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Research Article

Generalized Stability of Euler-Lagrange Quadratic Functional Equation

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The main goal of this paper is the investigation of the general solution and the generalized Hyers-Ulam stability theorem of the following Euler-Lagrange type quadratic functional equation $f(ax+by)+af(x-by)=(a+1)b^2f(y)+a(a+1)f(x)$, in (β,p)-Banach space, where a,b are fixed rational numbers such that $a\neq -1,0$ and $b\neq 0$.

1. Introduction

In 1940, Ulam [1] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

Let G be a group and let G' be a metric group with metric $\rho(\cdot,\cdot)$. Given $\epsilon>0$, does there exist a $\delta>0$ such that if $f:G\to G'$ satisfies $\rho(f(xy),f(x)f(y))<\delta$ for all $x,y\in G$, then a homomorphism $h:G\to G'$ exists with $\rho(f(x),h(x))<\epsilon$ for all $x\in G$?

In 1941, the first result concerning the stability of functional equations was presented by Hyers [2]. He has answered the question of Ulam for the case where G_1 and G_2 are Banach spaces.

Let E_1 and E_2 be real vector spaces. A function $f:E_1\to E_2$ is called a quadratic function if and only if f is a solution function of the quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y). \quad (1.1)$$

It is well known that a function f between real vector spaces is quadratic if and only if there exists a unique symmetric biadditive function B such that $f(x)=B(x,x)$ for all x, where the mapping B is given by $B(x,y) = (1/4)(f(x+y) - f(x-y))$. See [3, 4] for the details. The Hyers-Ulam stability of the quadratic functional equation (1.1) was first proved by Skof [5] for functions $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [6] demonstrated that Skof's theorem is also valid if E_1 is replaced by an Abelian group G. Assume that a function $f : G \to E$ satisfies the inequality

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \delta, \tag{1.2}$$

for some $\delta \geq 0$ and for all $x, y \in G$. Then there exists a unique quadratic function $Q : G \to E$ such that

$$\|f(x) - Q(x)\| \leq \frac{\delta}{2}, \tag{1.3}$$

for all $x \in G$. Czerwik [7] proved the Hyers-Ulam-Rassias stability of quadratic functional equation (1.1). Let E_1 and E_2 be a real normed space and a real Banach space, respectively, and let $p \neq 2$ be a positive constant. If a function $f : E_1 \to E_2$ satisfies the inequality

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p), \tag{1.4}$$

for some $\epsilon > 0$ and for all $x, y \in E_1$, then there exists a unique quadratic function $q : E_1 \to E_2$ such that

$$\|f(x) - q(x)\| \leq \frac{2\epsilon}{|4 - 2^p|}\|x\|^p, \tag{1.5}$$

for all $x \in E_1$. Furthermore, according to the theorem of Borelli and Forti [8], we know the following generalization of stability theorem for quadratic functional equation. Let G be an Abelian group and E a Banach space, and let $f : G \to E$ be a mapping with $f(0) = 0$ satisfying the inequality

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \varphi(x,y), \tag{1.6}$$

for all $x, y \in G$. Assume that one of the series

$$\Phi(x,y) := \begin{cases} \displaystyle\sum_{k=0}^{\infty} \frac{1}{2^{2(k+1)}} \varphi(2^k x, 2^k y) < \infty, \\ \displaystyle\sum_{k=0}^{\infty} 2^{2k} \varphi\left(\frac{x}{2^{(k+1)}}, \frac{y}{2^{(k+1)}}\right) < \infty, \end{cases} \tag{1.7}$$

then there exists a unique quadratic function $Q : G \to E$ such that

$$\|f(x) - Q(x)\| \leq \Phi(x,x), \tag{1.8}$$

for all $x \in G$. During the last three decades, a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability of several functional equations, and there are many interesting results concerning this problem [9–13].

The notion of quasi-β-normed space was introduced by Rassias and Kim in [14]. This notion is a generalization of that of quasi-normed space. We consider some basic concepts concerning quasi-β-normed space. We fix a real number β with $0 < \beta \leq 1$ and let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Let X be a linear space over $\mathbb{K}$. A quasi-β-norm $\|\cdot\|$ is a real-valued function on X satisfying the following:

(1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$,

(2) $\|\lambda x\| = |\lambda|^{\beta}\|x\|$ for all $\lambda \in \mathbb{K}$ and all $x \in X$,

(3) there is a constant $K \geq 1$ such that $\|x + y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a quasi-β-normed space if $\|\cdot\|$ is a quasi-β-norm on X. The smallest possible K is called the modulus of concavity of $\|\cdot\|$. A quasi-β-Banach space is a complete quasi-β-normed space. A quasi-β-norm $\|\cdot\|$ is called a (β, p)-norm $(0 < p \leq 1)$ if

$$\|x + y\|^{p} \leq \|x\|^{p} + \|y\|^{p}, \tag{1.9}$$

for all $x, y \in X$. In this case, the quasi-β-Banach space is called a (β, p)-Banach space. We observe that if $x_1, x_2, \ldots, x_n$ are nonnegative real numbers, then

$$\left(\sum_{i=1}^{n} x_i\right)^{p} \leq \sum_{i=1}^{n} x_i^{p}, \tag{1.10}$$

where $0 < p \leq 1$ [15].

