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Research Article

Parallel and Cyclic Algorithms for Quasi-Nonexpansives in Hilbert Space

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Let $\{T\}_{i=1}^{N}$ be N quasi-nonexpansive mappings defined on a closed convex subset C of a real Hilbert space H. Consider the problem of finding a common fixed point of these mappings and introduce the parallel and cyclic algorithms for solving this problem. We will prove the strong convergence of these algorithms.

1. Introduction

Throughout this paper, we always assume that C is a nonempty, closed, and convex subset of a real Hilbert space H. Let $A : C \to H$ be a nonlinear mapping. Recall the following definitions.

(1) A is said to be monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C. \tag{1.1}$$

(2) A is said to be strongly positive if there exists a constant $\overline{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in C. \tag{1.2}$$

(3) A is said to be strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in C. \tag{1.3}$$

For such a case, A is said to be α-strongly monotone.

(4) A is said to be inverse strongly if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{1.4}$$

For such a case, A is said to be α-inverse-strongly-monotone (α-ism).

Assume A is strongly positive operator, that is, there is a constant $\overline{\gamma}$ with the property

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in H. \tag{1.5}$$

Remark 1.1. Let $F = A - \gamma f$, where A is strongly positive operator, and f is contraction mapping with coefficient $\beta \in (0,1)$. It is a simple matter to see that the operator F is $(\overline{\gamma} - \gamma\beta)$-strongly monotone over C, that is,

$$\langle Fx - Fy, x - y\rangle \geq (\overline{\gamma} - \gamma\beta)\|x - y\|^2, \quad \forall (x, y) \in C \times C. \tag{1.6}$$

The classical variational inequality which is denoted by $\mathrm{VI}(A, C)$ is to find $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.7}$$

The variational inequality has been extensively studied in literature; see, for example, [1, 2] and the reference therein. A mapping $T : C \to C$ is said to be a strict pseudocontraction [3] if there exists a constant $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \tag{1.8}$$

for all $x, y \in C$ (If (1.8) holds, we also say that T is a k-strict pseudo-contraction). These mappings are extensions of nonexpansive mappings which satisfy the inequality (1.8) with $k = 0$. That is, $T : C \to C$ is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.9}$$

In [4], Xu proved that the sequence $\{x_n\}$ defined by the iterative method below with the initial guess $x_0 \in H$ chosen arbitrarily,

$$x_{n+1} = \alpha_n b + (I - \alpha_n A)Tx_n, \quad n \geq 0, \tag{1.10}$$

where the sequence $\{\alpha_n\}$ satisfies certain conditions, he proved the sequence $\{x_n\}$ converges strongly to the unique solution of the following minimization problem:

$$\min_{x\in C}\frac{1}{2}\langle Ax,x\rangle-\langle x,b\rangle. \tag{1.11}$$

In [5], Marino and Xu considered the following general iterative method:

$$x_{n+1}=\alpha_n\gamma f(x_n)+(I-\alpha_n A)Tx_n,\quad n\geq 0, \tag{1.12}$$

they proved that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.12) converges strongly to the unique solution of the variational inequality

$$\left\langle(\gamma f-A)x^*,x-x^*\right\rangle\leq 0,\quad x\in C, \tag{1.13}$$

which is the optimality condition for the minimization problem

$$\min_{x\in C}\frac{1}{2}\langle Ax,x\rangle-h(x), \tag{1.14}$$

where h is a potential function for γf (i.e., $h'(x)=\gamma f(x)$ for $x\in H$). Some people also study the applications of the iterative method (1.12) [6, 7].

Acedo and Xu [8] consider the following parallel and cyclic algorithms:

Parallel Algorithm

The sequence $\{x_n\}$ was generated by

$$x_{n+1}=\alpha_n x_n+(1-\alpha_n)\sum_{i=1}^{N}\lambda_i^{(n)}T_i x_n, \tag{1.15}$$

where $\{T_i\}_{i=1}^N$ are N strict pseudocontractions defined on a closed convex subset C of a Hilbert space H. Under the following assumptions on the sequences of the weights $\{\lambda_i^{(n)}\}_{i=1}^N$:

(a1) $\sum_{i=1}^N\lambda_i^{(n)}=1$ for all n and $\inf_{n\geq1}\lambda^{(n)}>0$, for all $1\leq i\leq N$,

(a2) $\sum_{i=1}^N\sqrt{\sum_{i=1}^N|\lambda_i^{(n+1)}-\lambda_i^{(n)}|}<\infty$.

By (1.15), they will prove the weak convergence to a solution of the problem $x\in\bigcap_{i=1}^N F_{ix}(T_i)$.

Cyclic Algorithm

They define the sequence $\{x_n\}$ cyclically by

$$\begin{aligned} x_1 &= \alpha_0 x_0 + (1-\alpha_0)T_0 x_0; \\ x_2 &= \alpha_1 x_1 + (1-\alpha_1)T_1 x_1; \\ &\vdots \\ x_N &= \alpha_{N-1} x_{N-1} + (1-\alpha_{N-1})T_{N-1} x_{N-1}; \\ x_{N+1} &= \alpha_N x_N + (1-\alpha_N)T_0 x_N; \\ &\vdots \end{aligned} \tag{1.16}$$

In a more compact form, they are rewritten x_{n+1} as

$$x_{N+1} = \alpha_N x_N + (1-\alpha_N)T_N x_n, \tag{1.17}$$

where $\{T_i\}_{i=1}^N$ are k_i-strict pseudo-contractions and $T_N = T_i$ with $i = n \pmod N$, $0 \leq i \leq N-1$. They show that this cyclic algorithm (1.17) is weakly convergent if the sequence $\{\alpha_n\}$ of parameters is appropriately chosen. On the other hand, Osilike and Shehu [9] also consider the cyclic algorithm (1.17), under appropriate assumptions on the sequences of $\{\alpha_n\}$, some strong convergence theorems are proved.

In this paper, we are concerned with the problem of finding a point x such that

$$x \in \bigcap_{i=1}^{N} F_{ix}(T_{\omega_i}), \quad N \geq 1, \tag{1.18}$$

where $T_{\omega_i} = (1-\omega_i)I + \omega_i T_i$, $\{\omega_i\}_{i=1}^N \in (0,1]$ and $\{T_i\}_{i=1}^N$ are quasi-nonexpansive mappings defined on a closed convex subset C of a Hilbert space H. Here $F_{ix}(T_{\omega_i}) = \{q \in C : T_{\omega_i} q = q\}$ is the set of fixed points of T_i, $1 \leq i \leq N$.

Let T be defined by

$$T = \sum_{i=1}^{N} \lambda_i T_{\omega_i}, \tag{1.19}$$

where $\lambda_i > 0$ for all $i \in (0,1)$ such that $\sum_{i=1}^N \lambda_i = 1$. Motivated and inspired by Acedo and Xu [8], we consider the following two general iterative algorithms for a family of quasi-nonexpansive mappings.

Algorithm 1.2.

$$T = \sum_{i=1}^{N} \lambda_i T_{\omega_i},$$
$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((I - \beta_n)I - \alpha_n A)T x_n. \tag{1.20}$$

Algorithm 1.3.

$$T = \sum_{i=1}^{N} \lambda_i^{(n)} T_{\omega_i},$$
$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((I - \beta_n)I - \alpha_n A)T x_n. \tag{1.21}$$

In (1.20), the weights $\{\lambda_i\}_{i=1}^N$ are constant in the sense that they are independent of n, the number of steps of the iteration process. In (1.21), we consider a more general case by allowing the weights $\{\lambda_i^{(n)}\}_{i=1}^N$. Under appropriate assumptions on the sequences of the wights $\{\lambda_i^{(n)}\}_{i=1}^N$, $\{\lambda_i\}_{i=1}^N$, $\{\alpha_n\}$ and $\{\beta_n\}$. From (1.20) and (1.21), we will prove some strong convergence to a solution of the problem (1.18). In addition, we can also know that the condition $\sum_{i=1}^N \sqrt{\sum_{i=1}^N |\lambda_i^{(n+1)} - \lambda_i^{(n)}|} < \infty$ in [8] is superfluous.

Another approach to the problem (1.18) is the cyclic algorithm (for convenience, we relabel the mappings $\{T_{\omega_i}\}_{i=1}^N$ as $\{T_{\omega_i}\}_{i=0}^{N-1}$). This means that beginning with an $x_0 \in C$, we define the sequence $\{x_n\}$ cyclically by

$$\begin{aligned}
x_1 &= \alpha_0 \gamma f(x_0) + \beta_0 x_0 + ((I - \beta_0)I - \alpha_0 A)T_{\omega_0} x_0,\\
x_2 &= \alpha_1 \gamma f(x_1) + \beta_1 x_1 + ((I - \beta_1)I - \alpha_1 A)T_{\omega_1} x_1,\\
&\vdots
\end{aligned} \tag{1.22}$$

$$\begin{aligned}
x_N &= \alpha_{N-1} \gamma f(x_{N-1}) + \beta_{N-1} x_{N-1} + ((I - \beta_{N-1})I - \alpha_{N-1} A)T_{\omega_{N-1}} x_{N-1},\\
x_{N+1} &= \alpha_N \gamma f(x_N) + \beta_N x_N + ((I - \beta_N)I - \alpha_N A)T_{\omega_N} x_N,\\
&\vdots
\end{aligned} \tag{1.23}$$

In a more compact from, x_{n+1} can be written as

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((I - \beta_n)I - \alpha_n A)T_{[n]} x_n, \tag{1.24}$$

where $T_{[n]} = T_{\omega_i}$, $T_{\omega_i} = (1 - \omega_i)I + \omega_i T_i$, $\{\omega_i\}_{i=1}^N \in (0,1]$, with $i = n \pmod N$, $0 \le i \le N - 1$.

We will show that this cyclic algorithm (1.24) is also strongly convergent if the sequence $\{\alpha_n\}$ of parameters is appropriately chosen.

2. Preliminaries

Throughout this paper, we write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. The following definitions and lemmas are useful for main results.

