

Research Article

Modified Hybrid Block Iterative Algorithm for Uniformly Quasi- ϕ -Asymptotically Nonexpansive Mappings

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Saewan and Kumam (2010) have proved the convergence theorems for finding the set of solutions of a general equilibrium problems and the common fixed point set of a family of closed and uniformly quasi- ϕ -asymptotically nonexpansive mappings in a uniformly smooth and strictly convex Banach space E with Kadec-Klee property. In this paper, authors prove the convergence theorems and do not need the Kadec-Klee property of Banach space and some other conditions used in the paper of S. Saewan and P. Kumam. Therefore, the results presented in this paper improve and extend some recent results.

1. Introduction

Let C be a nonempty closed convex subspace of a real Banach space E . A mapping $A : D(A) \subset E \rightarrow E^*$ is said to be monotone if for each $x, y \in D(A)$, the following inequality holds:

$$\langle x - y, Ax - Ay \rangle \geq 0. \quad (1.1)$$

A mapping $A : C \rightarrow E^*$ is called α -inverse-strongly monotone if there exists $\alpha > 0$ such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2. \quad (1.2)$$

A monotone mapping T is said to be maximal monotone if $R(J + rT) = E^*$, for all $r > 0$, where J is the normalized duality mapping. We denote by $T^{-1}(0) = \{x \in E : 0 \in Tx\}$ the set of zero points of T .

Remark 1.1. It is well known that if $A : C \rightarrow E^*$ is an α -inverse-strongly monotone mapping, then it is $(1/\alpha)$ -Lipschitzian, and hence uniformly continuous. Clearly, the class of monotone mappings includes the class of α -inverse strongly monotone mappings.

Let C be a nonempty closed convex subspace of a real Banach space E with dual E^* and $\langle \cdot, \cdot \rangle$ is the pairing between E and E^* . Let $f : C \times C \rightarrow R$ be a bifunction and $A : C \rightarrow E^*$ be a monotone mapping. The generalized equilibrium problem means that finding a $u \in C$ such that

$$f(u, y) + \langle Au, y - u \rangle \geq 0, \quad \forall y \in C. \quad (1.3)$$

The set of solutions of (1.3) is denoted by $\text{GEP}(f, A)$, that is,

$$\text{GEP}(f, A) = \{u \in C : f(u, y) + \langle Au, y - u \rangle \geq 0, \forall y \in C\}. \quad (1.4)$$

If $A = 0$, then the problem (1.3) is equivalent to that of finding a $u \in C$ such that

$$f(u, y) \geq 0, \quad \forall y \in C, \quad (1.5)$$

which is called the equilibrium problem. The solution of (1.5) is denoted by $\text{EP}(f)$. If $f = 0$, then the problem (1.3) is equivalent to that of finding a $u \in C$ such that

$$\langle Au, y - u \rangle \geq 0, \quad \forall y \in C, \quad (1.6)$$

which is called the variational inequality of Browder type. The solution of (1.6) is denoted by $\text{VI}(C, A)$.

The problem (1.3) was shown in [1] to cover variational inequality problems, monotone inclusion problems, vector equilibrium problems, numerous problems in physics, minimization problems, saddle point problems, and Nash equilibria in noncooperative games. In addition, there are several other problems, for example, fixed point problem, the complementarity problem, and optimization problem, which can also be written in the form of an $\text{EP}(f)$. In other words, the $\text{EP}(f)$ is a unifying model for several problems arising in physics, engineering, science, optimization, economics, and so on. In the past two decades, Some methods have been modified for solving the generalized equilibrium problem and the equilibrium problem in Hilbert space and Banach space, see [2–9].

The convex feasibility problem (CFP) is the problem for computing points that lay in the intersection of a finite family of closed convex subsets C_j , $j = 1, 2, \dots, N$, of a Banach space E . This problem appears in many fields of applied mathematics, such as the theory of optimization [1], Image Reconstruction from projections [10], and Game Theory [11] and plays an important role in these domains. There is a considerable investigation of (CFP) in the framework of Hilbert spaces which captures applications in various disciplines such as image restoration, computer tomograph, and radiation therapy treatment planning [12]. Also the projection methods have dominated in the iterative approaches to (CFP) in Hilbert spaces. In 1993, Kitahara and Takahashi [13] deal with the convex feasibility problem by convex and sunny nonexpansive retractions in a uniformly convex Banach space.

We note that the block iterative method is a common method by many authors to solve (CFP) [14]. In 2008, Plubtieng and Ungchittrakool [15] established block iterative methods

for a finite family of relatively nonexpansive mappings and got some strong convergence theorems in a Banach space by using the hybrid method.

