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Research Article

The Inequalities for Quasiarithmetic Means

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Overview and refinements of the results are given for discrete, integral, functional and operator variants of inequalities for quasiarithmetic means. The general results are applied to further refinements of the power means. Jensen's inequalities have been systematically presented, from the older variants, to the most recent ones for the operators without operator convexity.

1. Introduction

Quasiarithmetic means are very important because they are general and unavoidable in applications. This paper begins with the quasiarithmetic means of points, continues with the quasiarithmetic means of measurable function, through the quasiarithmetic means of functions with respect to linear functionals, and ends with the quasiarithmetic means of operators with respect to linear mappings. Conclusion of the paper is dedicated to the applications of operator quasiarithmetic means on power means with strictly positive operators. At this point, it should be emphasized that in all four of the next sections the basic and initial inequality was precisely the Jensen inequality (see Figure 1).

The applications of convexity often used strictly monotone continuous functions φ and ψ such that ψ is convex with respect to φ (ψ is φ convex); that is, $f = \psi \circ \varphi^{-1}$ is convex by [1, Definition 1.19]. Similar notation is used for concavity. We observe a monotonicity of quasiarithmetic means with these functions φ and ψ. Good results for the monotonicity of quasiarithmetic means are obtained in [2] for the basic and integral case. The first results for the operator case without operator convexity are obtained in [3, 4]. Among other things, the paper gives some generalizations of the mentioned results.

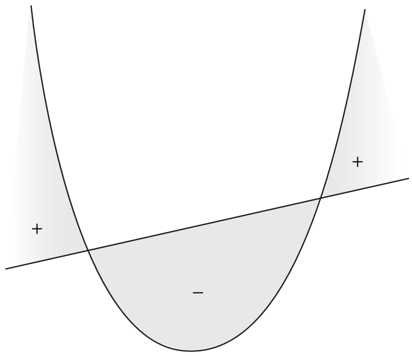

Figure 1: Graphic concept of Jensen's inequality.

Through this paper, we suppose that $I \subseteq \mathbb{R}$ is a nondegenerate interval, and $\varphi, \psi : I \to \mathbb{R}$ are strictly monotone continuous functions. It is assumed that the integer $n \geq 2$, wherever it appears in inequalities.

2. Results for Basic Case

For n-tuple $\mathbf{x} = (x_1, \ldots, x_n)$ with numbers $x_i \in \mathbb{R}$, sometimes we will write $\mathbf{x} > \mathbf{0}$ if all $x_i > 0$, and $\mathbf{x} \neq \mathbf{c}$ if $x_i \neq x_j$ for some $i \neq j$.

Below is a discrete basic form of Jensen's inequality for a convex function with respect to convex combinations points in interval.

Theorem A. *Let $f : I \to \mathbb{R}$ be a function. Let $\mathbf{x} = (x_1, \ldots, x_n)$ be n-tuple with points $x_i \in I$, and $\mathbf{p} = (p_1, \ldots, p_n)$ be n-tuple with numbers $p_i \in [0, 1]$ such that $\sum_{i=1}^{n} p_i = 1$.*

A function f is convex if and only if the following inequality

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i=1}^{n} p_i f(x_i), \tag{2.1}$$

holds for all above n-tuples $\mathbf{p}$ and $\mathbf{x}$.

Consequently, if $\sum_{i=1}^{n} p_i = p > 0$, not necessarily equals 1, then f is convex if and only if

$$f\left(\frac{1}{p}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{p}\sum_{i=1}^{n} p_i f(x_i). \tag{2.2}$$

A function f is concave if and only if the reverse inequality is valid in (2.1) *and* (2.2).

A function f is strictly convex if and only if the inequality in (2.1) and (2.2) is strict for all $\mathbf{p} > \mathbf{0}$ and $\mathbf{x} \neq \mathbf{c}$.

Let $\varphi : I \to \mathbb{R}$ be a strictly monotone continuous function. Let $\mathbf{x} = (x_1, \ldots, x_n)$ be n-tuple with points $x_i \in I$, and $\mathbf{p} = (p_1, \ldots, p_n)$ be n-tuple with numbers $p_i \in [0, 1]$ such that

$\sum_{i=1}^{n} p_i = 1$. The discrete basic φ-quasiarithmetic mean of points (particles) x_i with coefficients (weights) p_i is a number

$$\mathcal{M}_\varphi(\mathbf{px}) = \varphi^{-1}\left(\sum_{i=1}^{n} p_i\varphi(x_i)\right). \tag{2.3}$$

We understand that $\mathbf{px} = \sum_{i=1}^{n} p_i x_i$. The φ-quasiarithmetic mean resulting, first by moving the convex combination $\mathbf{px} \in I$ into convex combination $\mathbf{p}\varphi(\mathbf{x}) \in \varphi(I)$, then its return using φ^{-1} back in the interval I. So, the number $\mathcal{M}_\varphi(\mathbf{px})$ is in the interval I, in fact in the closed interval $[\min\{x_i\}, \max\{x_i\}]$. If φ is an identity function on I, that is, $\varphi(x) = \mathrm{id}(x) = x$ for $x \in I$, then the φ-quasiarithmetic mean is just a convex combination as follows:

$$\mathcal{M}_{\mathrm{id}}(\mathbf{px}) = \sum_{i=1}^{n} p_i x_i. \tag{2.4}$$

Basic quasiarithmetic means have the property

$$\mathcal{M}_{a\varphi+b}(\mathbf{px}) = \mathcal{M}_\varphi(\mathbf{px}), \tag{2.5}$$

for every pair of real numbers a and b with $a \neq 0$.

Suppose that all coefficients $p_i = 1/n$. If we take $\varphi_1(x) = x$, then $\mathcal{M}_{\varphi_1}(\mathbf{px})$ is the arithmetic mean of numbers x_i. If all $x_i > 0$ and we take $\varphi_0(x) = \ln x$, then $\mathcal{M}_{\varphi_0}(\mathbf{px})$ is the geometric mean of numbers x_i. If all $x_i > 0$ and we take $\varphi_{-1}(x) = 1/x$, then $\mathcal{M}_{\varphi_{-1}}(\mathbf{px})$ is the harmonic mean of numbers x_i.

Corollary 2.1. *Let $\varphi, \psi : I \to \mathbb{R}$ be strictly monotone continuous functions.*

A function ψ is either φ-convex and increasing or φ-concave and decreasing if and only if following the inequality:

$$\mathcal{M}_\varphi(\mathbf{px}) \leq \mathcal{M}_\psi(\mathbf{px}), \tag{2.6}$$

holds for all n-tuples $\mathbf{p}$ and $\mathbf{x}$ as in (2.3).

A function ψ is either φ-concave and increasing or φ-convex and decreasing if and only if the reverse inequality is valid in (2.6).

A function ψ is strictly φ-convex if and only if the inequality in (2.6) is strict for all $\mathbf{p} > \mathbf{0}$ and $\mathbf{x} \neq \mathbf{c}$ (see Figure 2).

Suppose that all $x_i > 0$. If we apply Corollary 2.1 on three strictly monotone functions $\varphi_{-1}(x) = 1/x$, $\varphi_0(x) = \ln x$ and $\varphi_1(x) = x$ (two by two in pairs), then we get the weighted harmonic-geometric-arithmetic inequality

$$\frac{1}{(p_1/x_1) + \cdots + (p_n/x_n)} \leq x_1^{p_1} \cdot \ldots \cdot x_n^{p_n} \leq p_1x_1 + \cdots + p_nx_n. \tag{2.7}$$

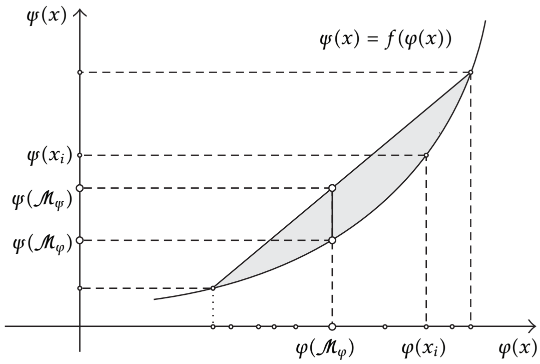

Figure 2: The ψ-order among quasiarithmetic means $\mathcal{M}_\varphi$ and $\mathcal{M}_\psi$ for φ-convex and increasing ψ.

Recall that a function $f : I \to \mathbb{R}$ is convex if and only if the following inequality:

$$\frac{f(y) - f(x)}{y - x} \leq \frac{f(z) - f(y)}{z - y}, \tag{2.8}$$

holds for all triples $x, y, z \in I$ such that $x < y < z$. A function f is strictly convex if and only if the above inequality is strict. So, the function ψ is φ-convex if and only if

$$\frac{\psi(y) - \psi(x)}{\varphi(y) - \varphi(x)} \leq \frac{\psi(z) - \psi(y)}{\varphi(z) - \varphi(y)}. \tag{2.9}$$

Let $u, v : [a_0, a_1] \to \mathbb{R}$, where $a_0 < a_1$, be nonnegative continuous functions so that v/u is a strictly monotone increasing positive function on an open interval $\langle a_0, a_1 \rangle$, with boundary conditions $u(a_0) = v(a_1) = 1$ and $u(a_1) = v(a_0) = 0$. Let both φ and ψ be strictly monotone increasing or decreasing. For any $t \in [a_0, a_1]$, we define a strictly monotone continuous function

$$\phi_t(x) = u(t)\varphi(x) + v(t)\psi(x) \quad \text{with } x \in I. \tag{2.10}$$

For example, we can take $u(t) = 1 - t$ and $v(t) = t$ for $t \in [0, 1]$, $u(t) = 1 - \sqrt{t}$ and $v(t) = t^2$ for $t \in [0, 1]$, $u(t) = \cos t$ and $v(t) = \sin t$ for $t \in [0, \pi/2]$.

Lemma 2.2. *Let a, b, α, β be real numbers.*

If $a \leq b$ and $\alpha \leq \beta$, then

$$\frac{1 + \beta a}{1 + \alpha a} \leq \frac{1 + \beta b}{1 + \alpha b}, \tag{2.11}$$

provided that denominators are the same sign. The inequality in (2.11) *is strict if $a < b$ and $\alpha < \beta$.*

If either $a \geq b$ and $\alpha \leq \beta$ or $a \leq b$ and $\alpha \geq \beta$, then the reverse inequality is valid in (2.11).

Proposition 2.3. *Let $\phi_t = u(t)\varphi + v(t)\psi : I \to \mathbb{R}$ for $t \in [a_0, a_1]$ be functions as in* (2.10)*. Let $t_0, t_1 \in [a_0, a_1]$ such that $t_0 \leq t_1$.*

If ψ is φ-convex (resp. φ-concave), then ϕ_{t_1} is ϕ_{t_0}-convex (resp. ϕ_{t_0}-concave).