Definition 2.1. An operator $T : H \to H$ is said to be quasi-nonexpansive if

$$F_{ix}(T) \neq 0 \text{ and if } \|Tx - z\| \leq \|x - z\|, \quad \forall z \in F_{ix}(T),\ \forall x \in H. \tag{2.1}$$

Iterative methods for quasi-nonexpansive mappings have been extensively investigated; see [10, 11].

Remark 2.2. From the above definitions, It is easy to see that

(i) a nonexpansive mapping is a quasi-nonexpansive mapping;

(ii) the set of fixed points of T is the set $F_{ix}(T) = \{x \in H : Tx = x\}$. We assume that $F_{ix}(T) \neq \emptyset$, it is well know that $F_{ix}(T)$ is closed and convex.

Remark 2.3 (see [10]). Let $T_\alpha = (1-\alpha)I + \alpha T$, where T is a quasi-nonexpansive on H, $F_{ix}(T) \neq \emptyset$ and $\alpha \in (0, 1]$. Then the following statements are reached:

(i) $F_{ix}(T) = F_{ix}(T_\alpha)$;

(ii) T_α is quasi-nonexpansive;

(iii) $\|T_\alpha x - q\|^2 \leq \|x - q\|^2 - \alpha(1-\alpha)\|Tx - x\|^2$, for all $(x, q) \in H \times F_{ix}(T)$;

(iv) $\langle x - T_\alpha x, x - q\rangle \geq (\alpha/2)\|Tx - x\|^2$, for all $(x, q) \in H \times F_{ix}(T)$.

Example 2.4. Let $X = l^2$ with the norm $\|\cdot\|$ defined by

$$\|X\| = \sqrt{\sum_{i=1}^{\infty} x_i^2}, \qquad \forall x = (x_1, x_2, \ldots, x_n, \ldots) \in X, \tag{2.2}$$

and $C = \{x = (x_1, x_2, \ldots, x_n, \ldots) \mid x_1 \leq 0, x_i \in R^1, i = 2, 3, \ldots\}$. Then C is a nonempty subset of X.

Now, for any $x = (x_1, x_2, \ldots, x_n, \ldots) \in C$, define a mapping $T : C \to C$ as follows:

$$T(x) = (0, 4x_1, 0, \ldots, 0, \ldots). \tag{2.3}$$

It is easy to see that T is a quasi-nonexpansive mapping. In fact, for any $x = (x_1, x_2, \ldots, x_n, \ldots) \in X$, taking $T(x) = x$, that is,

$$(0, 4x_1, 0, \ldots, 0, \ldots) = (x_1, x_2, \ldots, x_n, \ldots), \tag{2.4}$$

we have $F(T) = \{0\}$ and

$$\begin{aligned}\|T(x) - 0\| &= \|(0, 4x_1, 0, \dots, 0, \dots) - (0, 0, 0, \dots, 0, \dots)\| = 4|x_1| \\ &\leq 4\sqrt{\sum_{i=1}^{\infty} x_i^2} \\ &= \|(x_1, x_2, \dots, x_n, \dots) - (0, 0, 0, \dots, 0, \dots)\| \\ &= \|x - 0\|.\end{aligned} \tag{2.5}$$

Lemma 2.5. *Assume C is a closed convex subset of a Hilbert space H.*

(i) *Given an integer $N \geq 1$, assume, for all $1 \leq i \leq N$, $T_i : C \to C$ is a quasi-nonexpansive. Let $\{\lambda_i\}_{i=1}^N$ be a positive sequence such that $\sum_{i=1}^N \lambda_i = 1$. Then $\sum_{i=1}^N \lambda_i T_i$ is a quasi-nonexpansive.*

(ii) *Let $\{T_i\}_{i=1}^N$ and $\sum_{i=1}^N \lambda_i = 1$ be given as in (i) above. Suppose that $\{T_i\}_{i=1}^N$ has a common fixed point. Then*

$$F_{ix}\left(\sum_{i=1}^{N} \lambda_i T_i\right) = \bigcap_{i=1}^{N} F_{ix}(T_i). \tag{2.6}$$

(iii) *Assume $T_i : C \to C$ be quasi-nonexpansives, let $T_{\alpha_i} = (1 - \alpha_i)I + \alpha_i T_i$, $1 \leq i \leq N$. If $\bigcap_{i=1}^N F_{ix}(T_i) \neq \emptyset$, then*

$$F_{ix}(T_{\alpha_1} T_{\alpha_2} \cdots T_{\alpha_N}) = \bigcap_{i=1}^{N} F_{ix}(T_{\alpha_i}). \tag{2.7}$$

Proof. To prove (i) we only need to consider the case of $N = 2$ (the general case can be proved by induction). Set $T = (1 - \lambda)T_1 + \lambda T_2$, where $\lambda \in (0, 1)$ and for $i = 1, 2$, T_i is a quasi-nonexpansive. We verify directly the following inequality: for all $z \in F_{ix}(T_1) \cap F_{ix}(T_2)$,

$$\begin{aligned}\|Tx - z\| &= \|((1 - \lambda)T_1 + \lambda T_2)x - z\| \\ &\leq (1 - \lambda)\|T_1 x - z\| + \lambda \|T_2 x - z\| \\ &\leq (1 - \lambda)\|x - z\| + \lambda \|x - z\| \\ &\leq \|x - z\|,\end{aligned} \tag{2.8}$$

that is, T is a quasi-nonexpansive.

To prove (ii) again we can assume $N = 2$. It suffices to prove that $F_{ix}(T) \subset F_{ix}(T_1) \cap F_{ix}(T_2)$, where $T = (1 - \lambda)T_1 + \lambda T_2$ with $\lambda \in (0, 1)$. Let $x \in F_{ix}(T)$.

Taking $z \in F_{ix}(T_1) \cap F_{ix}(T_2)$ to deduce that

$$\begin{aligned} \|z - x\| &= \|(1-\lambda)(z - T_1 x) + \lambda(z - T_2 x)\| \\ &\le (1-\lambda)\|z - T_1 x\| + \lambda\|z - T_2 x\| \\ &\le (1-\lambda)\|z - x\| + \lambda\|z - x\| \\ &\le \|z - x\|. \end{aligned} \tag{2.9}$$

By the strict convexity of H, it follows that $T_1(x) - z = T_2(x) - z = x - z$; that is, $T_1(x) = T_2(x) = x$, hence $x \in F_{ix}(T_1) \cap F_{ix}(T_2)$. According to induction, we can easily claim that (2.6) is holds.

To prove (iii) by induction, for $N = 2$, set $T_{\alpha_i} = (1-\alpha_i)I + \alpha_i T_i$ for all $i = 1, 2$. Obviously

$$F_{ix}(T_{\alpha_1}) \bigcap F_{ix}(T_{\alpha_2}) \subset F_{ix}(T_{\alpha_1} T_{\alpha_2}). \tag{2.10}$$

Now we prove

$$F_{ix}(T_{\alpha_1} T_{\alpha_2}) \subset F_{ix}(T_{\alpha_1}) \bigcap F_{ix}(T_{\alpha_2}). \tag{2.11}$$

For all $q \in F_{ix}(T_{\alpha_1} T_{\alpha_2})$, $T_{\alpha_1} T_{\alpha_2} q = q$, if $T_{\alpha_2} q = q$, then $T_{\alpha_1} q = q$, the conclusion holds. In fact, we can claim that $T_{\alpha_2} q = q$. From Remark 2.3, we know that T_{α_2} is quasi-nonexpansive and $F_{ix}(T_{\alpha_1}) \bigcap F_{ix}(T_{\alpha_2}) = F_{ix}(T_1) \bigcap F_{ix}(T_2) \neq \emptyset$. Take $p \in F_{ix}(T_{\alpha_1}) \bigcap F_{ix}(T_{\alpha_2})$, then

$$\begin{aligned} \|p - q\|^2 &= \|p - T_{\alpha_1} T_{\alpha_2} q\|^2 \\ &= \left\|p - \left[(1-\alpha_1) T_{\alpha_2} q + \alpha_1 T_1 T_{\alpha_2} q\right]\right\|^2 \\ &= \left\|(1-\alpha_1)(p - T_{\alpha_2} q) + \alpha_1 (p - T_1 T_{\alpha_2} q)\right\|^2 \\ &= (1-\alpha_1)\|p - T_{\alpha_2} q\|^2 + \alpha_1 \|p - T_1 T_{\alpha_2} q\|^2 - \alpha_1(1-\alpha_1)\|T_{\alpha_2} q - T_1 T_{\alpha_2} q\|^2 \\ &\le (1-\alpha_1)\|p - T_{\alpha_2} q\|^2 + \alpha_1 \|p - T_{\alpha_2} q\|^2 - \alpha_1(1-\alpha_1)\|T_{\alpha_2} q - T_1 T_{\alpha_2} q\|^2 \\ &= \|p - T_{\alpha_2} q\|^2 - \alpha_1(1-\alpha_1)\|T_{\alpha_2} q - T_1 T_{\alpha_2} q\|^2. \end{aligned} \tag{2.12}$$

From (2.12), we have

$$\|T_{\alpha_2} q - T_1 T_{\alpha_2} q\|^2 \le 0, \tag{2.13}$$

namely, $T_{\alpha_2} q = T_1 T_{\alpha_2} q$, that is,

$$T_{\alpha_2} q \in F_{ix}(T_1) = F_{ix}(T_{\alpha_1}), \qquad T_{\alpha_2} q = T_{\alpha_1} T_{\alpha_2} q = q. \tag{2.14}$$

Suppose that the conclusion holds for $N = k$, we prove that

$$F_{ix}(T_{\alpha_1}T_{\alpha_2}\cdots T_{\alpha_{k+1}}) = \bigcap_{i=1}^{k+1} F_{ix}(T_{\alpha_i}). \tag{2.15}$$