In 2009, Takahashi and Zembayashi [16] introduced the following iterative scheme in the case that E is uniformly smooth and uniformly convex Banach space:

$$\begin{aligned}
 x_0 &= x \in C, \text{ arbitrarily,} \\
 y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)Jx_n); \\
 u_n \in C \quad \text{s.t.} \quad & f(u_n, y) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle, \quad \forall y \in C \\
 C_{n+1} &= \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}; \\
 x_{n+1} &= \Pi_{C_{n+1}} x, \quad n = 1, 2, \dots,
 \end{aligned} \tag{1.7}$$

where T is a relatively nonexpansive mapping and f is a bifunction from $C \times C$ into R . They prove that the sequence $\{x_n\}$ converges strongly to $q = \Pi_{F(T) \cap EP(f)}$ under appropriate conditions.

In the same year, Qin et al. [7] introduced a hybrid projection algorithm to two quasi- ϕ -nonexpansive mappings in Banach spaces as follows:

$$\begin{aligned}
 x_0 &= x \in C, \text{ arbitrarily,} \\
 C_1 &= C; \\
 x_1 &= \Pi_{C_1} x_0; \\
 y_n &= J^{-1}(\alpha_n Jx_n + \beta_n JT x_n + \gamma_n JS x_n); \\
 u_n \in C \quad \text{s.t.} \quad & f(u_n, y) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \quad \forall y \in C; \\
 C_{n+1} &= \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}; \\
 x_{n+1} &= \Pi_{C_{n+1}} x, \quad n = 1, 2, \dots,
 \end{aligned} \tag{1.8}$$

where $\Pi_{C_{n+1}}$ is the generalized projection from E onto C_{n+1} . They proved that the sequence $\{x_n\}$ converges strongly to $\Pi_{F(S) \cap F(T) \cap EP(f)} x_0$. Then Petrot et al. [17] improved the notion from a relatively nonexpansive mapping or a quasi- ϕ -nonexpansive mapping to two relatively quasi-nonexpansive mappings; they also proved some strong convergence theorems to find a common element of the set of fixed point of relatively quasi-nonexpansive mappings and the set of solutions of an equilibrium problem in the framework of Banach spaces.

In 2010, Saewan and Kumam [18] introduced the following iterative method to find a common element of the set of solutions of an equilibrium problem and the set of common fixed points of an infinite family of closed and uniformly quasi- ϕ -asymptotically

nonexpansive mappings in a uniformly smooth and strictly convex Banach space with Kadec-Klee property.

$$\begin{aligned}
 x_0 &= x \in E, \text{ arbitrarily,} \\
 C_1 &= C; \\
 x_1 &= \Pi_{C_1} x_0; \\
 y_n &= J^{-1}(\beta_n Jx_n + (1 - \beta_n) Jz_n); \\
 z_n &= J^{-1}\left(\alpha_{n,0} Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i} JS_i^n x_n\right); \\
 u_n &\in C \quad \text{s.t.} \quad f(u_n, y) + \langle Ay_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq 0; \\
 C_{n+1} &= \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + \theta_n\}; \\
 x_{n+1} &= \Pi_{C_{n+1}} x, \quad n = 1, 2, \dots,
 \end{aligned} \tag{1.9}$$

They proved that $\{x_n\}$ converges strongly to $\Pi_{\bigcap_{i=1}^{\infty} F(S_i) \cap \text{GEP}(f, A)}$ under the proper conditions. The same year, Chang et al. [19] proposed the modified block iterative algorithm for solving the convex feasibility problems for an infinite family of closed and uniformly quasi- ϕ -asymptotically nonexpansive mapping; they obtain the strong convergence theorems in a Banach space.

Motivated by Saewan and Kumam [18], in this paper we use some new conditions to prove strong convergence theorems for modified block hybrid projection algorithm for finding a common element of the set of solutions of the generalized equilibrium problems and the set of common fixed points of an infinite family of closed and uniformly quasi- ϕ -asymptotically nonexpansive mappings which is more general than closed quasi- ϕ -nonexpansive mappings in a uniformly smooth and strictly convex Banach space E . In (1.9) we find iterative step y_n is not essential, so we combine y_n with z_n of (1.9), and use an equally continuous mapping that is more weak than uniformly L -Lipschitz mapping in a uniformly smooth and strictly convex Banach space E , but the Banach space E does not have Kadec-Klee property, under the circumstances we prove strong convergence theorems and get some results same as the results of Saewan and Kumam [18]. The results presented in this paper improve some well-known results in the literature.

2. Preliminaries

The space E is said to be smooth if the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for all $x, y \in U = \{z \in E : \|z\| = 1\}$, and E is said to be uniformly smooth if the limit (2.1) exists uniformly for all $x, y \in U$. Then a Banach space E is said to be strictly convex if

$\|x + y\|/2 \leq 1$ for all $x, y \in U$ and $x \neq y$. It is said to be uniformly convex if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that $\|x + y\|/2 \leq 1 - \delta$ for all $x, y \in U$ with $\|x - y\| \geq \varepsilon$.