J. M. Rassias investigated the stability of Ulam for the Euler-Lagrange functional equation

$$f(ax + by) + f(bx - ay) = \left(a^2 + b^2\right)[f(x) + f(y)] \tag{1.11}$$

in the paper of [16]. Gordji and Khodaei investigated the generalized Hyers-Ulam stability of other Euler-Lagrange quadratic functional equations [17]. Jun et al. [18] introduced a new quadratic Euler-Lagrange functional equation

$$f(ax + y) + af(x - y) = (a + 1)f(y) + a(a + 1)f(x), \tag{1.12}$$

for any fixed $a \in \mathbb{Z}$ with $a \neq 0, -1$, which was a modified and instrumental equation for [19], and solved the generalized stability of (1.12). Now, we improve the functional equation (1.12) to the following functional equations:

$$f(ax + by) + af(x - by) = (a + 1)f(by) + a(a + 1)f(x), \tag{1.13}$$

$$f(ax + by) + af(x - by) = (a + 1)b^2 f(y) + a(a + 1)f(x), \tag{1.14}$$

for any fixed rational numbers $a, b \in \mathbb{Q}$ with $a \neq 0, -1$ and $b \neq 0$, which are generalized versions of (1.12). In this paper, we establish the general solution of (1.13) and (1.14) and then prove the generalized Hyers-Ulam stability of (1.13) and (1.14). We remark that there are some interesting papers concerning the stability of functional equations in quasi-Banach spaces [15, 20–23] and quasi-β-normed spaces [14, 24, 25].

2. General Solution of (1.13) and (1.14)

First, we present the general solution of (1.14) in the class of all functions between vector spaces.

Lemma 2.1. *Let X and Y be vector spaces over $\mathbb{K}$. Then a mapping $f : X \to Y$ is a solution of the functional equation* (1.12) *for any fixed rational number $a \in \mathbb{Q}$ with $a \neq 0, -1$ if and only if f is quadratic.*

Proof. See the same proof in [18]. □

Lemma 2.2. *Let X and Y be vector spaces over $\mathbb{K}$. Then a mapping $f : X \to Y$ is a solution of the functional equation* (1.13) *if and only if f is quadratic.*

Proof. We assume that a mapping $f : X \to Y$ satisfies the functional equation (1.13). Letting $by = u$ in (1.13), then (1.13) is equivalent to (1.12). Then by Lemma 2.1, f is quadratic. Conversely, if f is quadratic, then it is obvious that f satisfies (1.13). □

Theorem 2.3. *Let X and Y be vector spaces over $\mathbb{K}$. Then a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional equation* (1.14) *if and only if f is quadratic. In this case, $f(ax) = a^2 f(x)$ and $f(bx) = b^2 f(x)$ hold for all $x \in X$.*

Proof. We assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional equation (1.14). Then replacing y in (1.14) by 0, we also get the equality $f(ax) = a^2 f(x)$ for all $x \in X$. Now, we decompose f into the even part and the odd part by setting

$$f_e(x) = \frac{1}{2}\left(f(x) + f(-x)\right), \qquad f_o(x) = \frac{1}{2}\left(f(x) - f(-x)\right), \tag{2.1}$$

for all $x \in X$. Then f_e and f_o satisfy the functional equation (1.14). Therefore, we may assume without loss of generality that f is even and satisfies (1.14) for all $x, y \in X$. If we replace x in (1.14) by 0, then we get

$$f(by) + af(-by) = (a+1)b^2 f(y), \tag{2.2}$$

for all $y \in X$. From this equality, we have $f(by) = b^2 f(y)$ for all $y \in X$. Therefore, (1.14) implies (1.13) for all $x, y \in X$. By Lemma 2.2, f is quadratic.

Now, we assume that f is odd and satisfies (1.14) for all $x, y \in X$. For the case $a = 1$, we have

$$f(x + by) + f(x - by) = 2b^2 f(y) + 2f(x), \tag{2.3}$$

for all $x, y \in X$. Setting x by 0 in (2.3), one obtains $f \equiv 0$. Let $a \neq 1$. Replacing x by 0 in (1.14), we have

