Yao, "Implicit iteration scheme with perturbed mapping for common fixed points of a finite family of nonexpansive mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 64, no. 11, pp. 2507–2515, 2006.

[12] L.-C. Zeng, "A note on approximating fixed points of nonexpansive mappings by the Ishikawa iteration process," *Journal of Mathematical Analysis and Applications*, vol. 226, no. 1, pp. 245–250, 1998.

[13] S.-S. Chang, "Viscosity approximation methods for a finite family of nonexpansive mappings in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 323, no. 2, pp. 1402–1416, 2006.

[14] H. Zegeye and N. Shahzad, "Viscosity methods of approximation for a common fixed point of a family of quasi-nonexpansive mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 7, pp. 2005–2012, 2008.

[15] C. E. Chidume and C. O. Chidume, "Iterative approximation of fixed points of nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 318, no. 1, pp. 288–295, 2006.

[16] J. S. Jung, "Iterative approaches to common fixed points of nonexpansive mappings in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 302, no. 2, pp. 509–520, 2005.

[17] O. Scherzer, "Convergence criteria of iterative methods based on Landweber iteration for solving nonlinear problems," *Journal of Mathematical Analysis and Applications*, vol. 194, no. 3, pp. 911–933, 1995.

[18] F. E. Browder and W. V. Petryshyn, "Construction of fixed points of nonlinear mappings in Hilbert space," *Journal of Mathematical Analysis and Applications*, vol. 20, pp. 197–228, 1967.

[19] G. Marino and H.-K. Xu, "Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 329, no. 1, pp. 336–346, 2007.

[20] K. Q. Lan and J. H. Wu, "Convergence of approximants for demicontinuous pseudo-contractive maps in Hilbert spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 49, no. 6, pp. 737–746, 2002.

[21] S. Ishikawa, "Fixed points by a new iteration method," *Proceedings of the American Mathematical Society*, vol. 44, pp. 147–150, 1974.

[22] C. E. Chidume and S. A. Mutangadura, "An example of the Mann iteration method for Lipschitz pseudocontractions," *Proceedings of the American Mathematical Society*, vol. 129, no. 8, pp. 2359–2363, 2001.

[23] C. Martinez-Yanes and H.-K. Xu, "Strong convergence of the CQ method for fixed point iteration processes," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 64, no. 11, pp. 2400–2411, 2006.

[24] T.-H. Kim and H.-K. Xu, "Strong convergence of modified Mann iterations for asymptotically nonexpansive mappings and semigroups," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 64, no. 5, pp. 1140–1152, 2006.

[25] H. Zhou, "Convergence theorems of fixed points for Lipschitz pseudo-contractions in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 343, no. 1, pp. 546–556, 2008.

[26] K. Deimling, "Zeros of accretive operators," *Manuscripta Mathematica*, vol. 13, pp. 365–374, 1974.

[27] H. Zhou, "Convergence theorems of common fixed points for a finite family of Lipschitz pseudocontractions in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 10, pp. 2977–2983, 2008.

[28] M. O. Osilike and A. Udomene, "Demiclosedness principle and convergence theorems for strictly pseudocontractive mappings of Browder-Petryshyn type," *Journal of Mathematical Analysis and Applications*, vol. 256, no. 2, pp. 431–445, 2001.

[29] C. E. Chidume and H. Zegeye, "Approximate fixed point sequences and convergence theorems for Lipschitz pseudocontractive maps," *Proceedings of the American Mathematical Society*, vol. 132, no. 3, pp. 831–840, 2004.

[30] Y. Yao, Y.-C. Liou, and R. Chen, "Strong convergence of an iterative algorithm for pseudocontractive mapping in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 12, pp. 3311–3317, 2007.

[31] L.-C. Zeng, N.-C. Wong, and J.-C. Yao, "Strong convergence theorems for strictly pseudocontractive mappings of Browder-Petryshyn type," *Taiwanese Journal of Mathematics*, vol. 10, no. 4, pp. 837–849, 2006.

[32] A. Udomene, "Path convergence, approximation of fixed points and variational solutions of Lipschitz pseudocontractions in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 8, pp. 2403–2414, 2007.

[33] W. Takahashi, Y. Takeuchi, and R. Kubota, "Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 341, no. 1, pp. 276–286, 2008.

[34] Y. Yao, Y.-C. Liou, and G. Marino, "A hybrid algorithm for pseudo-contractive mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 10, pp. 4997–5002, 2009.

[35] K. Geobel and W. A. Kirk, *Topics on Metric Fixed-Point Theory*, Cambridge University, Cambridge, UK, 1990.

[36] H. K. Xu and T. H. Kim, "Convergence of hybrid steepest-descent methods for variational inequalities," *Journal of Optimization Theory and Applications*, vol. 119, no. 1, pp. 185–201, 2003.

[37] L. C. Ceng, D. S. Shyu, and J. C. Yao, "Relaxed composite implicit iteration process for common fixed points of a finite family of strictly pseudocontractive mappings," *Fixed Point Theory and Applications*, vol. 2009, Article ID 402602, 16 pages, 2009.

[38] L. C. Ceng, A. Petruşel, and J. C. Yao, "A hybrid method for lipschitz continuous monotone mappings and asymptotically strict pseudocontractive mappings in the intermediate sense," *Journal of Nonlinear and Convex Analysis*, vol. 11, no. 1, 2010.

[39] L.-C. Ceng, A. Petruşel, and J.-C. Yao, "Iterative approximation of fixed points for asymptotically strict pseudocontractive type mappings in the intermediate sense," *Taiwanese Journal of Mathematics*, vol. 15, no. 2, pp. 587–606, 2011.

[40] L.-C. Ceng, Q. H. Ansari, and J.-C. Yao, "Strong and weak convergence theorems for asymptotically strict pseudocontractive mappings in intermediate sense," *Journal of Nonlinear and Convex Analysis*, vol. 11, no. 2, pp. 283–308, 2010.

[41] L.-C. Ceng, A. Petruşel, S. Szentesi, and J.-C. Yao, "Approximation of common fixed points and variational solutions for one-parameter family of Lipschitz pseudocontractions," *Fixed Point Theory*, vol. 11, no. 2, pp. 203–224, 2010.

[42] D. R. Sahu, N. C. Wong, and J. C. Yao, "A unified hybrid iterative method for solving variational inequalities involving generalized pseudo-contractive mappings," *SIAM Journal on Control and Optimization*. In press.