Let E be a Banach space and let E^* be the topological dual of E . For all $x \in E$ and $x^* \in E^*$, we denote the value of x^* at x by $\langle x, x^* \rangle$. Then, the duality mapping $J : E \rightarrow 2^{E^*}$ is defined by

$$J(x) = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \right\}, \quad (2.2)$$

for every $x \in E$. By the Hahn-Banach theorem, $J(x)$ is nonempty.

The following basic properties can be found in Cioranescu [20].

- (i) If E is a uniformly smooth Banach space, then J is uniformly continuous on each bounded subset of E .
- (ii) If E is a reflexive and strictly convex Banach space, then J^{-1} is norm-weak *-continuous.
- (iii) If E is a smooth, strictly convex, and reflexive Banach space, then the normalized duality mapping $J : E \rightarrow 2^{E^*}$ is single-valued, one-to-one, and onto.
- (iv) A Banach space E is uniformly smooth if and only if E^* is uniformly convex.

Let E be a smooth, strictly convex, and reflexive Banach space and let C be a nonempty closed convex subset of E . Throughout this paper, we denote by ϕ the function defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E. \quad (2.3)$$

Following Alber [21], the generalized projection Π_C from E onto C is defined by

$$\Pi_C(x) = \arg \min_{u \in C} \phi(u, x), \quad \forall x \in E. \quad (2.4)$$

If E is a Hilbert space, then $\phi(x, y) = \|x - y\|^2$ and Π_C is the metric projection of H onto C . We know the following lemmas for generalized projections.

Lemma 2.1 (see Alber [21] and Kamimura and Takahashi [22]). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E . Then*

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y), \quad \forall x \in C, y \in E. \quad (2.5)$$

Lemma 2.2 (see Alber [21], Kamimura and Takahashi [22]). *Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space and let $x \in E$ and $z \in C$. Then*

$$z = \Pi_C x \iff \langle y - z, Jx - Jz \rangle \leq 0, \quad \forall y \in C. \quad (2.6)$$

Lemma 2.3 (see Kamimura and Takahashi [22]). *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n \rightarrow \infty} \phi(x_n, y_n) = 0$, then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.*

Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E and let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T , that is $F(T) = \{x : Tx = x\}$. A point $p \in C$ is said to be an asymptotic fixed point of T if there exists $\{x_n\}$ in C which converges weakly to p and $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$. We denote the set of all asymptotic fixed points of T by $\hat{F}(T)$.

A mapping T from C into itself is said to be relatively nonexpansive [23] if the following conditions are satisfied:

- (1) $F(T)$ is nonempty,
- (2) $\phi(u, Tx) \leq \phi(u, x)$, for all $u \in F(T)$, $x \in E$,
- (3) $F(T) = \hat{F}(T)$.

A mapping T from C into itself is said to be relatively quasi-nonexpansive if the following conditions are satisfied:

- (1) $F(T)$ is nonempty,
- (2) $\phi(u, Tx) \leq \phi(u, x)$, for all $u \in F(T)$, $x \in E$,

The asymptotic behavior of a relatively nonexpansive mapping was studied in [24]. T is said to be ϕ -nonexpansive, if $\phi(Tx, Ty) \leq \phi(x, y)$ for $x, y \in C$. T is said to be quasi- ϕ -asymptotically nonexpansive if the following conditions are satisfied:

- (1) $F(T)$ is nonempty,
- (2) $\phi(u, T^n x) \leq k_n \phi(u, x)$, for all $u \in F(T)$, $x \in E$ and $n \geq 1$,

where $\{k_n\}$ is a real sequence within $[1, \infty)$ and $k_n \rightarrow 1$ as $n \rightarrow \infty$.

A mapping T is said to be closed if for any sequence $\{x_n\} \subset C$ with $x_n \rightarrow x$ and $Tx_n \rightarrow y$, then $Tx = y$. It is easy to know that each relatively nonexpansive mapping is closed. The class of quasi- ϕ -asymptotically nonexpansive mappings contains properly the class of quasi- ϕ -nonexpansive mappings as a subclass and the class of quasi- ϕ -nonexpansive mappings contains properly the class of relatively nonexpansive mappings as a subclass, but the converse may be not true (see more details [24, 25]).

By using the similar method as in Su et al. [26], the following Lemma is not hard to prove.

Lemma 2.4. *Let E be a strictly convex and uniformly smooth real Banach space, let C be a closed convex subset of E , and let T be a closed and quasi- ϕ -asymptotically nonexpansive mapping from C into itself with a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \rightarrow 1$ as $n \rightarrow \infty$. Then $F(T)$ is a closed and convex subset of C .*

For solving the equilibrium problem, let us assume that a bifunction f satisfies the following conditions:

- (A1) $f(x, x) = 0$, for all $x \in E$,
- (A2) f is monotone, that is, $f(x, y) + f(y, x) \leq 0$, for all $x, y \in E$,
- (A3) for all $x, y, z \in E$, $\limsup_{t \downarrow 0} f(tz + (1-t)x, y) \leq f(x, y)$,
- (A4) for all $x \in C$, $y \mapsto f(x, y)$ is convex and lower semicontinuous.