$$(1-a)f(by) = (a+1)b^2 f(y), \tag{2.4}$$

for all $y \in X$. From (1.14) and (2.4), we get

$$f(ax+by) + af(x-by) = (1-a)f(by) + a(a+1)f(x), \tag{2.5}$$

for all $x, y \in X$. Putting $by = u$ in (2.5), then we obtain

$$f(ax+u) + af(x-u) = (1-a)f(u) + a(a+1)f(x), \tag{2.6}$$

for all $x, u \in X$. Replacing u by au in (2.6), we get

$$f(ax+au) + af(x-au) = (1-a)f(au) + a(a+1)f(x), \tag{2.7}$$

for all $x, u \in X$. Since $f(ax) = a^2 f(x)$, (2.7) yields

$$af(x+u) + f(x-au) = (1-a)af(u) + (a+1)f(x), \tag{2.8}$$

for all $x, u \in X$. Interchanging x and u in (2.8), we have by oddness of f

$$-f(ax-u) + af(x+u) = (1-a)af(x) + (a+1)f(u), \tag{2.9}$$

for all $x, u \in X$. Replacing u by $-u$ in (2.6), we get

$$f(ax-u) + af(x+u) = -(1-a)f(u) + a(a+1)f(x), \tag{2.10}$$

for all $x, u \in X$. Adding (2.9) and (2.10) side by side, this leads to

$$f(x+u) = f(x) + f(u), \tag{2.11}$$

for all $x, u \in X$. Therefore, f is additive and so $f(ax) = af(x)$ for all $x \in X$ and for any odd function satisfying (1.14). Using the equality $f(ax) = a^2 f(x)$, we obtain $f(x) = 0$ for all $x \in X$. Therefore, $f(x) = f_e(x) + f_o(x)$ is a quadratic mapping, as desired.

Conversely, if f is quadratic, then it is obvious that f satisfies (1.14). □

We note that $f(0) = 0$ if $a + b^2 \neq 1$ and f satisfies (1.14).

3. Generalized Stability of (1.14) for $a \neq 1$

For convenience, we use the following abbreviation: for any fixed rational numbers a and b with $a \neq -1, 0, 1$ and $b \neq 0$,

$$D_f(x, y) := f(ax + by) + af(x - by) - (a+1)b^2 f(y) - a(a+1)f(x), \tag{3.1}$$

for all $x, y \in X$, which is called the approximate remainder of the functional equation (1.14) and acts as a perturbation of the equation.

From now on, let X be a vector space, and let Y be a (β, p)-Banach space unless we give any specific reference. We will investigate the generalized Hyers-Ulam stability problem for the functional equation (1.14). Thus, we find some conditions such that there exists a true quadratic function near an approximate solution of (1.14).

Theorem 3.1. *Let $\varphi : X \times X \to [0, \infty)$ be a function such that*

$$\Phi(x) := \sum_{n=0}^{\infty} \frac{1}{|a|^{2\beta np}} (\varphi(a^n x, 0))^p < \infty, \tag{3.2}$$

$$\lim_{n \to \infty} \frac{1}{|a|^{2\beta n}} \varphi(a^n x, a^n y) = 0, \tag{3.3}$$

for all $x, y \in X$. Suppose that a function $f : X \to Y$ with $f(0) = 0$ satisfies

$$\| D_f(x, y) \|_Y \leq \varphi(x, y), \tag{3.4}$$

for all $x, y \in X$. Then there exists a unique quadratic function $Q : X \to Y$ satisfying

$$\| f(x) - Q(x) \|_Y \leq \frac{1}{|a|^{2\beta}} [\Phi(x)]^{1/p}, \tag{3.5}$$

for all $x \in X$. The function Q is given by

$$Q(x) = \lim_{k \to \infty} \frac{1}{a^{2k}} f\left(a^k x\right), \tag{3.6}$$

for all $x \in X$.

Proof. Letting y by 0 in (3.4), we get

$$\left\| f(ax) - a^2 f(x) \right\|_Y \leq \varphi(x, 0), \tag{3.7}$$

for all $x \in X$. Multiplying both sides by $1/|a|^{2\beta}$ in (3.7), we have

$$\left\| \frac{1}{a^2} f(ax) - f(x) \right\|_Y \leq \frac{1}{|a|^{2\beta}} \varphi(x, 0), \tag{3.8}$$

for all $x \in X$. Replacing x by $a^n x$ and multiplying both sides by $1/|a|^{2n\beta}$ in (3.8), we have

$$\left\| \frac{1}{a^{2(n+1)}} f\left(a^{n+1}x\right) - \frac{1}{a^{2n}} f(a^n x) \right\|_Y \leq \frac{1}{|a|^{2\beta(n+1)}} \varphi(a^n x, 0), \tag{3.9}$$

for all $x \in X$. Next we show that the sequence $\{(1/a^{2n}) f(a^n x)\}$ is a Cauchy sequence. For any $m, n \in \mathbb{N}$, $m > n \geq 0$, and $x \in X$, it follows from (3.9) that

$$\begin{aligned}
\left\| \frac{1}{a^{2(m+1)}} f\left(a^{m+1}x\right) - \frac{1}{a^{2n}} f(a^n x) \right\|_Y^p &= \left\| \sum_{i=n}^{m} \frac{1}{a^{2(i+1)}} f\left(a^{i+1}x\right) - \frac{1}{a^{2i}} f\left(a^i x\right) \right\|_Y^p \\
&\leq \sum_{i=n}^{m} \left\| \frac{1}{a^{2(i+1)}} f\left(a^{i+1}x\right) - \frac{1}{a^{2i}} f\left(a^i x\right) \right\|_Y^p \\
&\leq \sum_{i=n}^{m} \frac{1}{|a|^{2\beta p(i+1)}} \left(\varphi\left(a^i x, 0\right)\right)^p \\
&= \frac{1}{|a|^{2\beta p}} \sum_{i=n}^{m} \frac{1}{|a|^{2\beta p i}} \left(\varphi\left(a^i x, 0\right)\right)^p,
\end{aligned} \tag{3.10}$$