Proof. Suppose that ψ is φ-convex. Show that the function ϕ_{t_1} is ϕ_{t_0}-convex. If $t_0 = a_1$, then $t_0 = t_1$, so we can suppose that $t_0 < a_1$. Let $x, y, z \in I$ such that $x < y < z$. Let, with respect to (2.9) and definition of functions u and v,

$$a = \frac{\psi(y) - \psi(x)}{\varphi(y) - \varphi(x)} \leq \frac{\psi(z) - \psi(y)}{\varphi(z) - \varphi(y)} = b, \qquad \alpha = \frac{v(t_0)}{u(t_0)} \leq \frac{v(t_1)}{u(t_1)} = \beta. \tag{2.12}$$

Note that numbers a and b are positive because both φ and ψ are strictly monotone increasing or decreasing. Applying Lemma 2.2 with a, b, α, and β, we obtain that

$$\begin{aligned}
\frac{\phi_{t_1}(y) - \phi_{t_1}(x)}{\phi_{t_0}(y) - \phi_{t_0}(x)} &= \frac{u(t_1)[\varphi(y) - \varphi(x)] + v(t_1)[\psi(y) - \psi(x)]}{u(t_0)[\varphi(y) - \varphi(x)] + v(t_0)[\psi(y) - \psi(x)]} \\
&= \frac{u(t_1)}{u(t_0)} \frac{1 + \beta a}{1 + \alpha a} \leq \frac{u(t_1)}{u(t_0)} \frac{1 + \beta b}{1 + \alpha b} \\
&= \frac{u(t_1)[\varphi(z) - \varphi(y)] + v(t_1)[\psi(z) - \psi(y)]}{u(t_0)[\varphi(z) - \varphi(y)] + v(t_0)[\psi(z) - \psi(y)]} \\
&= \frac{\phi_{t_1}(z) - \phi_{t_1}(y)}{\phi_{t_0}(z) - \phi_{t_0}(y)},
\end{aligned} \tag{2.13}$$

which shows the required convexity by (2.9). Case of the concavity can be proved in a similar way. □

If ψ is strictly φ-convex (resp. φ-concave), then ϕ_{t_1} is strictly ϕ_{t_0}-convex (resp. ϕ_{t_0}-concave).

According to Proposition 2.3, we can express refinements of the basic quasiarithmetic means.

Theorem 2.4. *Let $\phi_t = u(t)\varphi + v(t)\psi : I \to \mathbb{R}$ for $t \in [a_0, a_1]$ be functions as in* (2.10)*. Let $t_0, t_1 \in [a_0, a_1]$ such that $t_0 \leq t_1$.*

If either ψ is φ-convex with both φ and ψ increasing or φ-concave with both φ and ψ decreasing, then the following inequality:

$$\mathcal{M}_{\varphi}(\mathbf{px}) \leq \mathcal{M}_{\phi_{t_0}}(\mathbf{px}) \leq \mathcal{M}_{\phi_{t_1}}(\mathbf{px}) \leq \mathcal{M}_{\psi}(\mathbf{px}), \tag{2.14}$$

holds for all n-tuples $\mathbf{p}$ and $\mathbf{x}$ as in (2.3)*.*

If either ψ is φ-concave with both φ and ψ increasing or φ-convex with both φ and ψ decreasing, then the reverse inequality is valid in (2.14)*.*

Proof. If ψ is φ-convex with both φ and ψ increasing, then the function ϕ_{t_1} is increasing, and ϕ_{t_0}-convex by Proposition 2.3, and according to Corollary 2.1 the inequality in (2.14) is valid. In the same way, we prove the concavity case. □

In other words, the above theorem says that a function

$$t \mapsto \mathcal{M}_{\phi_t}(\mathbf{px}) \quad \text{with } t \in [a_0, a_1], \tag{2.15}$$

is monotone increasing for any fixed $\mathbf{p}$ and $\mathbf{x}$ as in (2.3). In the case $n = 2$ it is proved in [2, Lemma 4] for functions $u(t) = 1 - t$ and $v(t) = t$ with $t \in [0, 1]$.

We emphasize that the inequality in (2.14) is strict for $a_0 < t_0 < t_1 < a_1$ if ψ is strictly φ-convex or φ-concave, $\mathbf{p} > \mathbf{0}$ and $\mathbf{x} \neq \mathbf{c}$.

Let us take strictly monotone decreasing functions $\varphi(x) = 1/x$ and $\psi(x) = -\ln x$ with $x > 0$. Then $(\psi \circ \varphi^{-1})(x) = \ln x$, so ψ is strictly φ-concave. Let

$$\phi_t^{hg}(x) = u(t)\frac{1}{x} - v(t)\ln x = \ln\frac{e^{u(t)/x}}{x^{v(t)}}. \tag{2.16}$$

If we apply the inequality in (2.14) with $t = t_0 = t_1$ on ϕ_t^{hg}, we get

$$\frac{1}{\sum_{i=1}^{n}(p_i/x_i)} \leq \left(\phi_t^{hg}\right)^{-1}\left(\sum_{i=1}^{n} p_i \ln\frac{e^{u(t)/x_i}}{x_i^{v(t)}}\right) \leq \prod_{i=1}^{n} x_i^{p_i}. \tag{2.17}$$

Let us take strictly monotone increasing functions $\varphi(x) = \ln x$ and $\psi(x) = x$ with $x > 0$. Then $(\psi \circ \varphi^{-1})(x) = e^x$, so ψ is strictly φ-convex. Let

$$\phi_t^{ga}(x) = u(t)\ln x + v(t)x = \ln\left(x^{u(t)}e^{v(t)x}\right). \tag{2.18}$$

If we apply the inequality in (2.14) with $t = t_0 = t_1$ on ϕ_t^{ga}, we get

$$\prod_{i=1}^{n} x_i^{p_i} \leq \left(\phi_t^{ga}\right)^{-1}\left(\sum_{i=1}^{n} p_i \ln\left(x_i^{u(t)}e^{v(t)x_i}\right)\right) \leq \sum_{i=1}^{n} p_i x_i. \tag{2.19}$$

Connecting two above inequalities results in

$$\begin{aligned}\frac{1}{\sum_{i=1}^{n}(p_i/x_i)} &\leq \left(\phi_t^{hg}\right)^{-1}\left(\sum_{i=1}^{n} p_i \ln\frac{e^{u(t)/x_i}}{x_i^{v(t)}}\right) \leq \prod_{i=1}^{n} x_i^{p_i} \\ &\leq \left(\phi_t^{ga}\right)^{-1}\left(\sum_{i=1}^{n} p_i \ln\left(x_i^{u(t)}e^{v(t)x_i}\right)\right) \leq \sum_{i=1}^{n} p_i x_i.\end{aligned} \tag{2.20}$$

The inequality in (2.20) is strict for $a_0 < t < a_1$ if all $p_i > 0$ and $x_i \neq x_j$ for some $i \neq j$, so in this case, we have refinements of the weighted harmonic-geometric-arithmetic inequality.

The weighted harmonic-geometric-arithmetic inequality is only the special case of a whole collection of inequalities which can be derived by applying of Corollary 2.1 on power means. As a special case of the basic quasiarithmetic mean in (2.3) with $I = \langle 0, +\infty\rangle$, $\varphi_r(x) = x^r$ for $r \neq 0$ and $\varphi_0(x) = \ln x$, we can observe the discrete basic power mean

$$\mathcal{M}_n^{[r]}(\mathbf{px}) = \begin{cases} \left(\sum_{i=1}^{n} p_i x_i^r\right)^{1/r} & \text{for } r \neq 0 \\ \exp\left(\sum_{i=1}^{n} p_i \ln x_i\right) & \text{for } r = 0. \end{cases} \tag{2.21}$$

Very useful consequence of Corollary 2.1 is a well-known property of monotonicity of basic power means.

Corollary 2.5. *If r and s are real numbers such that $r \leq s$, then the following inequality:*

$$\mathcal{M}_n^{[r]}(\mathbf{px}) \leq \mathcal{M}_n^{[s]}(\mathbf{px}), \tag{2.22}$$

holds for all n-tuples $\mathbf{p}$ and $\mathbf{x}$ as in (2.3) with $I = \langle 0, +\infty\rangle$.

The inequality in (2.22) is strict for $r < s$ if $\mathbf{p} > \mathbf{0}$ and $\mathbf{x} \neq \mathbf{c}$.

Let functions $\phi_t^{[r,s]} : \langle 0, +\infty\rangle \to \mathbb{R}$ for $t \in [a_0, a_1]$ be specially defined by

$$\phi_t^{[r,s]}(x) = \begin{cases} u(t)x^r + v(t)x^s & \text{for } r \neq 0,\ s \neq 0 \\ u(t)x^r - v(t)\ln x & \text{for } r \neq 0,\ s = 0 \\ u(t)\ln x + v(t)x^s & \text{for } r = 0,\ s \neq 0. \end{cases} \tag{2.23}$$

Then the functions $t \mapsto \mathcal{M}_{\phi_t^{[r,s]}}(\mathbf{px})$ with $\mathbf{x} > \mathbf{0}$ are monotone increasing in the next four cases.

Case $r < s < 0$.

Functions $\varphi(x) = x^r$ and $\psi(x) = x^s$ are strictly monotone decreasing with strictly concave $(\psi \circ \varphi^{-1})(x) = x^{s/r}$ because $0 < (s/r) < 1$.

Case $r < 0 = s$.

Functions $\varphi(x) = x^r$ and $\psi(x) = -\ln x$ are strictly monotone decreasing with strictly concave $(\psi \circ \varphi^{-1})(x) = -(1/r)\ln x$ because $-(1/r) > 0$.

Case $r = 0 < s$.

Functions $\varphi(x) = \ln x$ and $\psi(x) = x^s$ are strictly monotone increasing with strictly convex $(\psi \circ \varphi^{-1})(x) = e^{sx}$.

Case $0 < r < s$.

Functions $\varphi(x) = x^r$ and $\psi(x) = x^s$ are strictly monotone increasing with strictly convex $(\psi \circ \varphi^{-1})(x) = x^{s/r}$ because $(s/r) > 1$.

Given traditional signs of power means, we will mark $\mathcal{M}_{\phi_t^{[r,s]}}(\mathbf{px})$ with $\mathcal{M}_n^{\phi_t^{[r,s]}}(\mathbf{px})$. The inequality in (2.22) can be refined using Theorem 2.4 with functions $\phi_t^{[r,s]}$. The following are refinements of power means.

