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Research Article

Existence of Bounded Positive Solutions for Partial Difference Equations with Delays

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This paper deals with solvability of the third-order nonlinear partial difference equation with delays $\Delta_n(a_{m,n}\Delta_m^2(x_{m,n}+b_{m,n}x_{m-\tau_0,n-\sigma_0}))+f(m,n,x_{m-\tau_{1,m},n-\sigma_{1,n}},\ldots,x_{m-\tau_{k,m},n-\sigma_{k,n}})=c_{m,n}$, $m\geq m_0$, $n\geq n_0$. With the help of the Banach fixed-point theorem, the existence results of uncountably many bounded positive solutions for the partial difference equation are given; some Mann iterative schemes with errors are suggested, and the error estimates between the iterative schemes and the bounded positive solutions are discussed. Three nontrivial examples illustrating the results presented in this paper are also provided.

1. Introduction and Preliminaries

In the past twenty years many authors studied the oscillation, nonoscillation, asymptotic behavior, and solvability for various neutral delay difference and partial difference equations; see, for example, [1–14] and the references cited therein.

By using the Banach fixed-point theorem, Cheng [2] investigated the existence of a nonoscillatory solution for the second-order neutral delay difference equation with positive and negative coefficients

$$\Delta^2(x_n+px_{n-m})+p_nx_{n-k}-q_nx_{n-l}=0,\quad n\geq n_0 \tag{1.1}$$

under the condition $p \in \mathbb{R}\setminus\{-1\}$. Applying a nonlinear alternative of Leray-Schauder type for condensing operators, Agarwal et al. [1] discussed the existence of a bounded nonoscillatory solution for the discrete equation:

$$\Delta\left(a_n\Delta\left(x_n + px_{n-\tau}\right)\right) + F(n+1, x_{n+1-\sigma}) = 0, \quad n \geq 0. \tag{1.2}$$

Liu et al. [6] introduced the second-order nonlinear neutral delay difference equation

$$\Delta(a_n\Delta(x_n + bx_{n-\tau})) + f(n, x_{n-d_{1n}}, x_{n-d_{2n}}, \ldots, x_{n-d_{kn}}) = c_n, \quad n \geq 0 \tag{1.3}$$

with respect to all $b \in \mathbb{R}$ and gave the existence of uncountably many bounded nonoscillatory solutions for (1.3) by utilizing the Banach fixed-point theorem. Kong et al. [3] investigated a class of BVPs for the third-order functional difference equation

$$\Delta^3 x_n + a_n f\left(n, x_{w(n)}\right) = 0, \quad n \geq 0 \tag{1.4}$$

and established the existence of positive solutions for (1.4) under certain conditions. Using the Schauder fixed-point theorem, Yan and Liu [12] studied the existence of a bounded nonoscillatory solution for third order nonlinear delay difference equation

$$\Delta^3 x_n + f(n, x_n, x_{n-r}) = 0, \quad n \geq n_0 \tag{1.5}$$

and provided also a necessary and sufficient condition for the existence of a bounded nonoscillatory solution of (1.5).

Karpuz and Öcalan [4] discussed the first-order linear partial difference equation:

$$x_{m+1,n} + x_{m,n+1} - x_{m,n} + p_{m,n}x_{m-k,n-l} = 0, \quad (m,n) \in \mathbb{Z}_{0,0}, \tag{1.6}$$

where $\{p_{m,n}\}_{(m,n)\in\mathbb{Z}_{0,0}}$ is a nonnegative sequence and $k, l \in \mathbb{N}_1$ and obtained sufficient conditions under which every solution of (1.6) is oscillatory. Yang and Zhang [14] considered oscillations of the partial difference equation with several nonlinear terms of the form

$$x_{m+1,n} + x_{m,n+1} - x_{m,n} + \sum_{i=1}^{h} p_i(m,n)|x_{m-k_i,n-l_i}|^{\alpha_i} \operatorname{sgn} x_{m-k_i,n-l_i} = 0 \tag{1.7}$$

and established some new oscillatory criteria by making use of frequency measures. Wong and Agarwal [10] considered the partial difference equations

$$x_{m+1,n} + \beta_{m,n}x_{m,n+1} - \delta_{m,n}x_{m,n} + p(m,n,x_{m-k,n-l}) = Q(m,n,x_{m-k,n-l}), \quad m \geq m_0,\ n \geq n_0, \tag{1.8}$$