Lemma 2.5 (see Blum and Oettli [1]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E , let f be a bifunction from $C \times C$ to $\mathbb{R} = (-\infty, +\infty)$ satisfying (A1)–(A4), and let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C. \quad (2.7)$$

Lemma 2.6 (see Kumam [5]). *Let C be a closed convex subset of a uniformly smooth, strictly convex, and reflexive Banach space E . Let f be a bifunction from $C \times C$ to $\mathbb{R} = (-\infty, +\infty)$ satisfying (A1)–(A4) and let A be a monotone mapping from C into E^* . For $r > 0$, define a mapping $T_r : C \rightarrow C$ as follows:*

$$T_r(x) = \left\{ z \in C : f(z, y) + \langle Ax, y - z \rangle + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C \right\}, \quad (2.8)$$

for all $x \in E$. Then, the following hold:

- (1) T_r is single-valued,
- (2) T_r is a firmly nonexpansive-type mapping [6], that is, for all $x, y \in E$,

$$\langle T_r x - T_r y, JT_r x - JT_r y \rangle \leq \langle T_r x - T_r y, Jx - Jy \rangle, \quad (2.9)$$

- (3) $F(T_r) = \text{GEP}(f, A)$,
- (4) $\text{GEP}(f, A)$ is closed and convex.

Lemma 2.7 (see Kumam [5]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E . Let f be a bifunction from $C \times C$ to $\mathbb{R} = (-\infty, +\infty)$ satisfying (A1)–(A4) and let A be a monotone mapping from C into E^* . For $x \in E$, $q \in F(T_r)$, then the following holds:*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x). \quad (2.10)$$

Lemma 2.8 (see Chang et al. [19]). *Let E be a uniformly convex Banach space, $r > 0$ a positive number, and $B_r(0)$ a closed ball of E . Then, for any given sequence $\{x_i\}_{i=1}^\infty \subset B_r(0)$ and for any given sequence $\{\lambda_i\}_{i=1}^\infty$ of positive number with $\sum_{n=1}^\infty \lambda_n = 1$, there exists a continuous, strictly increasing, and convex function $g : [0, 2r) \rightarrow [0, \infty)$ with $g(0) = 0$ such that, for any positive integer i, j with $i \neq j$,*

$$\left\| \sum_{n=1}^\infty \lambda_n x_n \right\|^2 \leq \sum_{n=1}^\infty \lambda_n \|x_n\|^2 - \lambda_i \lambda_j g(\|x_i - x_j\|). \quad (2.11)$$

Definition 2.9. A mapping S from C into itself is said to be equally continuous if it follows that

$$\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0 \implies \lim_{n \rightarrow \infty} \|S^n x_n - S^n y_n\| = 0, \quad \forall x_n, y_n \in C. \quad (2.12)$$

A mapping S from C into itself is said to be uniformly L -Lipschitz continuous if there exists a constant $L > 0$ such that

$$\|S^n x - S^n y\| \leq L\|x - y\|, \quad \forall x, y \in C. \quad (2.13)$$

It is easy to know that each L -Lipschitz continuous mapping is equally continuous, but the converse may be not true.

Definition 2.10. Let $\{S_i\}_{i=1}^\infty : C \rightarrow C$ be a sequence of mapping. $\{S_i\}_{i=1}^\infty$ is said to be a family of uniformly quasi- ϕ -asymptotically nonexpansive mappings, if $\bigcap_{i=1}^\infty F(S_i) \neq \emptyset$, and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \rightarrow 1$ such that for each $i \geq 1$,

$$\phi(p, S_i^n x) \leq k_n \phi(p, x), \quad \forall p \in \bigcap_{i=1}^\infty F(S_i), \quad x \in C, \quad \forall n \geq 1. \quad (2.14)$$

3. Main Results

Theorem 3.1. Let E be a uniformly smooth and uniformly convex Banach space, and let C be a nonempty closed convex subset of E . Let f be a bifunction from $C \times C$ to $\mathbb{R} = (-\infty, +\infty)$ satisfying (A1)–(A4) and let A be a continuous monotone mapping of C into E^* . Let $\{S_i\}_{i=1}^\infty : C \rightarrow C$ be an infinite family of closed equally continuous and uniformly quasi- ϕ -asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1, \infty)$, $k_n \rightarrow 1$ such that $F := \bigcap_{i=1}^\infty F(S_i) \cap \text{GEP}(f, A)$ is a nonempty and bounded subset in C . Let $\{x_n\}$ be a sequence generated by

$$\begin{aligned} x_0 &\in E \quad \text{chosen arbitrarily,} \\ C_1 &= C, \quad x_1 = \Pi_{C_1} x_0, \\ y_n &= J^{-1} \left(\alpha_{n,0} Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i} JS_i^n x_n \right), \\ u_n &= T_{r_n} y_n, \\ C_{n+1} &= \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + \theta_n\}, \\ x_{n+1} &= \Pi_{C_{n+1}} x_0, \quad \forall n = 1, 2, 3, \dots, \end{aligned} \quad (3.1)$$