for all $x \in X$. It follows from (3.2) and (3.10) that the sequence $\{(1/a^{2n}) f(a^n x)\}$ is a Cauchy sequence in Y for all $x \in X$. Since Y is a (β, p)-Banach space, the sequence $\{(1/a^{2n}) f(a^n x)\}$ converges for all $x \in X$. Therefore, we can define a mapping $Q : X \to Y$ by

$$Q(x) = \lim_{n \to \infty} \frac{1}{a^{2n}} f(a^n x), \tag{3.11}$$

for all $x \in X$. Taking $m \to \infty$ and $n = 0$ in (3.10), we have

$$\|Q(x) - f(x)\|_Y^p \leq \frac{1}{|a|^{2\beta p}} \sum_{i=0}^{\infty} \frac{1}{|a|^{2\beta p i}} \left(\varphi\left(a^i x, 0\right)\right)^p = \frac{1}{|a|^{2\beta p}} \Phi(x), \tag{3.12}$$

for all $x \in X$. Therefore,

$$\|Q(x) - f(x)\|_Y \leq \frac{1}{|a|^{2\beta}} [\Phi(x)]^{1/p}, \tag{3.13}$$

for all $x \in X$, that is, the mapping Q satisfies (3.5). It follows from (3.3) and (3.4) that

$$\begin{aligned}
\|D_Q(x, y)\|_Y &= \lim_{n \to \infty} \left\| \frac{1}{a^{2n}} D_f(a^n x, a^n y) \right\|_Y \\
&= \lim_{n \to \infty} \frac{1}{|a|^{2\beta n}} \|D_f(a^n x, a^n y)\|_Y \\
&\leq \lim_{n \to \infty} \frac{1}{|a|^{2\beta n}} \varphi(a^n x, a^n y) = 0,
\end{aligned} \tag{3.14}$$

for all $x, y \in X$. Therefore, Q satisfies (1.14), and so the function Q is quadratic.

To prove the uniqueness of the quadratic function Q, let us assume that there exists a quadratic function $Q' : X \to Y$ satisfying the inequality (3.5). Then we have

$$\begin{aligned}
\|Q(x) - Q'(x)\|_Y^p &= \left\| \frac{1}{a^{2n}} Q(a^n x) - \frac{1}{a^{2n}} Q'(a^n x) \right\|_Y^p \\
&= \frac{1}{a^{2n\beta p}} \|Q(a^n x) - Q'(a^n x)\|_Y^p \\
&\leq \frac{1}{a^{2n\beta p}} \left(\|Q(a^n x) - f(a^n x)\|_Y^p + \|Q'(a^n x) - f(a^n x)\|_Y^p \right) \\
&\leq \frac{1}{|a|^{2n\beta p}} \frac{2}{|a|^{2\beta p}} \Phi(a^n x) \\
&= \frac{2}{|a|^{2\beta p(n+1)}} \sum_{i=0}^{\infty} \frac{1}{|a|^{2\beta p i}} \left(\varphi\left(a^{i+n} x, 0 \right) \right)^p \\
&= \frac{2}{|a|^{2\beta p}} \sum_{i=n}^{\infty} \frac{1}{|a|^{2\beta p i}} \left(\varphi\left(a^{i} x, 0 \right) \right)^p,
\end{aligned} \tag{3.15}$$

for all $x \in X$ and $n \in \mathbb{N}$. Therefore, letting $n \to \infty$, one has $Q(x) - Q'(x) = 0$ for all $x \in X$, completing the proof of uniqueness. □

In the following corollary, we get a stability result of (1.14).

Corollary 3.2. *Let X be a quasi-α-normed space for fixed real number α with $0 < \alpha \leq 1$. Let $\theta_1, \theta_2, \theta_3, \alpha_1, \alpha_2, \gamma_1, \gamma_2$ be positive reals such that either (1) $|a| > 1$, $(\alpha_1 + \alpha_2)\alpha < 2\beta$, and $\gamma_i \alpha < 2\beta$ or (2) $|a| < 1$, $(\alpha_1 + \alpha_2)\alpha > 2\beta$, and $\gamma_i \alpha > 2\beta$, for $i = 1, 2$. Assume that a function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\|D_f(x, y)\|_Y \leq \theta_1 \|x\|^{\alpha_1} \|y\|^{\alpha_2} + \theta_2 \|x\|^{\gamma_1} + \theta_3 \|y\|^{\gamma_2}, \tag{3.16}$$

for all $x, y \in X$. Then there exists a unique quadratic function $Q : X \to Y$ which satisfies the inequality

$$\|f(x) - Q(x)\|_Y \leq \frac{\theta_2 \|x\|^{\gamma_1}}{\left(|a|^{2\beta p} - |a|^{\gamma_1 \alpha p} \right)^{1/p}}, \tag{3.17}$$

for all $x \in X$. The function Q is given by

$$Q(x) = \lim_{k \to \infty} \frac{f(a^k x)}{a^{2k}}, \tag{3.18}$$

for all $x \in X$.