$$x_{m+1,n} + \beta_{m,n}x_{m,n+1} - \delta_{m,n}x_{m,n} + \sum_{i=1}^{\tau} p_i(m,n,x_{m-k,n-l}) = \sum_{i=1}^{\tau} Q(m,n,x_{m-k,n-l}), \quad m \geq m_0,\ n \geq n_0 \tag{1.9}$$

and offered sufficient conditions for the oscillation of all solutions for (1.8) and (1.9), respectively. Wong [9] established the existence of eventually positive and monotone decreasing solutions for the partial difference inequalities

$$\Delta_m \Delta_n x_{m,n} + \sum_{i=1}^{r} p_i\left(m,n,x_{g_i(m),h_i(n)}\right) \geq (\leq) \sum_{i=1}^{r} Q_i\left(m,n,x_{g_i(m),h_i(n)}\right), \quad m \geq m_0,\ n \geq n_0, \tag{1.10}$$

where $g_i(m)$ and $h_i(m)$ are some deviating arguments for $1 \leq i \leq \tau$.

However, to the best of our knowledge, there is no literature referred to the following third order nonlinear partial difference equation with delays:

$$\begin{aligned} &\Delta_n\left(a_{m,n}\Delta_m^2\left(x_{m,n} + b_{m,n}x_{m-\tau_0,n-\sigma_0}\right)\right) + f\left(m,n,x_{m-\tau_{1,m},n-\sigma_{1,n}},\ldots,x_{m-\tau_{k,m},n-\sigma_{k,n}}\right) \\ &\quad = c_{m,n}, \quad m \geq m_0,\ n \geq n_0, \end{aligned} \tag{1.11}$$

where $m_0, n_0 \in \mathbb{N}_0$, $k, \tau_0, \sigma_0 \in \mathbb{N}$, $\{a_{m,n}\}_{(m,n)\in\mathbb{N}_{m_0,n_0}}$, $\{b_{m,n}\}_{(m,n)\in\mathbb{N}_{m_0,n_0}}$, $\{c_{m,n}\}_{(m,n)\in\mathbb{N}_{m_0,n_0}}$ are real sequences with $a_{m,n} \neq 0$, $b_{m,n} \neq \pm 1$ for $(m,n) \in \mathbb{N}_{m_0,n_0}$, $f : \mathbb{N}_{m_0,n_0} \times \mathbb{R}^k \to \mathbb{R}$ and $\{\tau_{l,m}, \sigma_{l,n} : (m,n) \in \mathbb{N}_{m_0,n_0},\ l \in \{1,2,\ldots,k\}\} \subseteq \mathbb{Z}$ with

$$\lim_{m\to\infty}(m - \tau_{l,m}) = \lim_{n\to\infty}(n - \sigma_{l,n}) = +\infty, \quad l \in \{1,2,\ldots,k\}. \tag{1.12}$$

The aim of this paper is to establish three sufficient conditions of the existence of uncountably many bounded positive solutions for (1.11) by using the Banach fixed-point theorem, to suggest some Mann iterative methods with errors for these bounded positive solutions and to compute the error estimates between the bounded positive solutions and the sequences generated by the Mann iterative methods with errors. In order to explain the results presented in this paper, three nontrivial examples are constructed.

Throughout this paper, the forward partial difference operators Δ_m and Δ_n are defined by $\Delta_m x_{m,n} = x_{m+1,n} - x_{m,n}$ and $\Delta_n x_{m,n} = x_{m,n+1} - x_{m,n}$, respectively the second and third-order partial difference operators are defined by $\Delta_m^2 x_{m,n} = \Delta_m(\Delta_m x_{m,n})$ and $\Delta_n\Delta_m^2 x_{m,n} = \Delta_n(\Delta_m^2 x_{m,n})$, respectively. Let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{N}$ and $\mathbb{Z}$ denote the sets of all positive integers and integers, respectively,

$$\begin{aligned} \mathbb{N}_0 &= \{0\} \cup \mathbb{N},\ \mathbb{N}_s = \{n : n \in \mathbb{N}_0 \text{ with } n \geq s\}, \quad s \in \mathbb{N}_0, \\ \mathbb{N}_{s,t} &= \{(m,n) : m,n \in \mathbb{N}_0 \text{ with } m \geq s, n \geq t\}, \quad s,t \in \mathbb{N}_0, \\ \mathbb{Z}_{s,t} &= \{(m,n) :\ m,n \in \mathbb{Z} \text{ with } m \geq s, n \geq t\}, \quad s,t \in \mathbb{Z}, \\ \alpha &= \min\{m - \tau_0, m - \tau_{l,m} : 1 \leq l \leq k, m \in \mathbb{N}_{m_0}\}, \\ \beta &= \min\{n - \sigma_0, n - \sigma_{l,n} : 1 \leq l \leq k, n \in \mathbb{N}_{n_0}\}. \end{aligned} \tag{1.13}$$