Corollary 2.6. *Let $r, s \in \mathbb{R}$ such that $r < s$. Let $\phi_t^{[r,s]} : \langle 0, +\infty\rangle \to \mathbb{R}$ for $t \in [a_0, a_1]$ be functions as in* (2.23)*. Let $t_0, t_1 \in [a_0, a_1]$ such that $t_0 \leq t_1$.*

If $r < s < 0$ or $r < 0 = s$ or $r = 0 < s$ or $0 < r < s$, then the inequality

$$\mathcal{M}_n^{[r]}(\mathbf{px}) \leq \mathcal{M}_n^{\phi_{t_0}^{[r,s]}}(\mathbf{px}) \leq \mathcal{M}_n^{\phi_{t_1}^{[r,s]}}(\mathbf{px}) \leq \mathcal{M}_n^{[s]}(\mathbf{px}), \tag{2.24}$$

holds for all n-tuples $\mathbf{p}$ and $\mathbf{x}$ as in (2.3) *with $I = \langle 0, +\infty\rangle$.*

If $r < 0 < s$, then we can take the series of inequalities

$$\begin{aligned} \mathcal{M}_n^{[r]}(\mathbf{px}) &\leq \mathcal{M}_n^{\phi_{t_0}^{[r,0]}}(\mathbf{px}) \leq \mathcal{M}_n^{\phi_{t_1}^{[r,0]}}(\mathbf{px}) \leq \mathcal{M}_n^{[0]}(\mathbf{px}) \\ &\leq \mathcal{M}_n^{\phi_{t_0}^{[0,s]}}(\mathbf{px}) \leq \mathcal{M}_n^{\phi_{t_1}^{[0,s]}}(\mathbf{px}) \leq \mathcal{M}_n^{[s]}(\mathbf{px}). \end{aligned} \tag{2.25}$$

The inequalities in (2.24)-(2.25) are strict for $a_0 < t_0 < t_1 < a_1$ if $\mathbf{p} > \mathbf{0}$ and $\mathbf{x} \neq \mathbf{c}$.

The inequality in (2.20) is a special case of the collection of inequalities in (2.24).

3. Applications on Integral Case

In this section, (Ω, μ) is a probability measure space. It is assumed that every weighted function $w : \Omega \to \mathbb{R}$ is nonnegative almost everywhere on Ω, that is, $w(\omega) \geq 0$ for almost all $\omega \in \Omega$.

For n-tuple $\mathbf{g} = (g_1, \ldots, g_n)$ with functions $g_i : \Omega \to \mathbb{R}$, sometimes we will write $\mathbf{g} > \mathbf{0}$ if all $g_i > 0$ almost everywhere on Ω, and $\mathbf{g} \neq \mathbf{c}$ if $g_i \neq g_j$ almost everywhere on Ω for some $i \neq j$.

Here is an integral form of Jensen's inequality for a convex function with respect to measurable functions with weighted functions on the probability measure space.

Theorem B. *Let $f : I \to \mathbb{R}$ be a function. Let (Ω, μ) be a probability measure space, $g : \Omega \to I$ be a measurable function, and $w \in L^1(\Omega, \mu)$ be a weighted function with $\int_\Omega w\, d\mu = 1$ such that $w \cdot g, w \cdot (f \circ g) \in L^1(\Omega, \mu)$.*

If a function f is convex, then the inequality

$$f\left(\int_\Omega w \cdot g\, d\mu\right) \leq \int_\Omega w \cdot (f \circ g) d\mu, \tag{3.1}$$

holds for all above w, g and μ.

Consequently, if $\int_\Omega w\, d\mu = p > 0$, not necessarily equals 1, then

$$f\left(\frac{1}{p}\int_\Omega w \cdot g\, d\mu\right) \leq \frac{1}{p}\int_\Omega w \cdot (f \circ g) d\mu. \tag{3.2}$$

If a function f is concave, then the reverse inequality is valid in (3.1) *and* (3.2)*.*

The assumption $\int_\Omega w\, d\mu = 1$ with nonnegative w almost everywhere on Ω, for the inequality in (3.1), assures that

$$\int_\Omega w \cdot g\, d\mu \in I, \qquad \int_\Omega w \cdot (f \circ g) d\mu \in f(I). \tag{3.3}$$

Remark 3.1. The reverse of Theorem B is valid if for any $p \in [0,1]$ a measurable set $\Omega_p \subseteq \Omega$ exists so that $\mu(\Omega_p) = p$. In this case, we can determine a simple measurable function

$$g = x\chi_{\Omega_p} + y\chi_{\Omega\setminus\Omega_p}, \tag{3.4}$$

where χ is a characteristic set function, for every $x, y \in I$ and $p \in [0,1]$. If we take $w = 1$ at the same, then

$$\begin{gathered}\int_\Omega w \cdot g\, d\mu = px + (1-p)y, \\ \int_\Omega w \cdot (f \circ g) d\mu = \int_{\Omega_p} f(x) d\mu + \int_{\Omega\setminus\Omega_p} f(y) d\mu = pf(x) + (1-p)f(y).\end{gathered} \tag{3.5}$$

If we include these integrals in the inequality in (3.1), we have the convexity of the function f.

Theorem B can be generalized to n probability measures μ_i and n measurable functions g_i with weighted functions w_i. The following is a discrete integral form of Jensen's inequality.

Theorem 3.2. *Let $f : I \to \mathbb{R}$ be a function. Let $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ be n-tuple with probability measures μ_i on Ω, $\mathbf{g} = (g_1, \ldots, g_n)$ be n-tuple with measurable functions $g_i : \Omega \to I$, and $\mathbf{w} = (w_1, \ldots, w_n)$ be n-tuple with weighted functions $w_i \in L^1(\Omega, \mu_i)$ with $\sum_{i=1}^n \int_\Omega w_i\, d\mu_i = 1$ such that $w_i \cdot g_i, w_i \cdot (f \circ g_i) \in L^1(\Omega, \mu_i)$.*

A function f is convex if and only if the inequality

$$f\left(\sum_{i=1}^n \int_\Omega w_i \cdot g_i\, d\mu_i\right) \le \sum_{i=1}^n \int_\Omega w_i \cdot (f \circ g_i) d\mu_i, \tag{3.6}$$

holds for all above n-tuples $\mathbf{w}$, $\mathbf{g}$ and $\boldsymbol{\mu}$.

Consequently, if $\sum_{i=1}^n \int_\Omega w_i\, d\mu_i = p > 0$, not necessarily equals 1, then f is convex if and only if

$$f\left(\frac{1}{p}\sum_{i=1}^n \int_\Omega w_i \cdot g_i\, d\mu_i\right) \le \frac{1}{p}\sum_{i=1}^n \int_\Omega w_i \cdot (f \circ g_i) d\mu_i. \tag{3.7}$$

A function f is concave if and only if the reverse inequality is valid in (3.6) *and* (3.7).

In the proof of sufficiency theorem, we simply take $w_i = p_i$ and $g_i = x_i$ in which case the inequality in (3.6) and (3.7) becomes the basic inequality of convexity. The fact that $n \geq 2$ is coming to the fore.

A function f is strictly convex if and only if the inequality in (3.6) and (3.7) is strict for all $\mathbf{w} > \mathbf{0}$ and $\mathbf{g} \neq \mathbf{c}$.

Let $\varphi : I \to \mathbb{R}$ be a strictly monotone continuous function. Let $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ be n-tuple with probability measures μ_i on Ω, $\mathbf{g} = (g_1, \ldots, g_n)$ be n-tuple with measurable functions $g_i : \Omega \to I$, and $\mathbf{w} = (w_1, \ldots, w_n)$ be n-tuple with weighted functions $w_i \in L^1(\Omega, \mu_i)$ with $\sum_{i=1}^n \int_\Omega w_i \, d\mu_i = 1$ such that $w_i \cdot (\varphi \circ g_i) \in L^1(\Omega, \mu_i)$. The discrete integral φ-quasiarithmetic mean of measurable functions g_i with weighted functions w_i with respect to measures μ_i (namely, with respect to integrals $\int_\Omega \cdot \, d\mu_i$) is a number

$$\mathcal{M}_\varphi(\mathbf{wg}, \boldsymbol{\mu}) = \varphi^{-1}\left(\sum_{i=1}^n \int_\Omega w_i \cdot (\varphi \circ g_i) d\mu_i\right). \tag{3.8}$$

This number belongs to I because the integral convex combination $\sum_{i=1}^n \int_\Omega w_i \cdot (\varphi \circ g_i) d\mu_i \in \varphi(I)$. Integral quasiarithmetic means also satisfy the property

$$\mathcal{M}_{a\varphi+b}(\mathbf{wg}, \boldsymbol{\mu}) = \mathcal{M}_\varphi(\mathbf{wg}, \boldsymbol{\mu}), \tag{3.9}$$

for every pair of real numbers a and b with $a \neq 0$.

Bearing in mind Theorem 3.2, the following corollary is valid.

Corollary 3.3. *Let $\varphi, \psi : I \to \mathbb{R}$ be strictly monotone continuous functions.*

A function ψ is either φ-convex and increasing or φ-concave and decreasing if and only if the inequality

$$\mathcal{M}_\varphi(\mathbf{wg}, \boldsymbol{\mu}) \leq \mathcal{M}_\psi(\mathbf{wg}, \boldsymbol{\mu}), \tag{3.10}$$

holds for all n-tuples $\mathbf{w}$, $\mathbf{g}$, and $\boldsymbol{\mu}$ as in (3.8).

A function ψ is either φ-concave and increasing or φ-convex and decreasing if and only if the reverse inequality is valid in (3.10).

Combining basic and integral case by Corollaries 2.1 and 3.3, we get the following.

Proposition 3.4. *Let $\varphi, \psi : I \to \mathbb{R}$ be strictly monotone continuous functions. Then the inequality*

$$\mathcal{M}_\varphi(\mathbf{px}) \leq \mathcal{M}_\psi(\mathbf{px}), \tag{3.11}$$

holds for all n-tuples $\mathbf{p}$ and $\mathbf{x}$ as in (2.3) *if and only if the inequality*

$$\mathcal{M}_\varphi(\mathbf{wg}, \boldsymbol{\mu}) \leq \mathcal{M}_\psi(\mathbf{wg}, \boldsymbol{\mu}), \tag{3.12}$$

holds for all n-tuples $\mathbf{w}$, $\mathbf{g}$, and $\boldsymbol{\mu}$ as in (3.8).

The one direction of Proposition 3.4 is proved in [2, Theorem 1]. It is proved that the inequality for basic case implies the inequality for integral case with one function g.

The following integral analogy of Theorem 2.4.

Theorem 3.5. *Let $\phi_t = u(t)\varphi + v(t)\psi : I \to \mathbb{R}$ for $t \in [a_0, a_1]$ be functions as in* (2.10). *Let $t_0, t_1 \in [a_0, a_1]$ such that $t_0 \le t_1$.*

If either ψ is φ-convex with both φ and ψ increasing or φ-concave with both φ and ψ decreasing, then the inequality

$$\mathcal{M}_{\varphi}(\mathbf{wg}, \boldsymbol{\mu}) \le \mathcal{M}_{\phi_{t_0}}(\mathbf{wg}, \boldsymbol{\mu}) \le \mathcal{M}_{\phi_{t_1}}(\mathbf{wg}, \boldsymbol{\mu}) \le \mathcal{M}_{\psi}(\mathbf{wg}, \boldsymbol{\mu}), \tag{3.13}$$

holds for all n-tuples $\mathbf{w}$, $\mathbf{g}$, and $\boldsymbol{\mu}$ as in (3.8).