Proof. Let $\varphi(x,y) = \theta_1\|x\|^{\alpha_1}\|y\|^{\alpha_2} + \theta_2\|x\|^{\gamma_1} + \theta_3\|y\|^{\gamma_2}$. Then

$$\begin{aligned}\Phi(x) &= \sum_{n=0}^{\infty}\frac{1}{|a|^{2\beta np}}(\varphi(a^n x,0))^p = \sum_{n=0}^{\infty}\frac{1}{|a|^{2\beta np}}\theta_2^p\|a^n x\|^{\gamma_1 p}\\ &= \theta_2^p\|x\|^{\gamma_1 p}\sum_{n=0}^{\infty}|a|^{(\gamma_1\alpha-2\beta)np} < \infty,\end{aligned} \tag{3.19}$$

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{|a|^{2\beta n}}\varphi(a^n x,a^n y) &= \lim_{n\to\infty}\frac{1}{|a|^{2\beta n}}\left[\theta_1\left(\|a^n x\|^{\alpha_1}\left\|a^n y\right\|^{\alpha_2}\right) + \theta_2\|a^n x\|^{\gamma_1} + \theta_3\left\|a^n y\right\|^{\gamma_2}\right]\\ &= \theta_1\left(\|x\|^{\alpha_1}\left\|y\right\|^{\alpha_2}\right)\lim_{n\to\infty}|a|^{((\alpha_1+\alpha_2)\alpha-2\beta)n} + \theta_2\|x\|^{\gamma_1}\lim_{n\to\infty}|a|^{(\gamma_1\alpha-2\beta)n}\\ &\quad+\theta_3\left\|y\right\|^{\gamma_2}\lim_{n\to\infty}|a|^{(\gamma_2\alpha-2\beta)n} = 0.\end{aligned} \tag{3.20}$$

By Theorem 3.1, there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\begin{aligned}\left\|f(x) - Q(x)\right\|_Y &\le \frac{1}{|a|^{2\beta}}[\Phi(x)]^{1/p}\\ &= \frac{\theta_2\|x\|^{\gamma_1}}{|a|^{2\beta}}\left(\sum_{n=0}^{\infty}|a|^{(\gamma_1\alpha-2\beta)np}\right)^{1/p}\\ &= \frac{\theta_2\|x\|^{\gamma_1}}{\left(|a|^{2\beta p} - |a|^{\gamma_1\alpha p}\right)^{1/p}},\end{aligned} \tag{3.21}$$

for all $x \in X$. □

Theorem 3.3. *Let $\varphi : X \times X \to [0,\infty)$ be a function such that*

$$\Psi(x) := \sum_{n=0}^{\infty}|a|^{2\beta np}\left(\varphi\left(\frac{x}{a^{n+1}},0\right)\right)^p < \infty, \tag{3.22}$$

$$\lim_{n\to\infty}|a|^{2\beta n}\varphi\left(\frac{x}{a^n},\frac{y}{a^n}\right) = 0, \tag{3.23}$$

for all $x, y \in X$. Suppose that a function $f : X \to Y$ with $f(0) = 0$ satisfies

$$\left\|D_f(x,y)\right\|_Y \le \varphi(x,y), \tag{3.24}$$

for all $x, y \in X$. Then there exists a unique quadratic function $Q : X \to Y$ satisfying

$$\left\|f(x) - Q(x)\right\|_Y \le [\Psi(x)]^{1/p}, \tag{3.25}$$

for all $x \in X$. The function Q is given by

$$Q(x) = \lim_{k \to \infty} a^{2k} f\left(\frac{x}{a^k}\right), \tag{3.26}$$

for all $x \in X$.

Proof. Letting y by 0 in (3.24), we get

$$\left\| f(ax) - a^2 f(x) \right\|_Y \leq \varphi(x, 0), \tag{3.27}$$

for all $x \in X$. Replacing x by x/a in (3.27), we have

$$\left\| f(x) - a^2 f\left(\frac{x}{a}\right) \right\|_Y \leq \varphi\left(\frac{x}{a}, 0\right), \tag{3.28}$$

for all $x \in X$. Replacing x by x/a^n and multiplying both sides by $|a|^{2\beta n}$ in (3.28), we have

$$\left\| a^{2n} f\left(\frac{x}{a^n}\right) - a^{2(n+1)} f\left(\frac{x}{a^{n+1}}\right) \right\|_Y \leq |a|^{2\beta n} \varphi\left(\frac{x}{a^{n+1}}, 0\right), \tag{3.29}$$

for all $x \in X$. Next we show that the sequence $\{a^{2n} f(x/a^n)\}$ is a Cauchy sequence. For any $m, n \in \mathbb{N}$, $m > n \geq 0$, and $x \in X$, it follows from (3.29) that

$$\begin{aligned} \left\| a^{2n} f\left(\frac{x}{a^n}\right) - a^{2(m+1)} f\left(\frac{x}{a^{m+1}}\right) \right\|_Y^p &= \left\| \sum_{i=n}^{m} a^{2i} f\left(\frac{x}{a^i}\right) - a^{2(i+1)} f\left(\frac{x}{a^{i+1}}\right) \right\|_Y^p \\ &\leq \sum_{i=n}^{m} \left\| a^{2i} f\left(\frac{x}{a^i}\right) - a^{2(i+1)} f\left(\frac{x}{a^{i+1}}\right) \right\|_Y^p \\ &\leq \sum_{i=n}^{m} |a|^{2\beta p i} \left(\varphi\left(\frac{x}{a^{i+1}}, 0\right)\right)^p. \end{aligned} \tag{3.30}$$