where J is the duality mapping on E , $\{\alpha_{n,i}\}_{i=0}^\infty$ are sequences in $[0, 1]$ which satisfies $\sum_{i=0}^\infty \alpha_{n,i} = 1$, $\theta_n = \sup_{p \in F} (k_n - 1)\phi(p, x_n)$, and $r_n \in [a, +\infty)$ for some $a > 0$. If $\liminf_{n \rightarrow \infty} \alpha_{n,0} \alpha_{n,i} > 0$ for all $n \geq 0$, then $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection from C onto F .

Proof. We first show that C_n is closed and convex. It is obvious that C_n is closed. In addition, since

$$\phi(z, u_n) \leq \phi(z, x_n) + \theta_n \iff \|u_n\|^2 - \|x_n\|^2 - 2\langle z, Ju_n - Jx_n \rangle - \theta_n \leq 0, \quad (3.2)$$

so C_n is convex, therefore, C_n is a closed convex subset of E for all $n \geq 0$.

Next, we show that $F \subset C_n$ for all $n \geq 1$. It is clear that $F \subset C_1 = C$. Suppose $F \in C_n$ for $n > 1$, by the property of ϕ , $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$, Lemmas 2.6 and 2.8, and uniformly quasi- ϕ -asymptotically nonexpansive of S_n for each $u \in F \subset C_n$, then we have

$$\begin{aligned}
 \phi(u, u_n) &= \phi(u, T_{r_n} y_n) \\
 &\leq \phi(u, y_n) \\
 &= \phi\left(u, J^{-1}\left(\alpha_{n,0} Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i} JS_i^n x_n\right)\right) \\
 &= \|u\|^2 - 2\left\langle u, \left(\alpha_{n,0} Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i} JS_i^n x_n\right)\right\rangle \\
 &\quad + \left\|\alpha_{n,0} Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i} JS_i^n x_n\right\|^2 \\
 &\leq \|p\|^2 - 2\alpha_{n,0}\langle p, Jx_n \rangle - 2\sum_{i=1}^{\infty} \alpha_{n,i}\langle u, JS_i^n x_n \rangle + \alpha_{n,0}\|x_n\|^2 \\
 &\quad + \sum_{i=1}^{\infty} \alpha_{n,i}\|S_i^n x_n\|^2 - \alpha_{n,0}\alpha_{n,j}\mathcal{G}\left(\|Jx_n - JS_j^n x_n\|\right) \\
 &\leq \alpha_{n,0}\phi(u, x_n) + \sum_{i=1}^{\infty} \alpha_{n,i}\phi(u, S_i^n x_n) \\
 &\leq \alpha_{n,0}\phi(u, x_n) + \sum_{i=1}^{\infty} \alpha_{n,i}k_n\phi(u, x_n) \\
 &\leq k_n\phi(u, x_n) \\
 &\leq \phi(u, x_n) + \theta_n.
 \end{aligned} \tag{3.3}$$

This shows that $u \in C_{n+1}$ implies that $F \subset C_n$ for all $n \geq 1$ by induction. On the one hand, since $x_{n+1} = \Pi_{C_n} x_0$ and $C_{n+1} \subset C_n$ for all $n \geq 1$, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0). \tag{3.4}$$

Therefore $\{\phi(x_n, x_0)\}$ is nondecreasing. In the other hand, by Lemma 2.1, we have

$$\begin{aligned}
 \phi(x_n, x_0) &= \phi(\Pi_{C_n} x_0, x_0) \\
 &\leq \phi(u, x_0) - \phi(u, x_n) \\
 &\leq \phi(u, x_0),
 \end{aligned} \tag{3.5}$$

for each $u \in F(T) \subset C_n$ for all $n \geq 0$. Therefore, $\{\phi(x_n, x_0)\}$ is bounded; this together with (3.4) implies that the limit of $\{\phi(x_n, x_0)\}$ exists.

Since $\{\phi(x_n, x_0)\}$ is bounded, so $\{x_n\}$ is bounded by (1.7), together with $\lim_{n \rightarrow \infty} k_n = 1$, we have that

$$\lim_{n \rightarrow \infty} \theta_n = 0. \quad (3.6)$$

From Lemma 2.1, we have, for any positive integers n, m , that

$$\begin{aligned} \phi(x_{n+m}, x_n) &= \phi(x_{n+m}, \Pi_{C_n} x_0) \\ &\leq \phi(x_{n+m}, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\ &= \phi(x_{n+m}, x_0) - \phi(x_n, x_0). \end{aligned} \quad (3.7)$$

Because the limit of $\{\phi(x_n, x_0)\}$ exists, then we have

$$\lim_{n \rightarrow \infty} \phi(x_{n+m}, x_n) = 0 \quad (3.8)$$

uniformly for positive integers $m > 1$. Since $\{x_n\}$ is a bounded sequence, by using Lemma 2.4, we have

$$\lim_{n \rightarrow \infty} \|x_{n+m} - x_n\| = 0 \quad (3.9)$$

uniformly for positive integers $m > 1$. Hence $\{x_n\}$ is a Cauchy sequence, therefore, there exists a point $p \in C$ such that $\{x_n\}$ converges strongly to p .