If either ψ is φ-concave with both φ and ψ increasing or φ-convex with both φ and ψ decreasing, then the reverse inequality is valid in (3.13).

The inequality in (3.13) is strict for $a_0 < t_0 < t_1 < a_1$ if ψ is strictly φ-convex, $\mathbf{w} > \mathbf{0}$ and $\mathbf{g} \neq \mathbf{c}$.

An integral version of refinements of the harmonic-geometric-arithmetic inequality is also valid. So, the inequality

$$\begin{aligned}\frac{1}{\sum_{i=1}^{n} \int_{\Omega} (w_i/g_i)\, d\mu_i} &\le \left(\phi_t^{hg}\right)^{-1}\left(\sum_{i=1}^{n} \int_{\Omega} w_i \cdot \ln \frac{e^{u(t)/g_i}}{g_i^{v(t)}}\, d\mu_i\right) \le \prod_{i=1}^{n} \int_{\Omega} g_i^{w_i}\, d\mu_i \\ &\le \left(\phi_t^{ga}\right)^{-1}\left(\sum_{i=1}^{n} \int_{\Omega} w_i \cdot \ln\left(g_i^{u(t)} e^{v(t) g_i}\right) d\mu_i\right) \le \sum_{i=1}^{n} \int_{\Omega} w_i \cdot g_i\, d\mu_i,\end{aligned} \tag{3.14}$$

holds for all n-tuples $\mathbf{w} = (w_1, \ldots, w_n)$, $\mathbf{g} = (g_1, \ldots, g_n)$ and $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ as in (3.8) with $I = \langle 0, +\infty \rangle$. The above inequality is strict for $a_0 < t < a_1$ if all $w_i > 0$ almost everywhere on Ω and $g_i \neq g_j$ almost everywhere on Ω for some $i \neq j$.

As a special case of the integral quasiarithmetic mean in (3.8) with $I = \langle 0, +\infty \rangle$, $\varphi_r(x) = x^r$ for $r \neq 0$, and $\varphi_0(x) = \ln x$, we can observe the integral power mean

$$\mathcal{M}_n^{[r]}(\mathbf{wg}, \boldsymbol{\mu}) = \begin{cases} \left(\displaystyle\sum_{i=1}^{n} \int_{\Omega} w_i \cdot g_i^{r}\, d\mu_i\right)^{1/r} & \text{for } r \neq 0 \\ \exp\left(\displaystyle\sum_{i=1}^{n} \int_{\Omega} w_i \cdot \ln g_i\, d\mu_i\right) & \text{for } r = 0. \end{cases} \tag{3.15}$$

We quote the integral analogy of Corollary 2.6. The following is the property of monotonicity, with refinements, of integral power means.

Corollary 3.6. *Let $r, s \in \mathbb{R}$ such that $r < s$. Let $\phi_t^{[r,s]} : \langle 0, +\infty \rangle \to \mathbb{R}$ for $t \in [a_0, a_1]$ be functions as in* (2.23). *Let $t_0, t_1 \in [a_0, a_1]$ such that $t_0 \le t_1$.*

If $r < s < 0$ or $r < 0 = s$ or $r = 0 < s$ or $0 < r < s$, then the inequality

$$\mathcal{M}_n^{[r]}(\mathbf{wg}, \boldsymbol{\mu}) \le \mathcal{M}_n^{\phi_{t_0}^{[r,s]}}(\mathbf{wg}, \boldsymbol{\mu}) \le \mathcal{M}_n^{\phi_{t_1}^{[r,s]}}(\mathbf{wg}, \boldsymbol{\mu}) \le \mathcal{M}_n^{[s]}(\mathbf{wg}, \boldsymbol{\mu}), \tag{3.16}$$

holds for all n-tuples $\mathbf{w}$, $\mathbf{g}$, and $\boldsymbol{\mu}$ as in (3.8) *with $I = \langle 0, +\infty \rangle$.*

If $r < 0 < s$, then we can take the series of inequalities

$$\begin{aligned} \mathcal{M}_n^{[r]}(\mathbf{wg},\boldsymbol{\mu}) &\leq \mathcal{M}_n^{\phi_{t_0}^{[r,0]}}(\mathbf{wg},\boldsymbol{\mu}) \leq \mathcal{M}_n^{\phi_{t_1}^{[r,0]}}(\mathbf{wg},\boldsymbol{\mu}) \leq \mathcal{M}_n^{[0]}(\mathbf{wg}) \\ &\leq \mathcal{M}_n^{\phi_{t_0}^{[0,s]}}(\mathbf{wg},\boldsymbol{\mu}) \leq \mathcal{M}_n^{\phi_{t_1}^{[0,s]}}(\mathbf{wg},\boldsymbol{\mu}) \leq \mathcal{M}_n^{[s]}(\mathbf{wg},\boldsymbol{\mu}). \end{aligned} \tag{3.17}$$

The inequalities in (3.16)-(3.17) are strict for $a_0 < t_0 < t_1 < a_1$ if $\mathbf{w} > \mathbf{0}$ and $\mathbf{g} \neq \mathbf{c}$.

All the observed integral cases are reduced to the corresponding basic cases when we take constants $g_i = x_i$ and $w_i = p_i$.

4. Applications on Functional Case

Let S be a nonempty set and $\mathcal{S}$ be a vector space of real-valued functions $g : S \to \mathbb{R}$. Linear functional $\mathrm{P} : \mathcal{S} \to \mathbb{R}$ is positive (nonnegative) or monotone if $\mathrm{P}(g) \geq 0$ for every nonnegative function $g \in \mathcal{S}$. If a space $\mathcal{S}$ contains a unit function $\mathbf{1}$, by definition $\mathbf{1}(s) = 1$ for every $s \in S$, and $\mathrm{P}(\mathbf{1}) = 1$, we say that functional P is unital or normalized.

In this section, it is assumed that every weighted function $w : S \to \mathbb{R}$ is nonnegative, that is, $w(s) \geq 0$ for every $s \in S$.

Bellow is a functional form of Jensen's inequality for a convex function with respect to real-valued functions with weighted functions on the vector space of real-valued functions.

Theorem C. *Let $f : I \to \mathbb{R}$ be a continuous function where I is the closed interval. Let $\mathrm{P} : \mathcal{S} \to \mathbb{R}$ be a positive linear functional, $g : S \to I$ be a function, and $w \in \mathcal{S}$ be a weighted function with $\mathrm{P}(w) = 1$ such that $w \cdot g, w \cdot (f \circ g) \in \mathcal{S}$.*

If a function f is convex, then the inequality

$$f(\mathrm{P}(w \cdot g)) \leq \mathrm{P}(w \cdot (f \circ g)), \tag{4.1}$$

holds for all above w, g, and P.

Consequently, if $\mathrm{P}(w) = p > 0$, not necessarily equals 1, then

$$f\left(\frac{1}{p}\mathrm{P}(w \cdot g)\right) \leq \frac{1}{p}\mathrm{P}(w \cdot (f \circ g)). \tag{4.2}$$

If a function f is concave, then the reverse inequality is valid in (4.1) *and* (4.2).

The inequality in (4.1) with $w = 1$ (assuming $\mathbf{1} \in \mathcal{S}$ and $\mathrm{P}(\mathbf{1}) = 1$) is usually called the Jessen functional form of Jensen's inequality.

The interval I must be closed; otherwise, it could happen that $\mathrm{P}(w \cdot g) \notin I$ or $\mathrm{P}(w \cdot (f \circ g)) \notin f(I)$. The following example shows such an undesirable situation.

Example 4.1. Let $S = I = \langle 0, 1]$ and

$$\mathcal{S} = \left\{ g : I \longrightarrow \mathbb{R} \mid \lim_{x \to 0+} g(x) \text{ finite} \right\}. \tag{4.3}$$

If $\mathrm{P} : \mathcal{S} \to \mathbb{R}$ is defined by

$$\mathrm{P}(g) = \lim_{x \to 0+} g(x), \tag{4.4}$$

then P is positive linear functional. In that way, functional P is also unital because $\mathbf{1} \in \mathcal{S}$ and $\mathrm{P}(\mathbf{1}) = 1$. If we take $g(x) = x$ for $x \in I$, then $g \in \mathcal{S}$ and its image in I, but $\mathrm{P}(g) = 0 \notin I$.

Remark 4.2. Suppose that $\mathbf{1} \in \mathcal{S}$ and functional P is unital, that is, $\mathrm{P}(\mathbf{1}) = 1$. Then the reverse of Theorem C is valid if for any $p \in [0,1]$ a subset $S_p \subseteq S$ exists so that $\chi_{S_p} \in \mathcal{S}$ and $\mathrm{P}(\chi_{S_p}) = p$. If we take $g = x\chi_{S_p} + y\chi_{S\setminus S_p}$ and $w = 1$, then it follows that

$$\begin{gathered} \mathrm{P}(w \cdot g) = px + (1-p)y, \\ \mathrm{P}(w \cdot (f \circ g)) = \mathrm{P}\Big(f(x)\chi_{S_p} + f(y)\chi_{S\setminus S_p}\Big) = pf(x) + (1-p)f(y), \end{gathered} \tag{4.5}$$

for every $x, y \in I$ and $p \in [0,1]$. If we include these expressions in the inequality in (4.1), we get the convexity of f.

Theorem C can be generalized to n linear functionals P_i and n functions g_i with weighted functions w_i. The following is a discrete functional form of Jensen's inequality.

Theorem 4.3. *Let $f : I \to \mathbb{R}$ be a continuous function where I is the closed interval. Let $\mathbf{P} = (P_1, \ldots, P_n)$ be n-tuple with positive linear functionals $P_i : \mathcal{S} \to \mathbb{R}$, $\mathbf{g} = (g_1, \ldots, g_n)$ be n-tuple with functions $g_i : S \to I$, and $\mathbf{w} = (w_1, \ldots, w_n)$ be n-tuple with weighted functions $w_i \in \mathcal{S}$ with $\sum_{i=1}^{n} \mathrm{P}_i(w_i) = 1$ such that $w_i \cdot g_i$, $w_i \cdot (f \circ g_i) \in \mathcal{S}$.*

If a function f is convex, then the inequality

$$f\left(\sum_{i=1}^{n} \mathrm{P}_i(w_i \cdot g_i)\right) \le \sum_{i=1}^{n} \mathrm{P}_i(w_i \cdot (f \circ g_i)), \tag{4.6}$$

holds for all above n-tuples $\mathbf{w}$, $\mathbf{g}$, and $\mathbf{P}$.

Consequently, if $\sum_{i=1}^{n} \mathrm{P}_i(w_i) = p > 0$, not necessarily equals 1, then

$$\left(\frac{1}{p}\sum_{i=1}^{n} \mathrm{P}_i(w_i \cdot g_i)\right) \le \frac{1}{p}\sum_{i=1}^{n} \mathrm{P}_i(w_i \cdot (f \circ g_i)). \tag{4.7}$$

If a function f is concave, then the reverse inequality is valid in (4.6) *and* (4.7).