It follows from (3.22) and (3.30) that the sequence $\{a^{2n} f(x/a^n)\}$ is a Cauchy sequence in Y for all $x \in X$. Since Y is a (β, p)-Banach space, the sequence $\{a^{2n} f(x/a^n)\}$ converges for all $x \in X$. Therefore, we can define a mapping $Q : X \to Y$ by

$$Q(x) = \lim_{n \to \infty} a^{2n} f\left(\frac{x}{a^n}\right), \tag{3.31}$$

for all $x \in X$. The rest of the proof is similar to the corresponding proof of Theorem 3.1. □

Corollary 3.4. *Let X be a quasi-α-normed space for fixed real number α with $0 < \alpha \leq 1$. Let $\theta_1, \theta_2, \theta_3, \alpha_1, \alpha_2, \gamma_1, \gamma_2$ be positive reals such that either (1) $|a| > 1$, $(\alpha_1 + \alpha_2)\alpha > 2\beta$, and $\gamma_i \alpha > 2\beta$ or*

(2) $|a| < 1$, $(\alpha_1 + \alpha_2)\alpha < 2\beta$, and $\gamma_i\alpha < 2\beta$, for $i = 1, 2$. Assume that a function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$\|D_f(x,y)\|_Y \le \theta_1\|x\|^{\alpha_1}\|y\|^{\alpha_2} + \theta_2\|x\|^{\gamma_1} + \theta_3\|y\|^{\gamma_2}, \tag{3.32}$$

for all $x, y \in X$. Then there exists a unique quadratic function $Q : X \to Y$ which satisfies the inequality

$$\|f(x) - Q(x)\|_Y \le \frac{\theta_2\|x\|^{\gamma_1}}{\left(|a|^{\gamma_1\alpha p} - |a|^{2\beta p}\right)^{1/p}}, \tag{3.33}$$

for all $x \in X$. The function Q is given by

$$Q(x) = \lim_{k\to\infty} a^{2k} f\left(\frac{x}{a^k}\right), \tag{3.34}$$

for all $x \in X$.

Proof. Let $\varphi(x,y) = \theta_1\|x\|^{\alpha_1}\|y\|^{\alpha_2} + \theta_2\|x\|^{\gamma_1} + \theta_3\|y\|^{\gamma_2}$. Then φ satisfies the conditions (3.22) and (3.23). Applying Theorem 3.3, we obtain the results, as desired. □

4. Generalized Stability of (1.13)

For convenience, we use the following abbreviation: for any fixed rational numbers a and b with $a \ne -1, 0$ and $b \ne 0$,

$$E_f(x,y) := f(ax + by) + af(x - by) - (a+1)f(by) - a(a+1)f(x), \tag{4.1}$$

for all $x, y \in X$, which is called the approximate remainder of the functional equation (1.13) and acts as a perturbation of the equation.

We will investigate the generalized Hyers-Ulam stability problem for the functional equation (1.13).

Theorem 4.1. *Let $\varphi : X \times X \to [0, \infty)$ be a function such that*

$$\Phi(x) := \sum_{n=0}^{\infty} \frac{1}{|a+1|^{2\beta n p}} \left(\varphi\left((a+1)^n x, \frac{(a+1)^n x}{b}\right)\right)^p < \infty, \tag{4.2}$$

$$\lim_{n\to\infty} \frac{1}{|a+1|^{2\beta n}} \varphi\left((a+1)^n x, (a+1)^n y\right) = 0, \tag{4.3}$$

for all $x, y \in X$. Suppose that a function $f : X \to Y$ with $f(0) = 0$ satisfies

$$\|E_f(x,y)\|_Y \le \varphi(x,y), \tag{4.4}$$

for all $x, y \in X$. Then there exists a unique quadratic function $Q : X \to Y$ satisfying

$$\|f(x) - Q(x)\|_Y \le \frac{1}{|a+1|^{2\beta}}[\Phi(x)]^{1/p}, \tag{4.5}$$

for all $x \in X$. The function Q is given by

$$Q(x) = \lim_{k\to\infty}\frac{1}{(a+1)^{2k}}f\left((a+1)^k x\right), \tag{4.6}$$

for all $x \in X$.

Proof. Replacing x by by in (4.4), we get

$$\left\|f((a+1)by) - (a+1)^2 f(by)\right\|_Y \le \varphi(by, y), \tag{4.7}$$

for all $y \in X$. Letting by be x in (4.7), we have

$$\left\|f((a+1)x) - (a+1)^2 f(x)\right\|_Y \le \varphi\left(x, \frac{x}{b}\right), \tag{4.8}$$

for all $x \in X$. Multiplying both sides by $1/|a+1|^{2\beta}$ in (4.8), we have

$$\left\|\frac{1}{(a+1)^2}f((a+1)x) - f(x)\right\|_Y \le \frac{1}{|a+1|^{2\beta}}\varphi\left(x, \frac{x}{b}\right), \tag{4.9}$$

for all $x \in X$. Replacing x by $(a+1)^i x$ and multiplying both sides by $1/|a+1|^{2i\beta}$ in (4.9), we have