It suffices to verify

$$F_{ix}(T_{\alpha_1}T_{\alpha_2}\cdots T_{\alpha_{k+1}}) \subset \bigcap_{i=1}^{k+1} F_{ix}(T_{\alpha_i}), \tag{2.16}$$

for all $q \in F_{ix}(T_{\alpha_1}T_{\alpha_2}\cdots T_{\alpha_{k+1}})$, that is, $T_{\alpha_1}T_{\alpha_2}\cdots T_{\alpha_{k+1}}q = q$. Using Remark 2.3 again, take $p \in \bigcap_{i=1}^{k+1} F_{ix}(T_{\alpha_i})$, we obtain

$$\begin{aligned}
\|p-q\|^2 &= \|p - T_{\alpha_1}T_{\alpha_2}\cdots T_{\alpha_{k+1}}q\|^2 \\
&= \left\|p - \left[(1-\alpha_1)T_{\alpha_2}\cdots T_{\alpha_{k+1}}q - \alpha_1 T_1 T_{\alpha_2}\cdots T_{\alpha_{k+1}}q\right]\right\|^2 \\
&= \left\|\left[(1-\alpha_1)\left(p - T_{\alpha_2}\cdots T_{\alpha_{k+1}}q\right) - \alpha_1\left(p - T_1T_{\alpha_2}\cdots T_{\alpha_{k+1}}q\right)\right]\right\|^2 \\
&= (1-\alpha_1)\|p - T_{\alpha_2}\cdots T_{\alpha_{k+1}}q\|^2 + \alpha_1\|p - T_1T_{\alpha_2}\cdots T_{\alpha_{k+1}}q\|^2 \\
&\quad - \alpha_1(1-\alpha_1)\|T_{\alpha_2}\cdots T_{\alpha_{k+1}}q - T_1T_{\alpha_2}\cdots T_{\alpha_{k+1}}q\|^2 \\
&\le \|p - T_{\alpha_2}\cdots T_{\alpha_{k+1}}q\|^2 - \alpha_1(1-\alpha_1)\|T_{\alpha_2}\cdots T_{\alpha_{k+1}}q - T_1T_{\alpha_2}\cdots T_{\alpha_{k+1}}q\|^2.
\end{aligned} \tag{2.17}$$

From (2.17), we obtain

$$\|T_{\alpha_2}\cdots T_{\alpha_{k+1}}q - T_1T_{\alpha_2}\cdots T_{\alpha_{k+1}}q\|^2 \le 0, \tag{2.18}$$

this implies that

$$T_{\alpha_2}\cdots T_{\alpha_{k+1}}q \in F_{ix}(T_1) = F_{ix}(T_{\alpha_1}), \tag{2.19}$$

namely,

$$T_{\alpha_2}\cdots T_{\alpha_{k+1}}q = T_{\alpha_1}T_{\alpha_2}\cdots T_{\alpha_{k+1}}q = q. \tag{2.20}$$

From (2.20) and inductive assumption, we have

$$q \in F_{ix}(T_{\alpha_1}T_{\alpha_2}\cdots T_{\alpha_{k+1}}) \subset \bigcap_{i=2}^{k+1} F_{ix}(T_{\alpha_i}), \tag{2.21}$$

therefore

$$T_{\alpha_i}q = q, \quad i = 2,3,\ldots,k+1. \tag{2.22}$$

Substituting it into (2.20), we obtain $T_{\alpha_1}q = q$. Thus we assert that

$$q \in \bigcap_{i=1}^{k+1} F_{ix}(T_{\alpha_i}). \tag{2.23}$$

□

Definition 2.6. A mapping T is said to be demiclosed, if for any sequence $\{x_n\}$ weakly converges to y, and if the sequence $\{Tx_n\}$ strongly converges to z, then $T(y) = z$.

Lemma 2.7 (see [5]). *Assume A is a strong positive linear bounded operator on a Hilbert space H with coefficient $\overline{\gamma} > 0$ and $0 < \rho \le \|A\|^{-1}$, then $\|I - \rho A\| \le 1 - \rho\overline{\gamma}$.*

Lemma 2.8 (see [12]). *Let H be a Hilbert space, K a closed convex subset of H, and $T : K \to K$ a nonexpansive mapping with $F_{ix}(T) \neq \emptyset$, if $\{x_n\}$ is a sequence in K weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$.*

Lemma 2.9 (see [13]). *Let $\{\tau_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{\tau_{n_j}\}_{j\ge 0}$ of $\{\tau_n\}$ which satisfies $\tau_{n_j} < \tau_{n_j+1}$ for all $j \ge 0$. Also consider the sequence of integers $\{\delta(n)\}_{n\ge n_0}$ defined by*

$$\delta(n) = \max\{k \le n \mid \tau_k < \tau_{k+1}\}. \tag{2.24}$$

Then $\{\delta(n)\}_{n\ge n_0}$ is a nondecreasing sequence verifying $\lim_{n\to\infty}\delta(n) = \infty$, for all $n \ge n_0$, it holds that $\tau_{\delta(n)} < \tau_{\delta(n)+1}$ and one has

$$\tau_n < \tau_{\delta(n)+1}. \tag{2.25}$$

Lemma 2.10. *Let K be a closed convex subset of a real Hilbert space H, given $x \in H$ and $y \in K$. Then $y = P_K x$ if and only if there holds the inequality*

$$\langle x - y, y - z\rangle \ge 0, \quad \forall z \in K. \tag{2.26}$$

3. Parallel Algorithm

In this section, we discuss the parallel algorithm, respectively, for solving the variational inequality over the set of the common fixed points of finite quasi-nonexpansives.

Before stating our main convergence result, we establish the boundedness of the iterates given by following algorithm:

$$\begin{gathered} T = \sum_{i=1}^{N}\lambda_i T_{\omega_i}, \\ x_{n+1} = \alpha_n\gamma f(x_n) + \beta_n x_n + ((I - \beta_n)I - \alpha_n A)Tx_n. \end{gathered} \tag{3.1}$$

In (3.1), the weight $\{\lambda_i\}_{i=1}^{N}$ are constant in the sense that they are independent of n, the number of steps of the iteration process. Below we consider a more general case by allowing the weights $\{\lambda_i\}_{i=1}^N$ to be step dependent. That is, initializing with x_0, we define $\{x_n\}$ by the algorithm

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + \left((I-\beta_n)I - \alpha_n A\right)\sum_{i=1}^N \lambda_i^{(n)} T_{\omega_i} x_n. \tag{3.2}$$

From (3.1) and (3.2), the sequence $\{x_n\}$ which converges strongly to the unique solution of variational inequality problem VI$(\gamma f - A, \bigcap_{i=1}^N F_{ix}(T_{\omega_i}))$: find x^* in $\bigcap_{i=1}^N F_{ix}(T_{\omega_i})$ such that

$$\forall v \in \bigcap_{i=1}^N F_{ix}(T_{\omega_i}), \quad \langle (\gamma f - A)x^*, v - x^* \rangle \le 0, \tag{3.3}$$

or equivalently

$$x^* = \left(P_{\bigcap_{i=1}^N F_{ix}(T_{\omega_i})} \cdot C\right)(x^*), \tag{3.4}$$

where $P_{\bigcap_{i=1}^N F_{ix}(T_{\omega_i})}$ denotes the metric projection from H onto $\bigcap_{i=1}^N F_{ix}(T_{\omega_i})$ (see, [14] for more details on the metric projection).

Lemma 3.1. *The sequence $\{x_n\}$ is generated by* (3.2)*, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence in* $[0,1]$*, and $\{T_i\}_{i=1}^N$ is a quasi-nonexpansive mapping on H, is bounded and satisfies*

$$\|x_n - v\| \le \max\left\{ \|x_1 - v\|, \frac{\|\gamma f(v) - Av\|}{\overline{\gamma} - \gamma\beta} \right\}, \quad \forall n \ge 1, \tag{3.5}$$

where v is any element in $F_{ix}(T_i)$, $1 \le i \le N$.

Proof. Since $\lim_{n\to\infty}\alpha_n = 0$, we shall assume that $\alpha_n \le (1-\beta_n)\|A\|^{-1}$ and $1 - \alpha_n(\overline{\gamma} - \gamma\beta) > 0$. Observe that if $\|u\| = 1$, then

$$\langle ((I-\beta_n)I - \alpha_n A)u, u \rangle = (1-\beta_n) - \alpha_n \langle Au, u \rangle \ge (1 - \beta_n - \alpha_n\|A\|) \ge 0. \tag{3.6}$$

By Lemma 2.7, we obtain

$$\|(I-\beta_n)I - \alpha_n A\| \le 1 - \beta_n - \alpha_n\overline{\gamma}. \tag{3.7}$$

Let $B_n = \sum_{i=1}^N \lambda_i^{(n)} T_{\omega_i}$, for all $n \ge 1$. By Lemma 2.5, each B_n is a quasi-nonexpansive mapping on H, and in light of Remark 2.3. Taking $v \in F_{ix}(T)$, we have

$$\left\| \sum_{i=1}^N \lambda_i^{(n)} T_{\omega_i} x_n - v \right\| \le \left\| \sum_{i=1}^N \lambda_i^{(n)} T_{\omega_i}(x_n - v) \right\| \le \left\| \sum_{i=1}^N \lambda_i^{(n)} (x_n - v) \right\| \le \|x_n - v\|. \tag{3.8}$$

From (3.1), we have

$$\begin{aligned}\|x_{n+1}-v\| &= \left\|\alpha_n(\gamma f(x_n)-Av)+\beta_n(x_n-v)+((I-\beta_n)I-\alpha_n A)T(x_n-v)\right\| \\ &\le \alpha_n\|\gamma f(x_n)-Av\|+\beta_n\|x_n-v\|+(1-\beta_n-\alpha_n\overline{\gamma})\|x_n-v\| \\ &\le \alpha_n\gamma\|f(x_n)-f(v)\|+\alpha_n\|\gamma f(v)-Av\|+(1-\alpha_n\overline{\gamma})\|x_n-v\| \\ &= \left[1-\alpha_n(\overline{\gamma}-\gamma\beta)\right]\|x_n-v\|+\alpha_n\|\gamma f(v)-Av\|.\end{aligned} \tag{3.9}$$