Proof. Let us prove the inequality in (4.6). If $\mathrm{P}_i(w_i) = 0$ for some i, then $\mathrm{P}_i(w_i \cdot g_i) = 0$. Without loss of generality, suppose that all $p_i = \mathrm{P}_i(w_i) > 0$. Let $x_i = (1/p_i)\mathrm{P}_i(w_i \cdot g_i)$. All numbers

x_i belong to I. Then from the basic inequality in (2.1) and functional inequality in (4.2), it follows that

$$\begin{aligned}
f\left(\sum_{i=1}^{n} \mathrm{P}_i(w_i \cdot g_i)\right) &= f\left(\sum_{i=1}^{n} p_i x_i\right) \le \sum_{i=1}^{n} p_i f(x_i) \\
&= \sum_{i=1}^{n} \mathrm{P}_i(w_i) f\left(\frac{1}{\mathrm{P}_i(w_i)} \mathrm{P}_i(w_i \cdot g_i)\right) \\
&\le \sum_{i=1}^{n} \mathrm{P}_i(w_i \cdot (f \circ g_i)).
\end{aligned} \tag{4.8}$$

□

If f is strictly convex, then the inequality in (4.6) and (4.7) is strict for all $\mathbf{w} > \mathbf{0}$ and $\mathbf{g} \neq \mathbf{c}$.

Remark 4.4. Suppose that $\mathbf{1} \in \mathcal{S}$ and all functionals P_i are unital; that is, $\mathrm{P}_i(\mathbf{1}) = 1$ holds for all P_i. Then it is $c \cdot \mathbf{1} \in \mathcal{S}$ and $\mathrm{P}_i(c \cdot \mathbf{1}) = c\mathrm{P}_i(\mathbf{1}) = c$ for every constant $c \in \mathbb{R}$. With the above assumptions, the reverse of Theorem 4.3 follows trivially if we take $w_i = p_i$ and $g_i = x_i$.

Let $\varphi : I \to \mathbb{R}$ be a strictly monotone continuous function where I is the closed interval. Let $\mathbf{P} = (\mathrm{P}_1, \ldots, \mathrm{P}_n)$ be n-tuple with positive linear functionals $\mathrm{P}_i : \mathcal{S} \to \mathbb{R}$, $\mathbf{g} = (g_1, \ldots, g_n)$ be n-tuple with functions $g_i : S \to I$, and $\mathbf{w} = (w_1, \ldots, w_n)$ be n-tuple with weighted functions $w_i \in \mathcal{S}$ with $\sum_{i=1}^{n} \mathrm{P}_i(w_i) = 1$ such that $w_i \cdot (\varphi \circ g_i) \in \mathcal{S}$. The discrete functional φ-quasiarithmetic mean of functions g_i with weighted functions w_i with respect to functionals P_i is a number

$$\mathcal{M}_{\varphi}(\mathbf{wg}, \mathbf{P}) = \varphi^{-1}\left(\sum_{i=1}^{n} \mathrm{P}_i(w_i \cdot (\varphi \circ g_i))\right). \tag{4.9}$$

This number belongs to I because the functional convex combination $\sum_{i=1}^{n} \mathrm{P}_i(w_i \cdot (\varphi \circ g_i))$ belongs to $\varphi(I)$. Functional quasiarithmetic means also satisfy the property

$$\mathcal{M}_{a\varphi+b}(\mathbf{wg}, \mathbf{P}) = \mathcal{M}_{\varphi}(\mathbf{wg}, \mathbf{P}), \tag{4.10}$$

for every pair of real numbers a and b with $a \neq 0$. Indeed, if $\phi(x) = a\varphi(x) + b$, then $\phi^{-1}(x) = \varphi^{-1}(x - b/a)$, and we have

$$\begin{aligned}
\mathcal{M}_{\phi}(\mathbf{wg}, \mathbf{P}) &= \phi^{-1}\left(\sum_{i=1}^{n} \mathrm{P}_i(w_i \cdot (\phi \circ g_i))\right) \\
&= \varphi^{-1}\left(\frac{\sum_{i=1}^{n} \mathrm{P}_i(w_i \cdot (a\varphi \circ g_i + b)) - b}{a}\right) \\
&= \varphi^{-1}\left(\sum_{i=1}^{n} \mathrm{P}_i(w_i \cdot (\varphi \circ g_i))\right) = \mathcal{M}_{\varphi}(\mathbf{wg}, \mathbf{P}).
\end{aligned} \tag{4.11}$$

Corollary 4.5. *Let $\varphi, \psi : I \to \mathbb{R}$ be strictly monotone continuous functions where I is the closed interval.*

If a function ψ is either φ-convex and increasing or φ-concave and decreasing, then the inequality

$$\mathcal{M}_{\varphi}(\mathbf{wg}, \mathbf{P}) \le \mathcal{M}_{\psi}(\mathbf{wg}, \mathbf{P}), \tag{4.12}$$

holds for all n-tuples $\mathbf{w}$, $\mathbf{g}$, and $\mathbf{P}$ as in (4.9).

If a function ψ is either φ-concave and increasing or φ-convex and decreasing, then the reverse inequality is valid in (4.12).

Proof. Suppose that ψ is φ-convex and increasing. If we apply the inequality in (4.6) with $f = \psi \circ \varphi^{-1} : f(I) \to \mathbb{R}$, and $\varphi \circ g_i : S \to f(I)$ instead of g_i, we get

$$\left(\psi \circ \varphi^{-1}\right)\left(\sum_{i=1}^{n} \mathrm{P}_i(w_i \cdot (\varphi \circ g_i))\right) \le \sum_{i=1}^{n} \mathrm{P}_i(w_i \cdot (\psi \circ g_i)). \tag{4.13}$$

After taking ψ^{-1} of the both sides, it follows that

$$\mathcal{M}_{\varphi}(\mathbf{wg}, \mathbf{P}) \le \mathcal{M}_{\psi}(\mathbf{wg}, \mathbf{P}). \tag{4.14}$$

In the same way, we can prove the case when ψ is φ-concave and decreasing. $\square$

According to Remark 4.2, the reverse of Corollary 4.5 is valid if $\mathbf{1} \in \mathcal{S}$ and all functionals P_i are unital. Then we connect the basic and functional case in the following proposition.

Proposition 4.6. *Let $\varphi, \psi : I \to \mathbb{R}$ be strictly monotone continuous functions where I is the closed interval. Then the inequality*

$$\mathcal{M}_{\varphi}(\mathbf{px}) \le \mathcal{M}_{\psi}(\mathbf{px}), \tag{4.15}$$

holds for all $\mathbf{p}$ and $\mathbf{x}$ as in (2.3) *if and only if the inequality*

$$\mathcal{M}_{\varphi}(\mathbf{wg}, \mathbf{P}) \le \mathcal{M}_{\psi}(\mathbf{wg}, \mathbf{P}), \tag{4.16}$$

holds for all n-tuples $\mathbf{w}$, $\mathbf{g}$, and $\mathbf{P}$ as in (4.9) *with $\mathbf{1} \in \mathcal{S}$ and unital functionals P_i.*

Next in line is a functional analogy of refinements.

Theorem 4.7. *Let $\phi_t = u(t)\varphi + v(t)\psi : I \to \mathbb{R}$ for $t \in [a_0, a_1]$ be functions as in* (2.10) *where I is the closed interval. Let $t_0, t_1 \in [a_0, a_1]$ such that $t_0 \le t_1$.*

If either ψ is φ-convex with both φ and ψ increasing or φ-concave with both φ and ψ decreasing, then the inequality

$$\mathcal{M}_{\varphi}(\mathbf{wg}, \mathbf{P}) \le \mathcal{M}_{\phi_{t_0}}(\mathbf{wg}, \mathbf{P}) \le \mathcal{M}_{\phi_{t_1}}(\mathbf{wg}, \mathbf{P}) \le \mathcal{M}_{\psi}(\mathbf{wg}, \mathbf{P}), \tag{4.17}$$

holds for all n-tuples $\mathbf{w}$, $\mathbf{g}$, and $\mathbf{P}$ as in (4.9) *with $\mathbf{1} \in \mathcal{S}$ and unital functionals P_i.*

If either ψ is φ-concave with both φ and ψ increasing or φ-convex with both φ and ψ decreasing, then the reverse inequality is valid in (4.17).

The inequality in (4.17) is strict for $a_0 < t_0 < t_1 < a_1$ if ψ is strictly φ-convex, $\mathbf{w} > \mathbf{0}$ and $\mathbf{g} \neq \mathbf{c}$.

A functional version of refinements of the harmonic-geometric-arithmetic inequality is also valid. So, the inequality

$$\begin{aligned}\frac{1}{\sum_{i=1}^{n} \mathrm{P}_i(w_i/g_i)} &\leq \left(\phi_t^{hg}\right)^{-1}\left(\sum_{i=1}^{n}\mathrm{P}_i\left(w_i \cdot \ln \frac{e^{u(t)/g_i}}{g_i^{v(t)}}\right)\right) \leq \prod_{i=1}^{n}\mathrm{P}_i\left(g_i^{w_i}\right) \\ &\leq \left(\phi_t^{ga}\right)^{-1}\left(\sum_{i=1}^{n}\mathrm{P}_i\left(w_i \cdot \ln\left(g_i^{u(t)} e^{v(t)g_i}\right)\right)\right) \leq \sum_{i=1}^{n}\mathrm{P}_i(w_i \cdot g_i),\end{aligned} \tag{4.18}$$

holds for all n-tuples $\mathbf{w} = (w_1, \dots, w_n)$, $\mathbf{g} = (g_1, \dots, g_n)$ and, $\mathbf{P} = (\mathrm{P}_1, \dots, \mathrm{P}_n)$ as in (4.9) with $I = [a, +\infty\rangle$ where $a > 0$. The above inequality is strict for $a_0 < t < a_1$ if all $w_i > 0$ and $g_i \neq g_j$ for some $i \neq j$.

As a special case of the functional quasiarithmetic mean in (4.9) with $I = [a, +\infty\rangle$ where $a > 0$, $\varphi_r(x) = x^r$ for $r \neq 0$ and $\varphi_0(x) = \ln x$, we can observe the functional power mean

$$\mathcal{M}_n^{[r]}(\mathbf{wg}, \mathbf{P}) = \begin{cases} \left(\displaystyle\sum_{i=1}^{n}\mathrm{P}_i(w_i \cdot g_i^r)\right)^{1/r} & \text{for } r \neq 0 \\ \exp\left(\displaystyle\sum_{i=1}^{n}\mathrm{P}_i(w_i \cdot \ln g_i)\right) & \text{for } r = 0. \end{cases} \tag{4.19}$$

The following is the property of monotonicity, with refinements, of functional power means.