$$\left\|\frac{1}{(a+1)^{2(i+1)}}f\left((a+1)^{i+1}x\right) - \frac{1}{(a+1)^{2i}}f\left((a+1)^i x\right)\right\|_Y \le \frac{1}{|a+1|^{2\beta(i+1)}}\varphi\left((a+1)^i x, \frac{(a+1)^i x}{b}\right), \tag{4.10}$$

for all $x \in X$. Next we show that the sequence $\{(1/(a+1)^{2n})f((a+1)^n x)\}$ is a Cauchy sequence. For any $m, n \in \mathbb{N}$, $m > n \ge 0$, and $x \in X$, it follows from (4.10) that

$$\begin{aligned}
&\left\|\frac{1}{(a+1)^{2(m+1)}}f\left((a+1)^{m+1}x\right) - \frac{1}{(a+1)^{2n}}f((a+1)^n x)\right\|_Y^p \\
&= \left\|\sum_{i=n}^{m}\frac{1}{(a+1)^{2(i+1)}}f\left((a+1)^{i+1}x\right) - \frac{1}{(a+1)^{2i}}f\left((a+1)^i x\right)\right\|_Y^p \\
&\le \sum_{i=n}^{m}\left\|\frac{1}{(a+1)^{2(i+1)}}f\left((a+1)^{i+1}x\right) - \frac{1}{(a+1)^{2i}}f\left((a+1)^i x\right)\right\|_Y^p
\end{aligned}$$

$$\leq \sum_{i=n}^{m} \frac{1}{|a+1|^{2\beta p(i+1)}} \left(\varphi\left((a+1)^i x, \frac{(a+1)^i x}{b} \right) \right)^p$$
$$= \frac{1}{|a+1|^{2\beta p}} \sum_{i=n}^{m} \frac{1}{|a+1|^{2\beta p i}} \left(\varphi\left((a+1)^i x, \frac{(a+1)^i x}{b} \right) \right)^p, \tag{4.11}$$

for all $x \in X$. It follows from (4.2) and (4.11) that the sequence $\{f((a+1)^n x)/(a+1)^{2n}\}$ is a Cauchy sequence in Y for all $x \in X$. Since Y is a (β, p)-Banach space, the sequence $\{f((a+1)^n x)/(a+1)^{2n}\}$ converges for all $x \in X$. Therefore, we can define a mapping $Q : X \to Y$ by

$$Q(x) = \lim_{n\to\infty} \frac{1}{(a+1)^{2n}} f((a+1)^n x), \tag{4.12}$$

for all $x \in X$. The rest of the proof is similar to the corresponding proof of Theorem 3.1. □

In the following corollary, we get a stability result of (1.13).

Corollary 4.2. *Let X be a quasi-α-normed space for fixed real number α with* $0 < \alpha \leq 1$. *Let* $\theta_1, \theta_2, \theta_3, \alpha_1, \alpha_2, \gamma_1, \gamma_2$ *be positive reals such that either (1)* $|a+1| > 1$, $(\alpha_1+\alpha_2)\alpha < 2\beta$, *and* $\gamma_i\alpha < 2\beta$ *or (2)* $|a+1| < 1$, $(\alpha_1+\alpha_2)\alpha > 2\beta$, *and* $\gamma_i\alpha > 2\beta$, *for* $i = 1, 2$. *Assume that a function* $f : X \to Y$ *with* $f(0) = 0$ *satisfies the inequality*

$$\|E_f(x,y)\|_Y \leq \theta_1 \|x\|^{\alpha_1} \|y\|^{\alpha_2} + \theta_2 \|x\|^{\gamma_1} + \theta_3 \|y\|^{\gamma_2}, \tag{4.13}$$

for all $x, y \in X$. *Then there exists a unique quadratic function* $Q : X \to Y$ *which satisfies the inequality*

$$\|f(x) - Q(x)\|_Y \leq \left\{ \frac{\theta_1^p \|x\|^{(\alpha_1+\alpha_2)p}}{|b|^{\alpha\alpha_2 p}\left(|a+1|^{2\beta p} - |a+1|^{(\alpha_1+\alpha_2)\alpha p}\right)} + \frac{\theta_2^p \|x\|^{\gamma_1 p}}{|a+1|^{2\beta p} - |a+1|^{\gamma_1 \alpha p}} + \frac{\theta_3^p \|x\|^{\gamma_2 p}}{|b|^{\gamma_2 \alpha p}\left(|a+1|^{2\beta p} - |a+1|^{\gamma_2 \alpha p}\right)} \right\}^{1/p}, \tag{4.14}$$

for all $x \in X$. *The function* Q *is given by*

$$Q(x) = \lim_{k\to\infty} \frac{1}{(a+1)^{2k}} f\left((a+1)^k x\right), \tag{4.15}$$

for all $x \in X$.