By simple inductions, we obtain

$$\|x_n-v\|\le \max\left\{\|x_1-v\|,\frac{\|\gamma f(v)-Av\|}{\overline{\gamma}-\gamma\beta}\right\},\quad \forall n\ge 1, \tag{3.10}$$

which gives that the sequence $\{x_n\}$ is bounded. □

Lemma 3.2. *Assume that $\{x_n\}$ is defined by* (3.2)*, if x^* is solution of* (3.3) *with $T: C\to C$ demi-closed and $\{y_n\}\subset H$ is a bounded sequence such that $\|Ty_n-y_n\|\to 0$, then*

$$\liminf_{n\to\infty}\langle (A-\gamma f)x^*, y_n-x^*\rangle\ge 0. \tag{3.11}$$

Proof. Clearly, by $\|Ty_n-y_n\|\to 0$ and $T: H\to H$ demi-closed, we know that any weak cluster point of $\{y_n\}$ belongs to $F_{ix}(T)$. It is also a simple matter to see that there exist $\overline{y}$ and a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that $\lim_{j\to\infty}y_{n_j}\rightharpoonup\overline{y}$ (hence $\overline{y}\in F_{ix}(T)$) and such that

$$\liminf_{n\to\infty}\langle (A-\gamma f)x^*, y_n-x^*\rangle=\lim_{j\to\infty}\left\langle (A-\gamma f)x^*, y_{n_j}-x^*\right\rangle, \tag{3.12}$$

it follows from (3.3), we can derive that

$$\liminf_{n\to\infty}\langle (A-\gamma f)x^*, y_n-x^*\rangle=\langle (A-\gamma f)x^*, \overline{y}-x^*\rangle\ge 0, \tag{3.13}$$

that is the desired result. □

Theorem 3.3. *Let C be a closed convex subset of a Hilbert space H and let $T_i: C\to C$ be a quasi-non-expansive for $T_{\omega_i}=(1-\omega_i)I+\omega_i T_i$, $\omega_i\in(0,1)$, $i\in(1,\dots,N)$ such that $\bigcap_{i=1}^N F_{ix}(T_{\omega_i})\ne\emptyset$, f be a contraction with coefficient $\beta\in(0,1)$, and λ_i a positive constant such that $\sum_{i=1}^N\lambda_i^{(n)}=1$ for all n and $\inf_{n\ge1}\lambda_i^{(n)}>0$ for all $i\in[1,N]$. Let A be a strongly positive bounded linear operator with coefficient $\overline{\gamma}$. Given the initial guess $x_0\in H$ chosen arbitrarily and given sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $(0,1)$, satisfying the following conditions:*

(c1) $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=0}^{\infty}\alpha_n=\infty$,

(c2) $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$.

Let $\{x_n\}$ *be the sequence generated by* (3.2). *Then* $\{x_n\}$ *converges strongly to the unique a element* $x^* \in \bigcap_{i=1}^{N} F_{ix}(T_{\omega_i}),\ N \geq 1$ *verifying*

$$x^* = \left(P_{\bigcap_{i=1}^{N} F_{ix}(T_{\omega_i})} \cdot f\right)x^* \tag{3.14}$$

which equivalently solves the following variational inequality problem:

$$x^* \in \bigcap_{i=1}^{N} F_{ix}(T_{\omega_i}), \quad \langle (\gamma f - A)x^*, \hat{x} - x^* \rangle \leq 0, \quad \forall \hat{x} \in \bigcap_{i=1}^{N} F_{ix}(T_{\omega_i}). \tag{3.15}$$

Proof. Taking $B_n = \sum_{i=1}^{N} \lambda_i^{(n)} T_{\omega_i}$, for all $n \geq 1$. By Lemma 2.5(i), each B_n is a quasi-nonexpansive mapping on C, and (3.2) can be rewritten as

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((I - \beta_n)I - \alpha_n A)B_n x_n. \tag{3.16}$$

Denote by Ω the common fixed point of the mappings $\{T_{\omega_i}\}_{i=1}^{N}$ (by Lemma 2.5(ii), we can easily know that $\Omega = \bigcap_{i=1}^{N} F_{ix}(T_{\omega_i}) = \bigcap_{i=1}^{N} F_{ix}(T_i)$) and take $x^* \in \Omega$ and from (3.16) we deduce that

$$x_{n+1} - x_n + \alpha_n (Ax_n - \gamma f(x_n)) = (I - \beta_n - \alpha_n A)(B_n x_n - x_n), \tag{3.17}$$

and hence

$$\begin{aligned} \langle x_{n+1} - x_n + \alpha_n (Ax_n - \gamma f(x_n)), x_n - x^* \rangle &= \langle (1 - \beta_n - \alpha_n A)B_n x_n - x_n, x_n - x^* \rangle \\ &= (1 - \beta_n - \alpha_n)\langle B_n x_n - x_n, x_n - x^* \rangle \\ &\quad + \alpha_n \langle (I - A)(B_n - I)x_n, x_n - x^* \rangle. \end{aligned} \tag{3.18}$$

Moreover, by $x^* \in \bigcap_{i=1}^{N} F_{ix}(T_{\omega_i})$ and using Remark 2.3(iv), we obtain

$$\begin{aligned} \langle x_n - B_n x_n, x_n - x^* \rangle &\geq \left\langle x_n - \sum_{i=1}^{N} \lambda_i^{(n)} T_{\omega_i} x_n, x_n - x^* \right\rangle \\ &\geq \sum_{i=1}^{N} \lambda_i^{(n)} \langle x_n - T_{\omega_i} x_n, x_n - x^* \rangle \\ &\geq \sum_{i=1}^{N} \frac{\lambda_i^{(n)} \omega_i}{2} \|x_n - T_i x_n\|^2, \end{aligned} \tag{3.19}$$

which combined with the (3.18) entails

$$\begin{aligned}\langle x_{n+1}-x_n+\alpha_n(A-\gamma f)x_n, x_n-x^*\rangle \leq{}& \frac{-(1-\beta_n-\alpha_n)}{2}\sum_{i=1}^{N}\left(\lambda_i^{(n)}\omega_i\|x_n-T_ix_n\|^2\right)\\ &+\alpha_n\langle (I-A)(B_n-I)x_n, x_n-x^*\rangle,\end{aligned} \tag{3.20}$$

or equivalently

$$\begin{aligned}-\langle x_n-x_{n+1}, x_n-x^*\rangle \leq{}& -\alpha_n\langle (A-\gamma f)x_n, x_n-x^*\rangle\\ &-\frac{(1-\beta_n-\alpha_n)}{2}\sum_{i=1}^{N}\left(\lambda_i^{(n)}\omega_i\|x_n-T_ix_n\|^2\right)\\ &+\alpha_n\langle (I-A)(B_n-I)x_n, x_n-x^*\rangle.\end{aligned} \tag{3.21}$$

Furthermore, using the following classical equality:

$$\langle u, v\rangle = \frac{1}{2}\|u\|^2-\frac{1}{2}\|u-v\|^2+\frac{1}{2}\|v\|^2, \qquad \forall u, v\in C, \tag{3.22}$$

and setting $\mathcal{T}_n=(1/2)\|x_n-x^*\|^2$, we have

$$\langle x_n-x_{n+1}, x_n-x^*\rangle = \mathcal{T}_n-\mathcal{T}_{n+1}+\frac{1}{2}\|x_n-x_{n+1}\|^2. \tag{3.23}$$

So that (3.21) can be equivalently rewritten as

$$\begin{aligned}\mathcal{T}_{n+1}-\mathcal{T}_n-\frac{1}{2}\|x_n-x_{n+1}\|^2 \leq{}& -\alpha_n\langle (A-\gamma f)x_n, x_n-x^*\rangle\\ &-\frac{(1-\beta_n-\alpha_n)}{2}\sum_{i=1}^{N}\left(\lambda_i^{(n)}\omega_i\|x_n-T_ix_n\|^2\right)\\ &+\alpha_n\langle (I-A)(B_n-I)x_n, x_n-x^*\rangle.\end{aligned} \tag{3.24}$$

Now using (3.16) again, we have

$$\|x_{n+1}-x_n\|^2 = \left\|\alpha_n(\gamma f(x_n)-Ax_n)+(I-\beta_n-\alpha_n A)(B_nx_n-x_n)\right\|^2. \tag{3.25}$$

Since $A: H\to H$ is a strongly positive bounded linear operator with coefficient $\overline{\gamma}>0$, hence it is a classical matter to see that

$$\|x_{n+1}-x_n\|^2 \leq 2\alpha_n^2\left\|\gamma f(x_n)-Ax_n\right\|^2+2(1-\beta_n-\alpha_n\overline{\gamma})^2\|B_nx_n-x_n\|^2. \tag{3.26}$$

from

$$\begin{aligned}\|B_n x_n - x_n\|^2 &= \left\| \sum_{i=1}^{N} \lambda_i^{(n)} T_{\omega_i} x_n - x_n \right\|^2 \\ &= \left\| \sum_{i=1}^{N} \lambda_i^{(n)} (T_{\omega_i} x_n - x_n) \right\|^2 \\ &\le 2 \sum_{i=1}^{N} \left(\lambda_i^{(n)}\right)^2 \omega_i^2 \|x_n - T_i x_n\|^2 \\ &\le 2 \sum_{i=1}^{N} \lambda_i^{(n)} \omega_i \|x_n - T_i x_n\|^2\end{aligned} \tag{3.27}$$

and $(1-\beta_n-\alpha_n\overline{\gamma})^2 \le (1-\beta_n-\alpha_n\overline{\gamma})$ yields

$$\frac{1}{2}\|x_{n+1}-x_n\|^2 \le \alpha_n^2 \left\|\gamma f(x_n) - A x_n\right\|^2 + (1-\beta_n-\alpha_n\overline{\gamma}) \sum_{i=1}^{N} \left(\lambda_i^{(n)} \omega_i \|x_n - T_i x_n\|^2\right). \tag{3.28}$$

Then from (3.24) and (3.28), we have

$$\begin{aligned}&\mathcal{T}_{n+1} - \mathcal{T}_n + \left[\frac{(1-\beta_n-\alpha_n)}{2} - (1-\beta_n-\alpha_n\overline{\gamma})\right] \sum_{i=1}^{N} \left(\lambda_i^{(n)} \omega_i \|x_n - T_i x_n\|^2\right) \\ &\quad \le \alpha_n \left(\alpha_n \left\|\gamma f(x_n) - A x_n\right\|^2 - \langle (A-\gamma f) x_n, x_n - x^* \rangle + \langle (I-A)(B_n - I) x_n, x_n - x^* \rangle \right).\end{aligned} \tag{3.29}$$

The rest of the proof will be divided into two parts.