Corollary 4.8. *Let $r, s \in \mathbb{R}$ such that $r < s$. Let $\phi_t^{[r,s]} : [a, +\infty\rangle \to \mathbb{R}$ for $t \in [a_0, a_1]$ be functions as in* (2.23) *where $a > 0$. Let t_0, $t_1 \in [a_0, a_1]$ such that $t_0 \leq t_1$.*

If $r < s < 0$ or $r < 0 = s$ or $r = 0 < s$ or $0 < r < s$, then the inequality

$$\mathcal{M}_n^{[r]}(\mathbf{wg}, \mathbf{P}) \leq \mathcal{M}_n^{\phi_{t_0}^{[r,s]}}(\mathbf{wg}, \mathbf{P}) \leq \mathcal{M}_n^{\phi_{t_1}^{[r,s]}}(\mathbf{wg}, \mathbf{P}) \leq \mathcal{M}_n^{[s]}(\mathbf{wg}, \mathbf{P}), \tag{4.20}$$

holds for all n-tuples $\mathbf{w}$, $\mathbf{g}$, and $\mathbf{P}$ as in (4.9) *with $I = [a, +\infty\rangle$, $\mathbf{1} \in \mathcal{S}$ and unital functionals P_i.*

If $r < 0 < s$, then we can take the series of inequalities

$$\begin{aligned}\mathcal{M}_n^{[r]}(\mathbf{wg}, \mathbf{P}) &\leq \mathcal{M}_n^{\phi_{t_0}^{[r,0]}}(\mathbf{wg}, \mathbf{P}) \leq \mathcal{M}_n^{\phi_{t_1}^{[r,0]}}(\mathbf{wg}, \mathbf{P}) \leq \mathcal{M}_n^{[0]}(\mathbf{wg}, \mathbf{P}) \\ &\leq \mathcal{M}_n^{\phi_{t_0}^{[0,s]}}(\mathbf{wg}, \mathbf{P}) \leq \mathcal{M}_n^{\phi_{t_1}^{[0,s]}}(\mathbf{wg}, \mathbf{P}) \leq \mathcal{M}_n^{[s]}(\mathbf{wg}, \mathbf{P}).\end{aligned} \tag{4.21}$$

The inequalities in (4.20)-(4.21) are strict for $a_0 < t_0 < t_1 < a_1$ if $\mathbf{w} > \mathbf{0}$ and $\mathbf{g} \neq \mathbf{c}$.

All the observed functional cases are reduced to the corresponding integral cases when we take

$$\mathrm{P}_i(g_i) = \int_{\Omega} g_i \, d\mu_i, \tag{4.22}$$

for all functions $g_i \in L^1(\Omega, \mu_i)$ such that $g_i(\omega) \in I$ for almost all $\omega \in \Omega$.

5. Results for Operator Case

We recall some notations and definitions. Let H be a Hilbert space. We define the bounds of linear operator $A : H \to H$ with

$$m_A = \inf_{\|x\|=1} \langle Ax, x \rangle, \qquad M_A = \sup_{\|x\|=1} \langle Ax, x \rangle. \tag{5.1}$$

Let $\mathcal{B}(H)$ be the C^*-algebra of all bounded linear operators $A : H \to H$. If $\mathrm{Sp}(A)$ denotes the spectrum of a self-adjoint operator $A \in \mathcal{B}(H)$, then it is well-known that $\mathrm{Sp}(A)$ is a subset of $\mathbb{R}$ and $\mathrm{Sp}(A) \subseteq [m_A, M_A]$. If 1_H denotes the identity operator on H, then the following holds:

$$\begin{gathered} m_A 1_H \leq A \leq M_A 1_H, \\ \|A\| = \sup_{\|x\|=1} |\langle Ax, x \rangle| = \max\{|m_A|, |M_A|\}. \end{gathered} \tag{5.2}$$

A continuous function $f : I \to \mathbb{R}$ is said to be operator increasing on I if

$$A \leq B \text{ implies } f(A) \leq f(B), \tag{5.3}$$

for every pair of self-adjoint operators A, B on H with spectra in I. A function f is said to be operator decreasing if—f is operator increasing. A function f is operator monotone if it is operator increasing or decreasing.

For n-tuple $\mathbf{A} = (A_1, \ldots, A_n)$ with operators $A_i \in \mathcal{B}(H)$ sometimes, we will write $\mathbf{A} > \mathbf{0}$ if all $A_i > 0$, and $\mathbf{A} \neq \mathrm{C}$ if $A_i \neq A_j$ for some $i \neq j$.

In this section, it is assumed that every weighted operator $W \in \mathcal{B}(H)$ is positive.

From the second half of the last century, Jensen's inequality was formulated for operator convex functions, self-adjoint operators, and positive linear mappings (see [5–8]). Very recently, Jensen's inequality for operators without operator convexity is formulated in [3], and generalized in [4].

The following theorem essentially coincides with the main theorem in [3]. The only difference is that now we add the weighted operators. We also give a short proof of the theorem that relies on the geometric property of convexity and affinity of the chord line or support line. So, we start with an operator form of Jensen's inequality for a convex function with respect to self-adjoint operators with weighted operators on the Hilbert space, and positive linear mappings.

Theorem 5.1. *Let $f : I \to \mathbb{R}$ be a continuous function. Let $\mathbf{\Phi} = (\Phi_1, \ldots, \Phi_n)$ be n-tuple with positive linear mappings $\Phi_i : \mathcal{B}(H) \to \mathcal{B}(K)$, $\mathbf{A} = (A_1, \ldots, A_n)$ be n-tuple with self-adjoint operators $A_i \in \mathcal{B}(H)$ with bounds $m_i \leq M_i$ from I, and $\mathbf{W} = (W_1, \ldots, W_n)$ be n-tuple with weighted operators $W_i \in \mathcal{B}(H)$ with $\sum_{i=1}^n \Phi_i(W_i) = 1_K$. Let $m_B \leq M_B$ be bounds of an operator $B = \sum_{i=1}^n \Phi_i(W_i A_i)$.*

If a function f is convex, then the inequality

$$f\left(\sum_{i=1}^n \Phi_i(W_i A_i)\right) \leq \sum_{i=1}^n \Phi_i\big(W_i f(A_i)\big), \tag{5.4}$$

holds for all above n-tuples $\mathbf{W}$, $\mathbf{A}$, and $\mathbf{\Phi}$ provided spectral conditions

$$[m_B, M_B] \cap [m_i, M_i] = \emptyset \text{ or } \{endpoint\} \quad \text{for } i = 1, \ldots, n. \tag{5.5}$$

Consequently, if $\sum_{i=1}^n \Phi_i(W_i) = W_\Phi$ is strictly positive, not necessarily equals 1_K, then

$$f\left(W_\Phi^{-1} \sum_{i=1}^n \Phi_i(W_i A_i)\right) \leq W_\Phi^{-1} \sum_{i=1}^n \Phi\big(W_i f(A_i)\big). \tag{5.6}$$

If a function f is concave, then the reverse inequality is valid in (5.4) *and* (5.6).

Proof. If $m_B < M_B$, then we take the chord line $f^{\text{cho}}_{[m_B, M_B]}(x) = kx + l$ through the points $T_1(m_B, f(m_B))$, and $T_2(M_B, f(M_B))$. It follows:

$$\begin{aligned} f\left(\sum_{i=1}^n \Phi_i(W_i A_i)\right) &\leq f^{\text{cho}}_{[m_B, M_B]}\left(\sum_{i=1}^n \Phi_i(W_i A_i)\right) \\ &= k \sum_{i=1}^n \Phi_i(W_i A_i) + l 1_K \\ &= \sum_{i=1}^n \Phi_i(W_i(kA_i + l1_H)) \\ &= \sum_{i=1}^n \Phi_i\Big(W_i f^{\text{cho}}_{[m_B, M_B]}(A_i)\Big) \\ &\leq \sum_{i=1}^n \Phi_i\big(W_i f(A_i)\big). \end{aligned} \tag{5.7}$$

If $m_B = M_B$, then we take any support line $f^{\text{sup}}_{[m_B]}(x) = kx + l$ instead of the chord line. □

If f is strictly convex, then the inequality in (5.4) and (5.6) is strict for all $\mathbf{W} > \mathbf{0}$ and $\mathbf{A} \neq \mathbf{C}$.

Remark 5.2. The reverse of Theorem 5.1 is trivially valid if all Φ_i are unital. With this assumption, we can take $W_i = p_i 1_H$ and $A_i = x_i 1_H$.

Let $\varphi : I \to \mathbb{R}$ be a strictly monotone continuous function. Let $\mathbf{\Phi} = (\Phi_1, \ldots, \Phi_n)$ be n-tuple with positive linear mappings $\Phi_i : \mathcal{B}(H) \to \mathcal{B}(K)$, $\mathbf{A} = (A_1, \ldots, A_n)$ be n-tuple with self-adjoint operators $A_i \in \mathcal{B}(H)$ with spectra in I, and $\mathbf{W} = (W_1, \ldots, W_n)$ be n-tuple with weighted operators $W_i \in \mathcal{B}(H)$ with $\sum_{i=1}^{n} \Phi_i(W_i) = 1_K$. The discrete operator φ-quasiarithmetic mean of operators A_i with weighted operators W_i with respect to mappings Φ_i is an operator

$$\mathcal{M}_{\varphi}(\mathbf{WA}, \mathbf{\Phi}) = \varphi^{-1}\left(\sum_{i=1}^{n} \Phi_i\left(W_i \varphi(A_i)\right)\right). \tag{5.8}$$

The spectrum of operator $\mathcal{M}_{\varphi}(\mathbf{WA}, \mathbf{\Phi})$ is contained in I because the spectrum of operator $\sum_{i=1}^{n} \Phi_i(W_i \varphi(A_i))$ is contained in $\varphi(I)$. Operator quasiarithmetic means also have the property

$$\mathcal{M}_{a\varphi+b}(\mathbf{WA}, \mathbf{\Phi}) = \mathcal{M}_{\varphi}(\mathbf{WA}, \mathbf{\Phi}), \tag{5.9}$$

for every pair of real numbers a and b with $a \neq 0$. To verify this equality, let us take $\phi = a\varphi + b$, so $\phi^{-1}(B) = \varphi^{-1}((B - b1_K)/a)$ if $B \in \mathcal{B}(K)$, and we get

$$\begin{aligned}
\mathcal{M}_{\phi}(\mathbf{WA}, \mathbf{\Phi}) &= \phi^{-1}\left(\sum_{i=1}^{n} \Phi_i\left(W_i \phi(A_i)\right)\right) \\
&= \varphi^{-1}\left(\frac{\sum_{i=1}^{n} \Phi_i\left(W_i\left(a\varphi(A_i) + b1_H\right)\right) - b1_K}{a}\right) \\
&= \varphi^{-1}\left(\sum_{i=1}^{n} \Phi_i\left(W_i \varphi(A_i)\right)\right) = \mathcal{M}_{\varphi}(\mathbf{WA}, \mathbf{\Phi}).
\end{aligned} \tag{5.10}$$

Corollary 5.3. *Let $\varphi, \psi : I \to \mathbb{R}$ be strictly monotone continuous functions with operator monotone ψ^{-1}.*

Let $\mathbf{W}$, $\mathbf{A}$ and $\mathbf{\Phi}$ be as in (5.8). *Let $m_i \leq M_i$ and $m_{\varphi} \leq M_{\varphi}$ be bounds of operators A_i and $\mathcal{M}_{\varphi}(\mathbf{WA}, \mathbf{\Phi})$, respectively.*

If a function ψ is either φ-convex with operator increasing ψ^{-1} or φ-concave with operator decreasing ψ^{-1}, then the inequality

$$\mathcal{M}_{\varphi}(\mathbf{WA}, \mathbf{\Phi}) \leq \mathcal{M}_{\psi}(\mathbf{WA}, \mathbf{\Phi}), \tag{5.11}$$

holds for all above n-tuples $\mathbf{W}$, $\mathbf{A}$, and $\mathbf{\Phi}$ provided spectral conditions

$$\left[m_{\varphi}, M_{\varphi}\right] \cap [m_i, M_i] = \emptyset \text{ or } \{\text{endpoint}\} \quad \text{for } i = 1, \ldots, n. \tag{5.12}$$

If a function ψ is either φ-concave with operator increasing ψ^{-1} or φ-convex with operator decreasing ψ^{-1}, then the reverse inequality is valid in (5.11).