Proof. Let $\varphi(x,y) = \theta_1\|x\|^{\alpha_1}\|y\|^{\alpha_2} + \theta_2\|x\|^{\gamma_1} + \theta_3\|y\|^{\gamma_2}$. Then φ satisfies the conditions (4.2) and (4.3). By Theorem 4.1, there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\begin{aligned} \|f(x) - Q(x)\|_Y &\le \frac{1}{|a+1|^{2\beta}}\left[\sum_{n=0}^{\infty}\frac{1}{|a+1|^{2\beta np}}\left(\varphi\left((a+1)^n x, \frac{(a+1)^n x}{b}\right)\right)^p\right]^{1/p} \\ &\le \left\{\frac{\theta_1^p\|x\|^{(\alpha_1+\alpha_2)p}}{|b|^{\alpha\alpha_2 p}\left(|a+1|^{2\beta p} - |a+1|^{(\alpha_1+\alpha_2)\alpha p}\right)}\right. \\ &\quad \left. + \frac{\theta_2^p\|x\|^{\gamma_1 p}}{|a+1|^{2\beta p} - |a+1|^{\gamma_1\alpha p}} + \frac{\theta_3^p\|x\|^{\gamma_2 p}}{|b|^{\gamma_2\alpha p}\left(|a+1|^{2\beta p} - |a+1|^{\gamma_2\alpha p}\right)}\right\}^{1/p}, \end{aligned} \tag{4.16}$$

for all $x \in X$. □

Theorem 4.3. *Let $\varphi : X \times X \to [0,\infty)$ be a function such that*

$$\begin{aligned} \Psi(x) &:= \sum_{n=0}^{\infty}|a+1|^{2\beta np}\left(\varphi\left(\frac{x}{(a+1)^{n+1}}, \frac{x}{(a+1)^{n+1}b}\right)\right)^p < \infty, \\ &\lim_{n\to\infty}|a+1|^{2\beta n}\varphi\left(\frac{x}{(a+1)^n}, \frac{y}{(a+1)^n}\right) = 0, \end{aligned} \tag{4.17}$$

for all $x, y \in X$. Suppose that a function $f : X \to Y$ with $f(0) = 0$ satisfies

$$\|E_f(x,y)\|_Y \le \varphi(x,y), \tag{4.18}$$

for all $x, y \in X$. Then there exists a unique quadratic function $Q : X \to Y$ satisfying

$$\|f(x) - Q(x)\|_Y \le [\Psi(x)]^{1/p}, \tag{4.19}$$

for all $x \in X$. The function Q is given by

$$Q(x) = \lim_{k\to\infty}(a+1)^{2k} f\left(\frac{x}{(a+1)^k}\right), \tag{4.20}$$

for all $x \in X$.

Proof. Replacing x by $x/(a+1)$ in (4.8), we have

$$\left\|f(x) - (a+1)^2 f\left(\frac{x}{a+1}\right)\right\|_Y \le \varphi\left(\frac{x}{a+1}, \frac{x}{(a+1)b}\right), \tag{4.21}$$

for all $x \in X$. The rest of the proof is similar to the corresponding proof of Theorem 3.3. □

Corollary 4.4. *Let X be a quasi-α-normed space for fixed real number α with $0 < \alpha \leq 1$. Let $\theta_1, \theta_2, \theta_3$, $\alpha_1, \alpha_2, \gamma_1, \gamma_2$ be positive reals such that either (1) $|a+1| > 1$ and $(\alpha_1+\alpha_2)\alpha > 2\beta$, $\gamma_i\alpha > 2\beta$ or (2) $|a+1| < 1$ and $(\alpha_1+\alpha_2)\alpha < 2\beta$, $\gamma_i\alpha < 2\beta$, for $i = 1, 2$. Assume that a function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\left\| E_f(x,y) \right\|_Y \leq \theta_1 \|x\|^{\alpha_1} \left\|y\right\|^{\alpha_2} + \theta_2 \|x\|^{\gamma_1} + \theta_3 \left\|y\right\|^{\gamma_2}, \tag{4.22}$$

for all $x, y \in X$. Then there exists a unique quadratic function $Q : X \to Y$ which satisfies the inequality

$$\left\| f(x) - Q(x) \right\|_Y \leq \left\{ \frac{\theta_1^p \|x\|^{(\alpha_1+\alpha_2)p}}{|b|^{\alpha\alpha_2 p}\left(|a+1|^{(\alpha_1+\alpha_2)\alpha p} - |a+1|^{2\beta p}\right)} + \frac{\theta_2^p \|x\|^{\gamma_1 p}}{|a+1|^{\gamma_1 \alpha p} - |a+1|^{2\beta p}} + \frac{\theta_3^p \|x\|^{\gamma_2 p}}{|b|^{\alpha\gamma_2 p}\left(|a+1|^{\gamma_2 \alpha p} - |a+1|^{2\beta p}\right)} \right\}^{1/p}, \tag{4.23}$$

for all $x \in X$. The function Q is given by

$$Q(x) = \lim_{k\to\infty} (a+1)^{2k} f\left(\frac{x}{(a+1)^k} \right), \tag{4.24}$$

for all $x \in X$.

Proof. Let $\varphi(x,y) = \theta_1 \|x\|^{\alpha_1} \|y\|^{\alpha_2} + \theta_2 \|x\|^{\gamma_1} + \theta_3 \|y\|^{\gamma_2}$. Then φ satisfies the conditions (4.17). Applying Theorem 4.3, we obtain the results, as desired. □

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