Case 1. Suppose that there exists n_0 such that $\{\mathcal{T}_n\}_{n\ge n_0}$ is nonincreasing. In this situation, $\{\mathcal{T}_n\}$ is then convergent because it is also nonnegative (hence it is bounded from below), so that $\lim_{n\to\infty}(\mathcal{T}_{n+1} - \mathcal{T}_n) = 0$; hence, in light of (3.29) together with $\lim_{n\to\infty}\alpha_n = 0$, $0 < \liminf_{n\to\infty}\beta_n \le \liminf_{n\to\infty}\beta_n < 1$, and the boundedness of $\{x_n\}$, we obtain

$$\lim_{n\to\infty} \sum_{i=1}^{N} \left(\lambda_i^{(n)} \omega_i \|x_n - T_i x_n\|^2\right) = 0. \tag{3.30}$$

By (3.27) and (3.30), we can easily claim that

$$\lim_{n\to\infty} \|B_n x_n - x_n\| = 0. \tag{3.31}$$

It also follows from (3.29) that

$$\mathcal{T}_n - \mathcal{T}_{n+1} \ge \alpha_n \left(-\alpha_n \left\|\gamma f(x_n) - A x_n\right\|^2 + \langle (A-\gamma f) x_n, x_n - x^* \rangle + \langle (I-A)(B_n - I) x_n, x_n - x^* \rangle \right). \tag{3.32}$$

Then, by $\sum_{n=0}^{\infty} \alpha_n = \infty$, we obviously deduce that

$$\liminf_{n \to \infty} \left(-\alpha_n \| \gamma f(x_n) - A x_n \|^2 + \langle (A - \gamma f) x_n, x_n - x^* \rangle + \langle (I - A)(B_n - I) x_n, x_n - x^* \rangle \right) \leq 0. \tag{3.33}$$

Since $\{f(x_n)\}$ and $\{x_n\}$ are both bounded, $\lim_{n \to \infty} \alpha_n = 0$ and $\lim_{n \to \infty} \| B_n x_n - x_n \| = 0$, we obtain

$$\liminf_{n \to \infty} \langle (A - \gamma f) x_n, x_n - x^* \rangle \leq 0. \tag{3.34}$$

Moreover, by Remark 1.1, we have

$$2(\overline{\gamma} - \gamma \beta) \tau_n + \langle (A - \gamma f) x^*, x_n - x^* \rangle \leq \langle (A - \gamma f) x_n, x_n - x^* \rangle, \tag{3.35}$$

which by (3.34) entails

$$\liminf_{n \to \infty} (2(\overline{\gamma} - \gamma \beta) \tau_n + \langle (A - \gamma f) x^*, x_n - x^* \rangle) \leq 0, \tag{3.36}$$

hence, recalling that $\lim_{n \to \infty} \tau_n$ exists, we equivalently obtain

$$2(\overline{\gamma} - \gamma \beta) \lim_{n \to \infty} \tau_n + \liminf_{n \to \infty} \langle (A - \gamma f) x^*, x_n - x^* \rangle \leq 0, \tag{3.37}$$

namely,

$$2(\overline{\gamma} - \gamma \beta) \lim_{n \to \infty} \tau_n \leq -\liminf_{n \to \infty} \langle (A - \gamma f) x^*, x_n - x^* \rangle. \tag{3.38}$$

From (3.30) and invoking Lemma 3.2, we obtain

$$\liminf_{n \to \infty} \langle (A - \gamma f) x^*, \widehat{x} - x^* \rangle \geq 0, \qquad \widehat{x} \in \bigcap_{n=1}^{N} F_{ix}(T_{\omega_i}), \tag{3.39}$$

which by (3.38) yields $\lim_{n \to \infty} \tau_n = 0$, so that $\{x_n\}$ converges strongly to x^*.

Case 2. Suppose there exists a subsequence $\{\tau_{n_k}\}_{k \geq 0}$ of $\{\tau_n\}_{n \geq 0}$ such that $\tau_{n_k} \leq \tau_{n_{k+1}}$ for all $k \geq 0$. In this situation, we consider the sequence of indices $\{\delta(n)\}$ as defined in Lemma 2.9. It follows that $\tau_{\delta(n+1)} - \tau_{\delta(n)} > 0$, which by (3.29) amounts to

$$\begin{aligned} &\left[\frac{(1 - \beta_{\delta(n)} - \alpha_{\delta(n)})}{2} - (1 - \beta_{\delta(n)} - \alpha_{\delta(n)} \overline{\gamma}) \right] \sum_{i=1}^{N} \left(\lambda_i^{(n)} \omega_i \| x_{\delta(n)} - T_i x_{\delta(n)} \|^2 \right) \\ &\quad \leq \alpha_{\delta(n)} \left(\alpha_{\delta(n)} \| \gamma f(x_{\delta(n)}) - A x_{\delta(n)} \|^2 - \langle (A - \gamma f) x_{\delta(n)}, x_n - x^* \rangle \right), \end{aligned} \tag{3.40}$$

hence, by the boundedness of $\{x_n\}$ and $\lim_{n\to\infty}\alpha_n = 0$, we immediately obtain

$$\lim_{n\to\infty}\sum_{i=1}^{N}\left(\lambda_i^{(n)}\omega_i\left\|x_{\delta(n)} - T_i x_{\delta(n)}\right\|^2\right) = 0. \tag{3.41}$$

From (3.28) we have

$$\begin{aligned}
&\frac{1}{2}\left\|x_{\delta(n)+1} - x_{\delta(n)}\right\|^2 \\
&\quad\le \alpha_{\delta(n)}^2\left\|\gamma f\left(x_{\delta(n)}\right) - Ax_{\delta(n)}\right\|^2 + \left(1 - \beta_{\delta(n)} - \alpha_{\delta(n)}\overline{\gamma}\right)\sum_{i=1}^{N}\left(\lambda_i^{(n)}\omega_i\left\|x_{\delta(n)} - T_i x_{\delta(n)}\right\|^2\right) \\
&\quad\le \alpha_{\delta(n)}\left\|\gamma f\left(x_{\delta(n)}\right) - Ax_{\delta(n)}\right\|^2 + \left(1 - \beta_{\delta(n)} - \alpha_{\delta(n)}\overline{\gamma}\right)\sum_{i=1}^{N}\left(\lambda_i^{(n)}\left\|x_{\delta(n)} - T_i x_{\delta(n)}\right\|^2\right),
\end{aligned} \tag{3.42}$$

which together with (3.41), $\lim_{n\to\infty}\alpha_n = 0$ and $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$ yields

$$\lim_{n\to\infty}\left\|x_{\delta(n)+1} - x_{\delta(n)}\right\| = 0. \tag{3.43}$$

Now by (3.40), we clearly have

$$\alpha_{\delta(n)}\left\|\gamma f\left(x_{\delta(n)}\right) - \mu Bx_{\delta(n)}\right\|^2 \ge \left\langle (A - \gamma f)x_{\delta(n)}, x_{\delta(n)} - x^*\right\rangle, \tag{3.44}$$

which in the light of (3.38) yields

$$2(\overline{\gamma} - \gamma\beta)\tau_{\delta(n)} + \left\langle (A - \gamma f)x^*, x_{\delta(n)} - x^*\right\rangle \le \alpha_{\delta(n)}\left\|\gamma f\left(x_{\delta(n)}\right) - Ax_{\delta(n)}\right\|^2, \tag{3.45}$$

hence (as $\lim_{n\to\infty}\alpha_{\delta(n)}\|\gamma f(x_{\delta(n)}) - Ax_{\delta(n)}\|^2 = 0$) it follows that

$$2(\overline{\gamma} - \gamma\beta)\limsup_{n\to\infty}\tau_{\delta(n)} \le -\liminf_{n\to\infty}\left\langle (A - \gamma f)x^*, x_{\delta(n)} - x^*\right\rangle. \tag{3.46}$$

From (3.41) and invoking Lemma 3.2, we obtain

$$\lim_{n\to\infty}\left\langle (A - \gamma f)x^*, \widehat{x} - x^*\right\rangle \ge 0, \quad \widehat{x} \in \bigcap_{n=1}^{N} F_{ix}(T_{\omega_i}), \tag{3.47}$$

which by (3.46) yields $\limsup_{n\to\infty}\tau_{\delta(n)} = 0$, so that $\lim_{n\to\infty}\tau_{\delta(n)} = 0$. Combining (3.43), we have $\lim_{n\to\infty}\tau_{\delta(n)+1} = 0$. Then, recalling that $\tau_n < \tau_{\delta(n)+1}$ (by Lemma 2.9), we get

$\lim_{n\to\infty}\tau_n = 0$, so that $x_n \to x^*$ strongly. In addition, the variational inequality (3.39) and (3.47) can be written as

$$\left\langle (I - A + \gamma f)x^* - x^*, \widehat{x} - x^* \right\rangle \le 0, \qquad \widehat{x} \in \bigcap_{n=1}^{N} F_{ix}(T_{\omega_i}). \tag{3.48}$$

So, by the Lemma 2.10, it is equivalent to the fixed point equation

$$x^* = P_{\bigcap_{n=1}^{N} F_{ix}(T_{\omega_i})}(I - A + \gamma f)x^* = \left(P_{\bigcap_{n=1}^{N} F_{ix}(T_{\omega_i})} \cdot f\right)x^*. \tag{3.49}$$

□

If the sequences of the weights $\{\lambda_i^{(n)}\}_{i=1}^N = \{\lambda_i\}_{i=1}^N$ in (3.2), according to the proof of Theorem 3.3, we can obtain the following corollary.