In addition, from (3.7) we have $\lim_{n \rightarrow \infty} \phi(x_{n+1}, x_n) = 0$, this together with the fact $x_{n+1} \in C_n$ implies that

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n) + \theta_n. \quad (3.10)$$

Taking limit on both side of (3.10) and from (3.6), we get that

$$\lim_{n \rightarrow \infty} \phi(x_{n+1}, u_n) = 0. \quad (3.11)$$

By using Lemma 2.4, we have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - u_n\| = 0, \quad (3.12)$$

which implies that $\{u_n\}$ converges strongly to p .

From (3.3), we have $\phi(u, y_n) \leq (u, x_n) + \theta_n$, together with $u_n = T_{r_n}y_n$ and Lemma 2.7, we have

$$\begin{aligned} \phi(u_n, y_n) &= \phi(T_{r_n}y_n, y_n) \\ &\leq \phi(u, y_n) - \phi(u, T_{r_n}y_n) \\ &\leq \phi(u, x_n) - \phi(u, T_{r_n}y_n) + \theta_n \\ &= \phi(u, x_n) - \phi(u, u_n) + \theta_n \end{aligned} \tag{3.13}$$

for any $u \in F$. This implies that

$$\lim_{n \rightarrow \infty} \phi(u_n, y_n) = 0. \tag{3.14}$$

Therefore, we have

$$\lim_{n \rightarrow \infty} \|u_n - y_n\| = 0, \tag{3.15}$$

which implies that $\{y_n\}$ converges strongly to p . Thus we have proved that

$$x_n \rightarrow p, \quad u_n \rightarrow p, \quad y_n \rightarrow p, \tag{3.16}$$

as $n \rightarrow \infty$, where $p \in C$. From (3.1)

$$\begin{aligned} \|Jx_n - Jy_n\| &= \left\| Jx_n - \left(\alpha_{n,0}Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i}JS_i^n x_n \right) \right\| \\ &= \left\| \sum_{i=1}^{\infty} \alpha_{n,i}(Jx_n - JS_i^n x_n) \right\| \\ &\leq \sum_{i=1}^{\infty} \alpha_{n,i} \|Jx_n - JS_i^n x_n\|, \end{aligned} \tag{3.17}$$

and hence

$$\|Jx_n - JS_i^n x_n\| \leq \frac{1}{\sum_{i=1}^{\infty} \alpha_{n,i}} \|Jx_n - Jy_n\|. \tag{3.18}$$

Taking limit on both side of above inequality, by $\liminf_{n \rightarrow \infty} \sum_{i=1}^{\infty} \alpha_{n,i} > 0$ and from (3.16), we have

$$\lim_{n \rightarrow \infty} \|Jx_n - JS_i^n x_n\| = 0. \tag{3.19}$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n \rightarrow \infty} \|x_n - S_i^n x_n\| = 0, \quad (3.20)$$

for each $i \geq 1$, together with (3.16), we get that

$$\lim_{n \rightarrow \infty} S_i^n x_n = p, \quad (3.21)$$

for each $i \geq 1$. Since S_i is equally continuous, we have

$$\begin{aligned} \|S_i^{n+1} x_n - S_i^n x_n\| &= \|S_i^{n+1} x_n - S_i^{n+1} x_{n+1}\| + \|S_i^{n+1} x_{n+1} - x_{n+1}\| \\ &\quad + \|x_{n+1} - x_n\| + \|x_n - S_i^{n+1} x_n\| \\ &\leq (L_i + 1) \|x_{n+1} - x_n\| + \|S_i^{n+1} x_{n+1} - x_{n+1}\| + \|x_n - S_i^{n+1} x_n\|. \end{aligned} \quad (3.22)$$

Together with (3.16) and (3.20), we have $\lim_{n \rightarrow \infty} \|S_i^{n+1} x_n - S_i^n x_n\| = 0$. From (3.21), we have $S_i^{n+1} x_n \rightarrow p$, that is, $S_i S_i^n x_n \rightarrow p$. In view of closeness of S_i , we have $S_i p = p$, for all $i \geq 1$. This implies that $p \in \bigcap_{i=1}^{\infty} F(S_i)$.