The following is operator analogy of Theorem 2.4.

Theorem 5.4. *Let $\phi_t = u(t)\varphi + v(t)\psi : I \to \mathbb{R}$ for $t \in [a_0, a_1]$ be functions as in* (2.10) *with operator monotone ψ^{-1}. Let $t_0, t_1 \in [a_0, a_1]$ such that $t_0 \le t_1$.*

If either ψ is φ-convex with operator increasing $\phi_{t_0}^{-1}$, $\phi_{t_1}^{-1}$, and ψ^{-1} or φ-concave with operator decreasing $\phi_{t_0}^{-1}$, $\phi_{t_1}^{-1}$, and ψ^{-1}, then the inequality

$$\mathcal{M}_{\varphi}(\mathbf{WA}, \mathbf{\Phi}) \le \mathcal{M}_{\phi_{t_0}}(\mathbf{WA}, \mathbf{\Phi}) \le \mathcal{M}_{\phi_{t_1}}(\mathbf{WA}, \mathbf{\Phi}) \le \mathcal{M}_{\psi}(\mathbf{WA}, \mathbf{\Phi}), \tag{5.13}$$

holds for all n-tuples $\mathbf{W}$, $\mathbf{A}$, and $\mathbf{\Phi}$ as in (5.8) *that provided the following spectral conditions:*

$$\begin{gathered}
[m_{\varphi}, M_{\varphi}] \cap [m_i, M_i] = \emptyset \text{ or } \{\text{endpoint}\} \quad \text{for } i = 1, \ldots, n \\
\left[m_{\phi_{t_0}}, M_{\phi_{t_0}}\right] \cap [m_i, M_i] = \emptyset \text{ or } \{\text{endpoint}\} \quad \text{for } i = 1, \ldots, n \\
\left[m_{\phi_{t_1}}, M_{\phi_{t_1}}\right] \cap [m_i, M_i] = \emptyset \text{ or } \{\text{endpoint}\} \quad \text{for } i = 1, \ldots, n.
\end{gathered} \tag{5.14}$$

If either ψ is φ-concave with operator increasing $\phi_{t_0}^{-1}$, $\phi_{t_1}^{-1}$, and ψ^{-1} or φ-convex with operator decreasing $\phi_{t_0}^{-1}$, $\phi_{t_1}^{-1}$, and ψ^{-1}, then the reverse inequality is valid in (5.13).

Proof. Let us prove the middle part of the inequality in (5.13), one that refers to ϕ_{t_0} and ϕ_{t_1}. If ψ is φ-convex with both φ and ψ increasing, then ϕ_{t_1} is ϕ_{t_0}-convex by Proposition 2.3. If $\phi_{t_1}^{-1}$ is operator increasing, then by Corollary 5.3 the inequality

$$\mathcal{M}_{\phi_{t_0}}(\mathbf{WA}, \mathbf{\Phi}) \le \mathcal{M}_{\phi_{t_1}}(\mathbf{WA}, \mathbf{\Phi}), \tag{5.15}$$

is valid with spectral conditions

$$\left[m_{\phi_{t_0}}, M_{\phi_{t_0}}\right] \cap [m_i, M_i] = \emptyset \text{ or } \{\text{endpoint}\} \quad \text{for } i = 1, \ldots, n. \tag{5.16}$$

Any part of the series of inequalities in (5.13) is proved similarly. □

The inequality in (5.13) is strict for $a_0 < t_0 < t_1 < a_1$ if ψ is strictly φ-convex, $\phi_{t_0}^{-1}$, $\phi_{t_1}^{-1}$, andψ^{-1} are strictly operator increasing, $\mathbf{W} > \mathbf{0}$ and $\mathbf{A} \ne \mathbf{C}$.

We are interested in sufficient conditions under which the functions ϕ_t will be operator increasing.

Lemma 5.5. *Let $\phi = \alpha\varphi + \beta\psi$ be a convex combination $(\alpha, \beta \ge 0, \alpha + \beta = 1)$ of strictly monotone increasing or decreasing continuous functions $\varphi, \psi : I \to \mathbb{R}$ such that $\varphi(I) = \psi(I) = J$. Then*

$$\phi^{-1} = u \cdot \varphi^{-1} + v \cdot \psi^{-1}, \tag{5.17}$$

where $u, v : J \to \mathbb{R}$ are nonnegative continuous functions such that $u(y) + v(y) = 1$ for every $y \in J$.

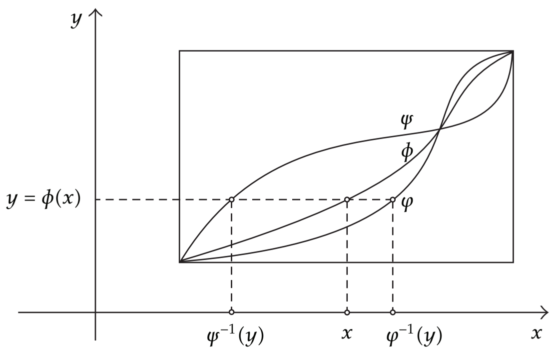

Figure 3: Convex combination of strictly monotone increasing functions φ and ψ.

Proof. Take any $x \in I$. If

$$y = \phi(x) = \alpha\varphi(x) + \beta\psi(x), \tag{5.18}$$

then

$$x = \phi^{-1}(y) = u(y) \cdot \varphi^{-1}(y) + v(y) \cdot \psi^{-1}(y), \tag{5.19}$$

for some nonnegative numbers $u(y)$ and $v(y)$ such that $u(y) + v(y) = 1$ (see Figure 3). Now, first replace $v(y)$ with $1 - u(y)$ in expression in (5.19), and then express $u(y)$. Realizing $u(y)$ as a function of the variable y, we obtain that

$$u(y) = \begin{cases} \dfrac{\phi^{-1}(y) - \psi^{-1}(y)}{\varphi^{-1}(y) - \psi^{-1}(y)} & \text{for } \varphi^{-1}(y) \neq \psi^{-1}(y) \\ \lim\limits_{y \to y_0} \dfrac{\phi^{-1}(y) - \psi^{-1}(y)}{\varphi^{-1}(y) - \psi^{-1}(y)} & \text{for } \varphi^{-1}(y_0) = \psi^{-1}(y_0). \end{cases} \tag{5.20}$$

The above limit is onesided if $\varphi^{-1} = \psi^{-1}$ on some subinterval of an interval J. The functions φ^{-1}, ψ^{-1}, and ϕ^{-1} are continuous on J, and the same is true for the function u. Thus, the expression in (5.19) is the required presentation of function ϕ^{-1} as the convex combination of functions φ^{-1} and ψ^{-1} with coefficient functions u and v. □

Theorem 5.4 can be simplified by using Lemma 5.5.

Corollary 5.6. *Let $\phi_t = u(t)\varphi + v(t)\psi : I \to \mathbb{R}$ for $t \in [a_0, a_1]$ be functions as in* (2.10) *with the addition of $u(t) + v(t) = 1$, $\varphi(I) = \psi(I)$ and operator monotone ψ^{-1}. Let $t_0, t_1 \in [a_0, a_1]$ such that $t_0 \leq t_1$.*

If either ψ is φ-convex with operator increasing ψ^{-1} or φ-concave with operator decreasing ψ^{-1}, then the inequality in (5.13) *holds for all n-tuples $\mathbf{W}$, $\mathbf{A}$, and $\mathbf{\Phi}$ as in* (5.8) *with spectral conditions as in Theorem 5.4.*

If either ψ is φ-concave with operator increasing ψ^{-1} or φ-convex with operator decreasing ψ^{-1}, then the reverse inequality is valid in (5.13).

Proof. According to Lemma 5.5 continuous functions u_t and v_t, with $u_t + v_t = 1$, exist for every $t \in [a_0, a_1]$ so that

$$\phi_t^{-1} = u_t \cdot \varphi^{-1} + v_t \cdot \psi^{-1} = u_t \cdot \left(\varphi^{-1} - \psi^{-1}\right) + \psi^{-1}. \tag{5.21}$$

Let $t > a_0$; otherwise, it is $\phi_{a_0}^{-1} = \varphi^{-1}$. Then $v_t \neq 0$ and ϕ_t^{-1} is operator increasing (resp. decreasing) if ψ^{-1} is operator increasing (resp. decreasing). □

A special case of the operator quasiarithmetic mean in (5.8) with $I = \langle 0, +\infty\rangle$, $\varphi_r(x) = x^r$ for $r \neq 0$ and $\varphi_0(x) = \ln x$, we can observe the operator power mean

$$\mathcal{M}_n^{[r]}(\mathbf{WA}, \boldsymbol{\Phi}) = \begin{cases} \left(\displaystyle\sum_{i=1}^{n} \Phi_i\left(W_i A_i^r\right)\right)^{1/r} & \text{for } r \neq 0 \\ \exp\left(\displaystyle\sum_{i=1}^{n} \Phi_i\left(W_i \ln A_i\right)\right) & \text{for } r = 0. \end{cases} \tag{5.22}$$

The consequence of Corollary 5.3 for operator power means the following.

Corollary 5.7. *Let r and s be real numbers such that $r \leq s$.*

Let $\mathbf{W}$, $\mathbf{A}$, and $\boldsymbol{\Phi}$ be as in (5.8) with strictly positive $\mathbf{A}$. Let $m_i \leq M_i$, $m^{[r]} \leq M^{[r]}$, and $m^{[s]} \leq M^{[s]}$ be bounds of operators A_i, $\mathcal{M}_n^{[r]}(\mathbf{WA}, \boldsymbol{\Phi})$, and $\mathcal{M}_n^{[s]}(\mathbf{WA}, \boldsymbol{\Phi})$, respectively.