Corollary 3.4. *Let C be a closed convex subset of a Hilbert space H and let $T_i : C \to C$ be a quasi-nonexpansive for $T_{\omega_i} = (1-\omega_i)I + \omega_i T_i$, $\omega_i \in (0,1)$, $i \in (1,\dots,N)$ such that $\bigcap_{i=1}^N F_{ix}(T_{\omega_i}) \neq \emptyset$, f is a contraction with coefficient $\beta \in (0,1)$ and λ_i is a positive constant such that $\sum_{i=1}^N \lambda_i = 1$. Let A be a strongly positive bounded linear operator with coefficient $\overline{\gamma}$. Given the initial guess $x_0 \in H$ chosen arbitrarily and given sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $(0,1)$, satisfying the following conditions:*

(c1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(c2) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$.

Let $\{x_n\}$ be the sequence generated by (3.1). *Then $\{x_n\}$ converges strongly to the unique a element $x^* \in \bigcap_{i=1}^N F_{ix}(T_{\omega_i})$, $N \ge 1$ verifying*

$$x^* = \left(P_{\bigcap_{i=1}^{N} F_{ix}(T_{\omega_i})} \cdot f\right)x^*, \tag{3.50}$$

which equivalently solves the following variational inequality problem:

$$x^* \in \bigcap_{i=1}^{N} F_{ix}(T_{\omega_i}), \quad \left\langle (\gamma f - A)x^*, \widehat{x} - x^* \right\rangle \le 0, \quad \forall \widehat{x} \in \bigcap_{i=1}^{N} F_{ix}(T_{\omega_i}). \tag{3.51}$$

4. Cyclic Algorithm

In this section, we discuss the cyclic algorithm, respectively, for solving the variational inequality over the set of the common fixed points of finite quasi-nonexpansives and introduce quasi-shrinking mapping and quoted its definition from [11]. Hereafter, for nonempty closed set $S \subset H$ and $r \ge 0$, we use the notations $d_S : H \ni u \mapsto d(u,S) := \inf_{x\in S}\|u - x\|$, $\diamond(S,r) := \{u \in H \mid d(u,S) = r\}$, Ł$(S,r) := \{u \in H \mid d(u,S) \le r\}$ and Ł$(S,r) := \{u \in H \mid d(u,S) \ge r\}$. In this case, by the upper semicontinuity of d_s (see e.g., [14, Theorem 1.3.3]), Ł(S,r) is closed. Moreover, for a nonempty bounded closed convex set $C \subset H$ and $r \ge 0$, it is not hard to verify that (i) $\diamond(C,r)$ and Ł(C,r) are also closed; (ii) $\diamond(C,r)$ and Ł(C,r) are bounded; (iii) Ł(C,r) is convex.

Definition 4.1 (see [11]). Suppose that $T : H \to H$ is quasi-nonexpansive with $F_{ix}(T) \bigcap C \neq \emptyset$ for some closed convex set C. Then $T : H \to H$ is called quasi-shrinking on C if

$$D : r \in [0,\infty) \longmapsto \begin{cases} \inf\limits_{u \in Ł(F_{ix}(T),r)\cap C} d(u, F_{ix}(T)) - d(T(u), F_{ix}(T)), \\ \text{if } u \in Ł(F_{ix}(T), r) \cap C \neq \emptyset, \\ \infty \qquad \text{otherwise} \end{cases} \tag{4.1}$$

satisfies $D(r) = 0 \Leftrightarrow r = 0$. In particular, if T is quasi-shrinking on H, then T is just called quasi-shrinking.

Let C be a closed convex subset of a Hilbert space H and let $\{T_i\}_{i=1}^{N-1}$ be quasi-nonexpansives defined on C such that the common fixed point set

$$F := \bigcap_{i=1}^{N} F_{ix}(T_{\omega_i}), \quad N \geq 1, \tag{4.2}$$

where $T_{\omega_i} = (1-\omega_i)I + \omega_i T_i$, $\{\omega_i\}_{i=1}^{N} \in (0,1)$. Let $x_0 \in C$, let $\{\alpha_n\}_{n=0}^{\infty}$, and $\{\beta_n\}_{n=0}^{\infty}$ sequences in $(0,1)$. The cyclic algorithm generates a sequence $\{x_n\}_{n=1}^{\infty}$ in the following way:

$$\begin{aligned} x_1 &= \alpha_0 \gamma f(x_0) + \beta_0 x_0 + ((I-\beta_0)I - \alpha_0 A)T_{\omega_0} x_0, \\ x_2 &= \alpha_1 \gamma f(x_1) + \beta_1 x_1 + ((I-\beta_1)I - \alpha_1 A)T_{\omega_1} x_1, \\ &\vdots \\ x_N &= \alpha_{N-1} \gamma f(x_{N-1}) + \beta_{N-1} x_{N-1} + ((I-\beta_{N-1})I - \alpha_{N-1} A)T_{\omega_{N-1}} x_{N-1}, \\ x_{N+1} &= \alpha_{N-1} \gamma f(x_{N-1}) + \beta_N x_N + ((I-\beta_N)I - \alpha_N A)T_{\omega_N} x_N, \\ &\vdots \end{aligned} \tag{4.3}$$

In general, x_{n+1} is defined by

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((I-\beta_n)I - \alpha_n A)T_{[n]} x_n, \tag{4.4}$$

where $T_{[n]} = T_{\omega_i} = (1-\omega_i)I + \omega_i T_i$, with $i = n \pmod N$, $0 \leq i \leq N-1$.

Lemma 4.2 (see [11]). *Let $\varphi(x) : [0,\infty) \to [0,\infty)$ satisfy*

(i) $x_1 > x_2 \Rightarrow \varphi(x_1) > \varphi(x_2)$,

(ii) $\varphi(x) = 0 \Leftrightarrow x = 0$.

Let $\{z_n\}_{n\geq 1} \subset [0,\infty)$ satisfy $\lim_{n\to\infty} z_n = 0$. Then any sequence $\{b_n\}_{n\geq 1} \subset [0,\infty)$ satisfying

$$b_{n+1} \leq b_n - \varphi(b_n) + z_{n+1}, \quad n = 0, 1, 2, \ldots \tag{4.5}$$

converges to 0.

Lemma 4.3 (see [15]). *Assume that* $\{\alpha_n\}_{n=0}^{\infty}$ *is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \le (1-\gamma_n)\alpha_n + \gamma_n\delta_n, \quad n \ge 0, \tag{4.6}$$

where $\{\gamma_n\}_{n=0}^{\infty} \subset [0,1]$ *and* $\{\delta_n\}_{n=0}^{\infty}$ *satisfy the following conditions:*

(i) $\sum_{n=0}^{\infty}\gamma_n = \infty$ *and* $\lim_{n\to\infty}\gamma_n = 0$,

(ii) $\limsup_{n\to\infty}\delta_n \le 0$ *or* $\sum_{n=0}^{\infty}|\gamma_n\delta_n| < \infty$.

Then $\lim_{n\to\infty}\alpha_n = 0$.

Theorem 4.4. *Let* C *be a closed convex subset of a Hilbert space* H *and let* $T_i : H \to H$ *be quasi-nonexpansives for* $T_{\omega_i} = (1-\omega_i)I+\omega_i T_i$, $\omega_i \in (0,1)$, $i \in (1,2,\dots,N)$ *such that* $F := \bigcap_{i=1}^{N} F_{ix}(T_{\omega_i}) \neq \emptyset$ *and* f *a contraction with coefficient* $\beta \in (0,1)$. *Let* A *be a strongly positive bounded linear operator with coefficient* $\overline{\gamma}$. *Given the initial guess* $x_0 \in H$ *chosen arbitrarily and given sequences* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *in* $(0,1)$, *satisfying the following conditions:*

(4.1a) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(4.1b) $\sum_{n=0}^{\infty}\|\alpha_{n+1}-\alpha_n\| < \infty$ *or* $\lim_{n\to\infty}\alpha_n/\alpha_{n+1} = 1$;

(4.1c) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$.

Let $\{x_n\}$ *be the sequence generated by* (4.4). *Then* $\{x_n\}$ *converges strongly to the unique a element* $x^* \in F := \bigcap_{i=1}^{N} F_{ix}(T_{\omega_i})$, $N \ge 1$ *verifying*

$$x^* = \left(P_{\bigcap_{i=1}^{N} F_{ix}(T_{\omega_i})} \cdot f\right)x^*, \tag{4.7}$$

which equivalently solves the following variational inequality problem:

$$x^* \in \bigcap_{i=1}^{N} F_{ix}(T_{\omega_i}), \quad \langle(\gamma f - A)x^*, \widehat{x} - x^*\rangle \le 0, \quad \forall \widehat{x} \in \bigcap_{i=1}^{N} F_{ix}(T_{\omega_i}). \tag{4.8}$$

Proof. Take a $p \in F := \bigcap_{i=1}^{N} F_{ix}(T_{\omega_i})$. We break the proof process into several steps.
Step 1. $\{x_n\}$ is bounded. In light of the Remark 2.3, we obtain

$$\|T_{[n]}(x_n - p)\| = \|T_{\omega_i}(x_n - p)\| \le \|x_n - p\|. \tag{4.9}$$

From (4.4), we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n(\gamma f(x_n) - Ap) + \beta_n(x_n - p) + ((I-\beta_n)I - \alpha_n A)T_{[n]}(x_n - p)\| \\
&\le \alpha_n\|\gamma f(x_n) - Ap\| + \beta_n\|x_n - p\| + (1-\beta_n-\alpha_n\overline{\gamma})\|x_n - p\| \\
&\le \alpha_n\gamma\|f(x_n) - f(p)\| + \alpha_n\|\gamma f(p) - Ap\| + (1-\alpha_n\overline{\gamma})\|x_n - p\| \\
&= [1-\alpha_n(\overline{\gamma}-\gamma\beta)]\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\|.
\end{aligned} \tag{4.10}$$

By simple inductions, we obtain

$$\|x_n - p\| \leq \max\left\{ \|x_1 - p\|, \frac{\|\gamma f(p) - Ap\|}{\overline{\gamma} - \gamma\beta} \right\}, \quad \forall n \geq 1, \tag{4.11}$$

which gives that the sequence $\{x_n\}$ is bounded; we also know that $\{T_{[n]}x_n\}$ and $\{f(x_n)\}$ are bounded.