Next we show $p \in \text{GEP}(f, A)$. By $u_n = T_{r_n} y_n$, we have

$$f(u_n, y) + \langle Ay_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \quad \forall y \in C. \quad (3.23)$$

From (A2), we get that

$$\langle Ay_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq -f(u_n, y) \geq f(y, u_n), \quad \forall y \in C, \quad (3.24)$$

and hence

$$0 \geq -\langle Ay_n, y - u_n \rangle - \left\langle y - u_n, \frac{Ju_n - Jy_n}{r_n} \right\rangle + f(y, u_n), \quad \forall y \in C. \quad (3.25)$$

For t with $0 < t < 1$ and $y \in C$, let $y_t = ty + (1-t)p$, then $y_t \in C$, from (3.25) we have

$$\begin{aligned} \langle Ay_t, y_t - u_n \rangle &\geq \langle Ay_t, y_t - u_n \rangle - \langle Ay_n, y_t - u_n \rangle - \left\langle y_t - u_n, \frac{Ju_n - Jy_n}{r_n} \right\rangle + f(y_t, u_n) \\ &= \langle Ay_t - Au_n, y_t - u_n \rangle + \langle Au_n - Ay_n, y_t - u_n \rangle \\ &\quad - \left\langle y_t - u_n, \frac{Ju_n - Jy_n}{r_n} \right\rangle + f(y_t, u_n). \end{aligned} \quad (3.26)$$

Since J is uniformly norm-to-norm continuous on bounded sets, A is monotone and (3.16), we have

$$\langle Ay_t, y_t - u_n \rangle \geq 0. \quad (3.27)$$

It follows from (A4) that

$$\begin{aligned} f(y_t, p) &\leq \liminf_{n \rightarrow \infty} f(y_t, u_n) \\ &\leq \lim_{n \rightarrow \infty} \langle Ay_t, y_t - u_n \rangle \\ &= \langle Ay_t, y_t - p \rangle \\ &= t \langle Ay_t, y - p \rangle. \end{aligned} \quad (3.28)$$

From the conditions (A1) and (A4), we have

$$\begin{aligned} 0 &= f(y_t, y_t) \\ &\leq t f(y_t, y) + (1-t) f(y_t, p) \\ &\leq t f(y_t, y) + (1-t) t \langle Ay_t, y - p \rangle \\ &\leq f(y_t, y) + (1-t) \langle Ay_t, y - p \rangle. \end{aligned} \quad (3.29)$$

Letting $t \rightarrow 0$, we get

$$f(p, y) + \langle Ap, y - p \rangle \geq 0, \quad \forall y \in C. \quad (3.30)$$

This implies that $p \in \text{GEP}(f, A)$.

Finally, we show that $p = \Pi_F x_0$. Let $w = \Pi_F x_0$, from $p \in F$, we have

$$\phi(p, x_0) \geq \phi(w, x_0). \quad (3.31)$$

Since $x_n = \Pi_{C_n} x_0$ and $w \in F \subset C_n$,

$$\phi(x_n, x_0) \leq \phi(w, x_0), \quad (3.32)$$

together with above two hands and $\lim_{n \rightarrow \infty} x_n = p$, we obtain

$$\phi(p, x_0) = \phi(w, x_0). \quad (3.33)$$

that is $p = w = \Pi_F x_0$. The proof is completed. \square

By using the similar method of proof as in Theorem 3.1, the following theorem is not hard to prove.

Theorem 3.2. Let E be a uniformly smooth and uniformly convex Banach space, and let C be a nonempty closed convex subset of E . Let f be a bifunction from $C \times C$ to $\mathbb{R} = (-\infty, +\infty)$ satisfying (A1)–(A4). Let $\{S_i\}_{i=1}^{\infty} : C \rightarrow C$ be an infinite family of closed equally continuous and uniformly quasi- ϕ -asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1, \infty)$, $k_n \rightarrow 1$ such that $F := \bigcap_{i=1}^{\infty} F(S_i) \cap \text{EP}(f)$ is a nonempty and bounded subset in C . Let $\{x_n\}$ be a sequence generated by

$$\begin{aligned} x_0 &\in E \quad \text{chosen arbitrarily,} \\ C_1 &= C, \quad x_1 = \Pi_{C_1} x_0, \\ y_n &= J^{-1} \left(\alpha_{n,0} Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i} JS_i^n x_n \right), \\ u_n &\in C \quad \text{s.t.} \quad f(u_n, y) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} &= \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + \theta_n\}, \\ x_{n+1} &= \Pi_{C_{n+1}} x_0, \quad \forall n = 1, 2, 3, \dots, \end{aligned} \tag{3.34}$$

where J is the duality mapping on E , $\{\alpha_{n,i}\}_{i=0}^{\infty}$ are sequences in $[0, 1]$ which satisfies $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$, $\theta_n = \sup_{p \in F} (k_n - 1)\phi(p, x_n)$, and $r_n \in [a, +\infty)$ for some $a > 0$. If $\liminf_{n \rightarrow \infty} \alpha_{n,0} \alpha_{n,i} > 0$ for all $n \geq 0$, then $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection from C onto F .