If $s \leq -1$ or $s \geq 1$, then the inequality

$$\mathcal{M}_n^{[r]}(\mathbf{WA}, \boldsymbol{\Phi}) \leq \mathcal{M}_n^{[s]}(\mathbf{WA}, \boldsymbol{\Phi}), \tag{5.23}$$

holds for all above n-tuples $\mathbf{W}$, $\mathbf{A}$, and $\boldsymbol{\Phi}$ provided spectral conditions

$$\left[m^{[r]}, M^{[r]}\right] \cap [m_i, M_i] = \emptyset \text{ or } \{\text{endpoint}\} \quad \text{for } i = 1, \ldots, n. \tag{5.24}$$

If $r \leq -1$ or $r \geq 1$, then the inequality in (5.23) holds provided spectral conditions

$$\left[m^{[s]}, M^{[s]}\right] \cap [m_i, M_i] = \emptyset \text{ or } \{\text{endpoint}\} \quad \text{for } i = 1, \ldots, n. \tag{5.25}$$

The inequality in (5.23) is strict for $r < s$ if $\mathbf{W} > \mathbf{0}$ and $\mathbf{A} \neq \mathbf{C}$.

The proof of Corollary 5.7 is the same as the proof of [3, Corollary 7].

An operator version of the harmonic-geometric-arithmetic inequality is the consequence of Corollary 5.7. The inequality $\mathcal{M}_n^{[-1]}(\mathbf{WA}, \boldsymbol{\Phi}) \leq \mathcal{M}_n^{[0]}(\mathbf{WA}, \boldsymbol{\Phi}) \leq \mathcal{M}_n^{[1]}(\mathbf{WA}, \boldsymbol{\Phi})$, that is,

$$\left(\sum_{i=1}^{n} \Phi_i\left(W_i A_i^{-1}\right)\right)^{-1} \leq \exp\left(\sum_{i=1}^{n} \Phi_i\left(W_i \ln A_i\right)\right) \leq \sum_{i=1}^{n} \Phi_i\left(W_i A_i\right), \tag{5.26}$$

holds for all n-tuples $\mathbf{W} = (W_1, \ldots, W_n)$, $\mathbf{A} = (A_1, \ldots, A_n)$ and $\mathbf{\Phi} = (\Phi_1, \ldots, \Phi_n)$ as in (5.8) with strictly positive A_i provided spectral conditions

$$\left[m^{[0]}, M^{[0]}\right] \cap [m_i, M_i] = \emptyset \text{ or } \{\text{endpoint}\} \quad \text{for } i = 1, \ldots, n. \tag{5.27}$$

There remain only the refinements of the operator power means by using Corollary 5.6.

Corollary 5.8. *Let $r, s \in \mathbb{R} \setminus \{0\}$ such that $r < s$. Let $\phi_t^{[r,s]} : \langle 0, +\infty\rangle \to \mathbb{R}$ for $t \in [a_0, a_1]$ be functions as in* (2.23) *with the addition of $u(t) + v(t) = 1$. Let $t_0, t_1 \in [a_0, a_1]$ such that $t_0 \le t_1$.*

Let $\mathbf{W}$, $\mathbf{A}$, and $\mathbf{\Phi}$ be as in (5.8) *with strictly positive $\mathbf{A}$. Let $m_i \le M_i$, $m^{[r]} \le M^{[r]}$, $m^{[s]} \le M^{[s]}$, $m^{[\phi_{t_0}^{[r,s]}]} \le M^{[\phi_{t_0}^{[r,s]}]}$, and $m^{[\phi_{t_1}^{[r,s]}]} \le M^{[\phi_{t_1}^{[r,s]}]}$ be bounds of operators A_i, $\mathcal{M}_n^{[r]}(\mathbf{WA}, \mathbf{\Phi})$, $\mathcal{M}_n^{[s]}(\mathbf{WA}, \mathbf{\Phi})$, $\mathcal{M}_n^{\phi_{t_0}^{[r,s]}}(\mathbf{WA}, \mathbf{\Phi})$, and $\mathcal{M}_n^{\phi_{t_1}^{[r,s]}}(\mathbf{WA}, \mathbf{\Phi})$, respectively.*

If $r > 0$, $s \ge 1$ or $s = -1$, then the inequality

$$\mathcal{M}_n^{[r]}(\mathbf{WA}, \mathbf{\Phi}) \le \mathcal{M}_n^{\phi_{t_0}^{[r,s]}}(\mathbf{WA}, \mathbf{\Phi}) \le \mathcal{M}_n^{\phi_{t_1}^{[r,s]}}(\mathbf{WA}, \mathbf{\Phi}) \le \mathcal{M}_n^{[s]}(\mathbf{WA}, \mathbf{\Phi}), \tag{5.28}$$

holds for all above n-tuples $\mathbf{W}$, $\mathbf{A}$, and $\mathbf{\Phi}$ provided spectral conditions:

$$\begin{gathered}
\left[m^{[r]}, M^{[r]}\right] \cap [m_i, M_i] = \emptyset \text{ or } \{\text{endpoint}\} \quad \text{for } i = 1, \ldots, n, \\
\left[m^{[\phi_{t_0}^{[r,s]}]}, M^{[\phi_{t_0}^{[r,s]}]}\right] \cap [m_i, M_i] = \emptyset \text{ or } \{\text{endpoint}\} \quad \text{for } i = 1, \ldots, n, \\
\left[m^{[\phi_{t_1}^{[r,s]}]}, M^{[\phi_{t_1}^{[r,s]}]}\right] \cap [m_i, M_i] = \emptyset \text{ or } \{\text{endpoint}\} \quad \text{for } i = 1, \ldots, n.
\end{gathered} \tag{5.29}$$

If $r = 1$, then the inequality

$$\mathcal{M}_n^{[1]}(\mathbf{WA}, \mathbf{\Phi}) \le \mathcal{M}_n^{\phi_{t_0}^{[s,1]}}(\mathbf{WA}, \mathbf{\Phi}) \le \mathcal{M}_n^{\phi_{t_1}^{[s,1]}}(\mathbf{WA}, \mathbf{\Phi}) \le \mathcal{M}_n^{[s]}(\mathbf{WA}, \mathbf{\Phi}), \tag{5.30}$$

holds for all above n-tuples $\mathbf{W}$, $\mathbf{A}$, and $\mathbf{\Phi}$ provided spectral conditions:

$$\begin{gathered}
\left[m^{[s]}, M^{[s]}\right] \cap [m_i, M_i] = \emptyset \text{ or } \{\text{endpoint}\} \quad \text{for } i = 1, \ldots, n, \\
\left[m^{[\phi_{t_0}^{[s,1]}]}, M^{[\phi_{t_0}^{[s,1]}]}\right] \cap [m_i, M_i] = \emptyset \text{ or } \{\text{endpoint}\} \quad \text{for } i = 1, \ldots, n, \\
\left[m^{[\phi_{t_1}^{[s,1]}]}, M^{[\phi_{t_1}^{[s,1]}]}\right] \cap [m_i, M_i] = \emptyset \text{ or } \{\text{endpoint}\} \quad \text{for } i = 1, \ldots, n.
\end{gathered} \tag{5.31}$$

Proof. Recall that $\phi_t^{[r,s]}(x) = u(t)x^r + v(t)x^s$ for $r \ne 0$, $s \ne 0$. The next is $\varphi(x) = x^r$ and $\psi(x) = x^s$ with $x \in I = \langle 0, +\infty\rangle$, so $\varphi(I) = \psi(I) = I$.

Case $r > 0$, $s \geq 1$.

We have $(\psi \circ \varphi^{-1})(x) = x^{s/r}$ with $(s/r) > 1$, and $\psi^{-1}(x) = x^{1/s}$ with $0 < (1/s) \leq 1$. The function ψ is strictly φ-convex with strictly operator increasing ψ^{-1}. Then the inequality in (5.28) is valid with associated spectral conditions by Corollary 5.6.

Case $s = -1$.

We have $(\psi \circ \varphi^{-1})(x) = x^{-1/r}$ with $0 < -(1/r) < 1$, and $\psi^{-1}(x) = x^{-1}$. The function ψ is strictly φ-concave with strictly operator decreasing ψ^{-1}. In this case, the inequality in (5.28) is also valid with associated spectral conditions by Corollary 5.6.

Case $r = 1$.

We use functions $\phi_t^{[s,1]}(x) = u(t)x^s + v(t)x$. In this case $\varphi(x) = x^s$ and $\psi(x) = x$, thus $(\psi \circ \varphi^{-1})(x) = x^{1/s}$ with $0 < (1/s) < 1$, and $\psi^{-1}(x) = x$. The function ψ is strictly φ-concave with strictly operator increasing ψ^{-1}. Then the inequality in (5.30) is valid with associated spectral conditions by Corollary 5.6. □

The inequalities in (5.28)–(5.30) are strict for $a_0 < t_0 < t_1 < a_1$ if $\mathbf{W} > \mathbf{0}$ and $\mathbf{A} \neq \mathbf{C}$. Unfortunately, we cannot use a logarithmic function because $\ln(I) = \mathbb{R} \neq I$.

Remark 5.9. Let r, s, ϕ_t, t_0, t_1,$\mathbf{W}$, $\mathbf{A}$, and $\mathbf{\Phi}$ be as in Corollary 5.8.

If $r \leq -1$ and $s < 0$, then the problem remains the inequality

$$\mathcal{M}_n^{[r]}(\mathbf{WA}, \mathbf{\Phi}) \leq \mathcal{M}_n^{\phi_{t_0}^{[s,r]}}(\mathbf{WA}, \mathbf{\Phi}) \leq \mathcal{M}_n^{\phi_{t_1}^{[s,r]}}(\mathbf{WA}, \mathbf{\Phi}) \leq \mathcal{M}_n^{[s]}(\mathbf{WA}, \mathbf{\Phi}), \tag{5.32}$$

valid for all above n-tuples $\mathbf{W}$, $\mathbf{A}$, and $\mathbf{\Phi}$ provided spectral conditions:

$$\begin{aligned} &\left[m^{[s]}, M^{[s]}\right] \cap [m_i, M_i] = \emptyset \text{ or } \{\text{endpoint}\} \quad \text{for } i = 1, \ldots, n, \\ &\left[m^{[\phi_{t_0}^{[s,r]}]}, M^{[\phi_{t_0}^{[s,r]}]}\right] \cap [m_i, M_i] = \emptyset \text{ or } \{\text{endpoint}\} \quad \text{for } i = 1, \ldots, n, \\ &\left[m^{[\phi_{t_1}^{[s,r]}]}, M^{[\phi_{t_1}^{[s,r]}]}\right] \cap [m_i, M_i] = \emptyset \text{ or } \{\text{endpoint}\} \quad \text{for } i = 1, \ldots, n. \end{aligned} \tag{5.33}$$

The inequality in (5.32) is valid for $r = -1$ with associated spectral conditions because $(\psi \circ \varphi^{-1})(x) = x^{-1/s}$ with $-(1/s) > 1$, that is, ψ is φ-convex, and $\psi^{-1}(x) = x^{-1}$, that is, ψ^{-1} is operator decreasing.

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