Step 2. Moreover if $T_{[n]} : H \to H$ is quasi-shrinking on the set C, we obtain the following statements:

(a) $\lim_{n\to\infty} d(x_n, F) = 0$;

(b) $\lim_{n\to\infty} \|T_{[n]}x_n - x_n\| = 0$;

(c) $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

By the boundedness of $\{x_n\}$, $\{T_{[n]}x_n\}$, and $\{f(x_n)\}$, there exists $M > 0$ satisfying

$$\max_{n\geq 0}\{\|x_n\|, \|T_{[n]}x_n\|, \|f(x_n)\|\} \leq M. \tag{4.12}$$

By a simple inspection, we deduce

$$\begin{aligned} d(x_{n+1}, F) &\leq \|x_{n+1} - P_F(T_{[n]}x_n)\| \\ &= \|\alpha_n\gamma f(x_n) + \beta_n x_n + ((I - \beta_n)I - \alpha_n A)T_{[n]}x_n - P_F(T_{[n]}x_n)\| \\ &\leq \alpha_n\|\gamma f(x_n) - AT_{[n]}x_n\| + \beta_n\|x_n - T_{[n]}x_n\| + \|T_{[n]}x_n - P_F(T_{[n]}x_n)\| \\ &\leq \alpha_n(\gamma\|f(x_n)\| + \|AT_{[n]}x_n\|) + \beta_n(\|x_n\| + \|T_{[n]}x_n\|) + \|T_{[n]}x_n - P_F(T_{[n]}x_n)\| \\ &\leq d(T_{[n]}x_n, F) + 2(\alpha_n + \beta_n)M. \end{aligned} \tag{4.13}$$

By $\{x_n\}_{n\geq 0} \subset C$, we can assume the boundedness of the sequence $b_n := d(x_n, F) \geq 0$ $(n \in \mathbb{N})$. Moreover, by Definition 4.1 and (4.13), it follows that

$$\begin{aligned} D(b_n) &\leq b_n - d(T_{[n]}x_n, F) \\ &\leq b_n - b_{n+1} + 2(\alpha_n + \beta_n)M, \quad \forall n \geq 0. \end{aligned} \tag{4.14}$$

Now application of Lemma 4.2 to (4.14) yields $\lim_{n\to\infty} b_n = 0$, hence (a) is proved.

The statements (b) and (c) are verified by

$$\begin{aligned} \|T_{[n]}x_n - x_n\| &= \|T_{[n]}x_n - P_F(x_n) + P_F(x_n) - x_n\| \\ &\leq \|T_{[n]}x_n - P_F(x_n)\| + \|P_F(x_n) - x_n\| \\ &\leq \|x_n - P_F(x_n)\| + \|P_F(x_n) - x_n\| \\ &= 2d(x_n, F) \longrightarrow 0, \quad (n \longrightarrow \infty), \end{aligned} \tag{4.15}$$

$$\begin{aligned}\|x_{n+1}-x_n\| &= \left\|\alpha_n\gamma f(x_n)+\beta_n x_n+\left((I-\beta_n)I-\alpha_n A\right)T_{[n]}x_n-x_n\right\| \\ &\le \alpha_n\left\|\gamma f(x_n)-Ax_n\right\|+\left(1-\beta_n-\alpha_n\overline{\gamma}\right)\left\|T_{[n]}x_n-x_n\right\| \\ &\to 0, \quad (n\longrightarrow\infty).\end{aligned} \tag{4.16}$$

Step 3. $\lim_{n\to\infty}\|x_{n+N}-x_n\|=0$. From (4.4) and (4.16), we obtain

$$\begin{aligned}\|x_{n+N+1}-x_{n+1}\| &= \big\|\alpha_{n+N}\gamma f(x_{n+N})+\beta_{n+N}x_{n+N}+\left((I-\beta_{n+N})I-\alpha_{n+N}A\right)T_{[n+N]}x_{n+N} \\ &\quad -\alpha_n\gamma f(x_n)-\beta_n x_n-\left((I-\beta_n)I-\alpha_n A\right)T_{[n]}x_n\big\| \\ &= \big\|\alpha_{n+N}\gamma f(x_{n+N})-\alpha_{n+N}\gamma f(x_n)+\beta_{n+N}\left(x_{n+N}-T_{[n+N]}x_{n+N}\right) \\ &\quad +(I-\alpha_{n+N}A)T_{[n+N]}x_{n+N}-(I-\alpha_{n+N}A)T_{[n]}x_n \\ &\quad +(I-\alpha_{n+N}A)T_{[n]}x_n-(I-\alpha_n A)T_{[n]}x_n \\ &\quad +\alpha_{n+N}\gamma f(x_n)-\alpha_n\gamma f(x_n)+\beta_n\left(x_n-T_{[n]}x_n\right)\big\| \\ &\le \alpha_{n+N}\gamma\beta\|x_{n+N}-x_n\|+\beta_{n+N}\left\|x_{n+N}-T_{[n+N]}x_{n+N}\right\| \\ &\quad +\left(1-\alpha_{n+N}\overline{\gamma}\right)\left\|T_{[n]}x_n-T_{[n+N]}x_{n+N}\right\|+|\alpha_n+\alpha_{n+N}|\overline{\gamma}\left\|T_{[n]}x_n\right\| \\ &\quad +|\alpha_n-\alpha_{n+N}|\gamma\beta\|x_n\|-\beta_n\left\|x_n-T_{[n]}x_n\right\| \\ &= \alpha_{n+N}\gamma\beta\|x_{n+N}-x_n\|+\beta_{n+N}\left\|x_{n+N}-T_{[n+N]}x_{n+N}\right\| \\ &\quad +\left(1-\alpha_{n+N}\overline{\gamma}\right)\sum_{j=n}^{N-1}\left\|T_{[j]}x_j-T_{[j+1]}x_{j+1}\right\|+|\alpha_n-\alpha_{n+N}|\overline{\gamma}\left\|T_{[n]}x_n\right\| \\ &\quad +|\alpha_n-\alpha_{n+N}|\gamma\beta\|x_n\|-\beta_n\left\|x_n-T_{[n]}x_n\right\| \\ &\le \alpha_{n+N}\gamma\beta\|x_{n+N}-x_n\|+\beta_{n+N}\left\|x_{n+N}-T_{[n+N]}x_{n+N}\right\| \\ &\quad +\left(1-\alpha_{n+N}\overline{\gamma}\right)\sum_{j=n}^{N-1}\left(\left\|x_{j+1}-T_{[j+1]}x_{j+1}\right\|+\left\|x_j-x_{j+1}\right\|+\left\|x_j-T_{[j]}x_j\right\|\right) \\ &\quad +|\alpha_n-\alpha_{n+N}|\overline{\gamma}\left\|T_{[n]}x_n\right\|+|\alpha_n-\alpha_{n+N}|\gamma\beta\|x_n\|+\beta_n\left\|x_n-T_{[n]}x_n\right\|.\end{aligned} \tag{4.17}$$

By conditions (4.1a), (4.1b), (4.1c), (4.15), and (4.16), $\{x_n\}$ and $\{T_{[n]}x_n\}$ are bounded we obtain that

$$\lim_{n\to\infty}\|x_{n+N}-x_n\|=0. \tag{4.18}$$

Step 4. $\lim_{n\to\infty}\|x_n-T_{[n+N]}\cdots T_{[n+1]}x_n\|=0$.

From (4.4), we observe that

$$\left\|x_{n+1}-T_{[n]}x_n\right\|=\alpha_n\gamma\left\|f(x_n)-AT_{[n]}x_n\right\|+\beta_n\left\|x_n-T_{[n]}x_n\right\|. \tag{4.19}$$

It follows from the condition (4.1a), (4.1c), (4.15), and the boundedness of $\{f(x_n)\}$ and $\{T_{[n]}x_n\}$ that

$$\left\|x_{n+1} - T_{[n]}x_n\right\| \longrightarrow 0, \quad (n \longrightarrow \infty). \tag{4.20}$$

Recursively,

$$\begin{aligned} \left\|x_{n+N} - T_{[n+N]}x_{n+N-1}\right\| \longrightarrow 0, \quad (n \longrightarrow \infty), \\ \left\|x_{n+N-1} - T_{[n+N-1]}x_{n+N-2}\right\| \longrightarrow 0, \quad (n \longrightarrow \infty). \end{aligned} \tag{4.21}$$

By Remark 2.3, $T_{[n+N]}$ is quasi-nonexpansive, we obtain

$$\left\|T_{[n+N]}x_{n+N-1} - T_{[n+N]}T_{[n+N-1]}x_{n+N-2}\right\| \longrightarrow 0, \quad (n \longrightarrow \infty). \tag{4.22}$$

Proceeded accordingly, we obtain

$$\begin{gathered} \left\|T_{[n+N]}T_{[n+N-1]}x_{n+N-2} - T_{[n+N]}T_{[n+N-1]}T_{[n+N-2]}x_{n+N-3}\right\| \longrightarrow 0, \quad (n \longrightarrow \infty), \\ \vdots \\ \left\|T_{[n+N]}\cdots T_{[n+2]}x_{n+1} - T_{[n+N]}\cdots T_{[n+1]}x_n\right\| \longrightarrow 0, \quad (n \longrightarrow \infty). \end{gathered} \tag{4.23}$$