Proof. In Theorem 3.1, put $A = 0$ we can obtain the conclusion of Theorem 3.2. \square

Theorem 3.3. Let E be a uniformly smooth and uniformly convex Banach space, and let C be a nonempty closed convex subset of E . Let f be a bifunction from $C \times C$ to $\mathbb{R} = (-\infty, +\infty)$ satisfying (A1)–(A4) and let A be a continuous monotone mapping of C into E^* . Let $S : C \rightarrow C$ be an infinite family of closed equally continuous and quasi- ϕ -asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1, \infty)$, $k_n \rightarrow 1$ such that $F := F(S) \cap \text{GEP}(f, A)$ is a nonempty and bounded subset in C . Let $\{x_n\}$ be a sequence generated by

$$\begin{aligned} x_0 &\in E \quad \text{chosen arbitrarily,} \\ C_1 &= C, \quad x_1 = \Pi_{C_1} x_0, \\ y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) JS_n x_n), \\ u_n &= T_{r_n} y_n, \\ C_{n+1} &= \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + \theta_n\}, \\ x_{n+1} &= \Pi_{C_{n+1}} x_0, \quad \forall n = 1, 2, 3, \dots, \end{aligned} \tag{3.35}$$

where J is the duality mapping on E , $\{\alpha_n\}$ are sequences in $[0, 1]$ which satisfies $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$ for all $n \geq 0$, $\theta_n = \sup_{p \in F} (k_n - 1)\phi(p, x_n)$ and $r_n \in [a, +\infty)$ for some $a > 0$. Then $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection from C onto F .

Proof. In Theorem 3.1, put $S_i = S$, for $i = 1, 2, \dots$, we can obtain the conclusion of Theorem 3.3. \square

4. Application for Optimization Problem

In this section, we study a kind of optimization problem by using the result of this paper. that is, we will give an iterative algorithm of solution for the following optimization problem with nonempty set of solutions:

$$\max h(x), \quad x \in C, \tag{4.1}$$

where $h(x)$ is a convex and lower semicontinuous functional defined on a closed convex subset C of a Banach space H . We denoted by S the set of solutions of (4.1). Let F be a bifunction from $C \times C$ to R defined by $f(x, y) = h(x) - h(y)$. We consider the following equilibrium problem, that is, to find $x \in C$ such that

$$f(x, y) \geq 0, \quad \forall y \in C. \tag{4.2}$$

It is obvious that $EP(F) = S$, where $EP(F)$ denote the set of solutions of equilibrium problem (4.2). In addition, it is easy to see that $f(x, y)$ satisfies the conditions (A1)–(A4) in the Section 2. Therefore, from the Theorem 3.1, we can obtain the following theorem.

Theorem 4.1. *Let E be a uniformly smooth and uniformly convex Banach space, and let C be a nonempty closed convex subset of E . Let f be a bifunction from $C \times C$ to $R = (-\infty, +\infty)$ satisfying (A1)–(A4) and let A be a continuous monotone mapping of C into E^* . Let $\{S_i\}_{i=1}^\infty : C \rightarrow C$ be an infinite family of closed equally continuous and uniformly quasi- ϕ -asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1, \infty), k_n \rightarrow 1$ such that $F := \bigcap_{i=1}^\infty F(S_i) \cap S$ is a nonempty and bounded subset in C . Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} x_0 &\in E \quad \text{chosen arbitrarily,} \\ C_1 &= C, \quad x_1 = \Pi_{C_1} x_0, \\ y_n &= J^{-1} \left(\alpha_{n,0} Jx_n + \sum_{i=1}^\infty \alpha_{n,i} JS_i^n x_n \right), \\ u_n &\in C, \quad \text{s.t.} \quad h(u_n) - h(y) + \frac{1}{r_n} \langle y - u_n, u_n - y_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} &= \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + \theta_n\}, \\ x_{n+1} &= \Pi_{C_{n+1}} x_0 \quad \forall n = 1, 2, 3, \dots, \end{aligned} \tag{4.3}$$

where J is the duality mapping on E , $\{\alpha_{n,i}\}_{i=0}^\infty$ are sequences in $[0, 1]$ which satisfies $\sum_{i=0}^\infty \alpha_{n,i} = 1$, $\theta_n = \sup_{p \in F} (k_n - 1)\phi(p, x_n)$, and $r_n \in [a, +\infty)$ for some $a > 0$. If $\liminf_{n \rightarrow \infty} \alpha_{n,0} \alpha_{n,i} > 0$ for all $n \geq 0$, then $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection from C onto F .

Proof. By the proof of Theorem 3.2, we can obtain Theorem 4.1. □

It is easy to see that, this paper has some new methods and conditions than the conditions of Takahashi and Zembayashi [16]. In this paper, we prove the convergence theorems for uniformly quasi- ϕ -asymptotically nonexpansive mappings and do not need the

Kadec-Klee property of Banach space and use the condition of equally continuous that is more weak different from the condition of uniformly L -Lipschitz.

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