Note that

$$\begin{aligned} \left\|x_{[n+N]} - T_{[n+N]}\cdots T_{[n+1]}x_n\right\| \leq {} & \left\|x_{n+N} - T_{[n+N]}x_{n+N-1}\right\| \\ & + \left\|T_{[n+N]}x_{n+N-1} - T_{[n+N]}T_{[n+N-1]}x_{n+N-2}\right\| \\ & \vdots \\ & + \left\|T_{[n+N]}\cdots T_{[n+2]}x_{n+1} - T_{[n+N]}\cdots T_{[n+1]}x_n\right\|. \end{aligned} \tag{4.24}$$

From all the expressions above, we have

$$\left\|x_{[n+N]} - T_{[n+N]}\cdots T_{[n]}x_n\right\| \longrightarrow 0, \quad (n \longrightarrow \infty). \tag{4.25}$$

Since

$$\left\|x_n - T_{[n+N]}\cdots T_{[n+1]}x_n\right\| \leq \left\|x_n + x_{n+N}\right\| + \left\|x_{n+N} - T_{[n+N]}\cdots T_{[n]}x_n\right\|, \tag{4.26}$$

it is concluded that

$$\lim_{n\to\infty}\left\|x_n - T_{[n+N]}\cdots T_{[n+1]}x_n\right\| = 0. \tag{4.27}$$

Step 5. $\omega_w(x_n) \subset \bigcap_{n=1}^N F_{ix}(T_{[n]}) = \bigcap_{n=1}^N F_{ix}(T_{\omega_i}), i = 1, 2, \ldots, N$. Assume that $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \rightharpoonup \widehat{x}$, we prove $\widehat{x} \in \bigcap_{n=1}^N F_{ix}(T_{[n]})$. By the conclusion of step 4, we obtain

$$\lim_{j\to\infty} \left\| x_{n_j} - T_{[n_j+N]} \cdots T_{[n_j+1]} x_{n_j} \right\| = 0. \tag{4.28}$$

Observe that, for each n_j, $T_{[n_j+N]}, \ldots, T_{[n_j+1]}$ is some permutation of the mappings $T_{[1]}, \ldots, T_{[N]}$, since $T_{[1]}, \ldots, T_{[N]}$ are finite, all the full permutation are $N!$, there must be some permutation that appears infinite times. Without loss of generality, suppose that this permutation is $T_{[1]}, \ldots, T_{[N]}$, we can take a subsequence $\{x_{n_{j_k}}\} \subset \{x_{n_j}\}$ such that

$$\lim_{j\to\infty} \left\| x_{n_j} - T_{[1]} \cdots T_{[N]} x_{n_j} \right\| = 0. \tag{4.29}$$

It is easy to prove that $T_{[1]}, \ldots, T_{[N]}$ is quasi-nonexpansive. By Lemma 2.5, we have

$$\widehat{x} = T_{[1]} \cdots T_{[N]} \widehat{x}. \tag{4.30}$$

Using Remark 2.3 and Lemma 2.5, we obtain

$$\widehat{x} \in F_{ix}\left(T_{[1]} \cdots T_{[N]}\right) = \bigcap_{n=1}^N F_{ix}\left(T_{[n]}\right) = \bigcap_{n=1}^N F_{ix}(T_n). \tag{4.31}$$

Step 6. $\liminf_{n\to\infty} \langle (\gamma f - A)x^*, x_n - x^* \rangle \le 0$. Indeed, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that

$$\liminf_{n\to\infty} \langle (\gamma f - A)x^*, x_n - x^* \rangle = \lim_{j\to\infty} \left\langle (\gamma f - A)x^*, x_{n_j} - x^* \right\rangle. \tag{4.32}$$

Without loss of generality, we may further assume that $x_{n_j} \rightharpoonup \widehat{x}$. It follows from (4.31) that $\widehat{x} \in \bigcap_{n=1}^N F(T_n)$. Since x^* is the unique solution of (4.8), we have

$$\begin{aligned} \liminf_{n\to\infty} \langle (\gamma f - A)x^*, x_n - x^* \rangle &= \lim_{j\to\infty} \left\langle (\gamma f - A)x^*, x_{n_j} - x^* \right\rangle \\ &= \langle (\gamma f - A)x^*, \widehat{x} - x^* \rangle \le 0. \end{aligned} \tag{4.33}$$

In addition, the variational inequality (4.33) can be written as

$$\langle (I - A + \gamma f)x^* - x^*, \widehat{x} - x^* \rangle \le 0, \quad \widehat{x} \in \bigcap_{n=1}^N F_{ix}(T_{\omega_i}). \tag{4.34}$$

So, by the Lemma 2.10, it is equivalent to the fixed point equation

$$x^* = P_{\bigcap_{n=1}^N F_{ix}(T_{\omega_i})} (I - A + \gamma f)x^* = \left(P_{\bigcap_{n=1}^N F_{ix}(T_{\omega_i})} \cdot f\right) x^*. \tag{4.35}$$

Step 7. $\lim_{n\to\infty}\|x_n - x^*\| = 0$. From (4.4), we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \left\|\alpha_n\gamma f(x_n) + \beta_n x_n + ((I-\beta_n)I - \alpha_n A)T_{[n]}x_n - x^*\right\|^2 \\
&= \left\langle \alpha_n(\gamma f(x_n) - Ax^*) + \beta_n(x_n - x^*) + ((I-\beta_n)I - \alpha_n A)(T_{[n]}x_n - x^*), x_{n+1} - x^* \right\rangle \\
&\le \alpha_n\langle \gamma f(x_n) - Ax^*, x_{n+1} - x^*\rangle + \beta_n\langle x_n - x^*, x_{n+1} - x^*\rangle \\
&\quad + (1 - \beta_n - \alpha_n)\langle T_{[n]}x_n - x^*, x_{n+1} - x^*\rangle + \alpha_n\langle (I-A)(T_{[n]}x_n - x^*), x_{n+1} - x^*\rangle \\
&\le \alpha_n\big(\langle \gamma f(x_n) - f(x^*), x_{n+1} - x^*\rangle + \langle \gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle \\
&\qquad + \langle (I-A)(T_{[n]}x_n - x^*), x_{n+1} - x^*\rangle\big) + \beta_n\|x_n - x^*\|\|x_{n+1} - x^*\| \\
&\quad + (1 - \beta_n - \alpha_n\overline{\gamma})\left\|T_{[n]}x_n - x^*\right\|\|x_{n+1} - x^*\| \\
&\le \alpha_n\gamma\beta\|x_n - x^*\|\|x_{n+1} - x^*\| + \alpha_n\langle \gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle \\
&\quad \alpha_n(1 - \overline{\gamma})\left\|T_{[n]}x_n - x^*\right\|\|x_{n+1} - x^*\| + \beta_n\|x_n - x^*\|\|x_{n+1} - x^*\| \\
&\quad + (1 - \beta_n - \alpha_n\overline{\gamma})\|x_n - x^*\|\|x_{n+1} - x^*\| \\
&\le \frac{1 - (\overline{\gamma} - \gamma\beta)}{2}\left(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2\right) + \alpha_n M \\
&\le \frac{1 - (\overline{\gamma} - \gamma\beta)}{2}\|x_n - x^*\|^2 + \frac{1}{2}\|x_{n+1} - x^*\|^2 + \alpha_n M,
\end{aligned} \tag{4.36}$$

where $M = (1-\overline{\gamma})\|T_{[n]}x_n - x^*\|\|x_{n+1} - x^*\| + \langle \gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle$. It follows that

$$\|x_{n+1} - x^*\|^2 \le (1 - (\overline{\gamma} - \gamma\beta))\|x_n - x^*\|^2 + 2\alpha_n M. \tag{4.37}$$

By using Lemma 4.3, we can obtain the desired conclusion easily. □

5. Application

In this section, we constructed a numerical example to compare the parallel algorithm and cyclic algorithm which is simple.

Let $x = (x_1, x_2) \in R^2$ and $f(x) = (1/2)(\sin(x_1), \cos(x_2))$ be a contraction mapping with coefficient $1/2$. Let $T_1(x) = (0, 4x_1)$ and $T_2(x) = (4x_2, 0)$ be quasi-nonexpansive mappings. Let $\alpha_n = \beta_n = 1/3$, $A = I$ and $\gamma = \lambda_1 = \lambda_2 = 1/2$. According to (1.20) and (1.24), we can obtain the following parallel algorithm and cyclic algorithm:

Parallel Algorithm

$$x_{n+1} = \frac{1}{6}f(x_n) + \frac{1}{2}x_n + \frac{1}{12}(T_1 + T_2)x_n. \tag{5.1}$$

Cyclic Algorithm

$$\begin{aligned}
x_1 &= \frac{1}{6}f(x_0) + \frac{1}{2}x_0 + \frac{1}{6}(T_1)x_0;\\
x_2 &= \frac{1}{6}f(x_1) + \frac{1}{2}x_1 + \frac{1}{6}(T_2)x_1;\\
x_3 &= \frac{1}{6}f(x_2) + \frac{1}{2}x_2 + \frac{1}{6}(T_1)x_2;\\
&\vdots\\
x_{n-1} &= \frac{1}{6}f(x_{n-2}) + \frac{1}{2}x_{n-2} + \frac{1}{6}(T_2)x_{n-2};\\
x_n &= \frac{1}{6}f(x_{n-1}) + \frac{1}{2}x_{n-1} + \frac{1}{6}(T_1)x_{n-1}.
\end{aligned} \tag{5.2}$$

Form Theorems 3.3 and 4.4, we can easily know that parallel algorithm (5.1) and cyclic algorithm (5.2) are converge to the unique point in R^2. Let $x_0 = (5, 2)$ and $|x_{n+1} - x_n|^2 \leq 10^{-9}$, and let x_P^* and x_X^* be the fixed point of the parallel algorithm and cyclic algorithm. Using the software of MATLAB, we obtain $x_P^* = x_{133} = (0.6821, 0.7080)$ and $x_X^* = x_{166} = (1.9325, 0.8729)$. From the computed results of x_P^* and x_X^*, we can easily know that parallel algorithm (5.1) is simpler than cyclic algorithm (5.2). On the other hand, we need to explain that those algorithms do not converge a common fixed point, because parallel algorithm (5.1) and cyclic algorithm (5.2) have the different algorithm structure.

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