$l^{\infty}_{\alpha,\beta}$ represents the Banach space of all bounded sequences on $\mathbb{Z}_{\alpha,\beta}$ with the norm

$$\|x\| = \sup_{m,n\in\mathbb{Z}_{\alpha,\beta}} |x_{m,n}| \quad \text{for } x = \{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in l^{\infty}_{\alpha,\beta},$$

$$A(N,M) = \left\{ x = \{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in l^{\infty}_{\alpha,\beta} : N \le x_{m,n} \le M, (m,n) \in \mathbb{Z}_{\alpha,\beta} \right\} \quad \text{for } M > N > 0. \tag{1.14}$$

It is not difficult to see that $A(N,M)$ is a bounded closed and convex subset of the Banach space $l^{\infty}_{\alpha,\beta}$. By a solution of (1.11), we mean a sequence $\{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}}$ with positive integers $m_1 \ge m_0 + \tau_0 + |\alpha|$ and $n_1 \ge n_0 + \sigma_0 + |\beta|$ such that (1.11) is satisfied for all $m \ge m_1$ and $n \ge n_1$.

Lemma 1.1 (see [15]). *Let $\{\alpha(n)\}_{n\in\mathbb{N}_0}$, $\{\beta(n)\}_{n\in\mathbb{N}_0}$, $\{\gamma(n)\}_{n\in\mathbb{N}_0}$, and $\{t(n)\}_{n\in\mathbb{N}_0}$ be nonnegative sequences satisfying the inequality*

$$\alpha(n+1) \le (1 - t(n))\alpha(n) + t(n)\beta(n) + \gamma(n), \quad n \in \mathbb{N}_0, \tag{1.15}$$

where $\{t(n)\}_{n\in\mathbb{N}_0} \subset [0,1]$ with $\sum_{n=0}^{\infty} t(n) = +\infty$, $\lim_{n\to\infty}\beta(n) = 0$ and $\sum_{n=0}^{\infty}\gamma(n) < +\infty$. Then $\lim_{n\to\infty}\alpha(n) = 0$.

2. Existence of Uncountably Many Bounded Positive Solutions and Mann Iterative Schemes with Errors

Utilizing the Banach fixed-point theorem, we now investigate the existence of uncountably many bounded positive solutions for (1.11), suggest the Mann type iterative schemes with errors and discuss the error estimates between the bounded positive solutions and the sequences generated by the Mann iterative schemes.

Theorem 2.1. *Assume that there exists positive constants M and N, nonnegative constants b_1 and b_2, and nonnegative sequences $\{P_{m,n}\}_{(m,n)\in\mathbb{N}_{m_0,n_0}}$ and $\{Q_{m,n}\}_{(m,n)\in\mathbb{N}_{m_0,n_0}}$ satisfying*

$$b_1 + b_2 < 1, \quad N < [1 - (b_1 + b_2)]M, \tag{2.1}$$

$$-b_2 \le b_{m,n} \le b_1, \quad \textit{eventually}, \tag{2.2}$$

$$\begin{aligned} &\left|f(m,n,u_1,u_2,\ldots,u_k) - f(m,n,\overline{u}_1,\overline{u}_2,\ldots,\overline{u}_k)\right| \le P_{m,n}\max\{|u_l - \overline{u}_l| : 1 \le l \le k\}, \\ &(m,n,u_l,\overline{u}_l) \in \mathbb{N}_{m_0,n_0} \times [N,M]^2, \quad 1 \le l \le k, \end{aligned} \tag{2.3}$$

$$\left|f(m,n,u_1,u_2,\ldots,u_k)\right| \le Q_{m,n}, \quad (m,n,u_l) \in \mathbb{N}_{m_0,n_0} \times [N,M],\ 1 \le l \le k; \tag{2.4}$$

$$\sum_{j=m_0}^{\infty}\sum_{i=j}^{\infty}\sup_{n\in\mathbb{N}_{n_0}}\left\{\frac{1}{|a_{i,n}|}\sum_{t=n}^{\infty}\max\{P_{i,t}, Q_{i,t}, |c_{i,t}|\}\right\} < +\infty. \tag{2.5}$$

Then

(a) *for any* $L \in (N + b_1 M, (1 - b_2)M)$, *there exist* $\theta \in (0,1)$, $m_1 \geq m_0 + \tau_0 + |\alpha|$ *and* $n_1 \geq n_0 + \sigma_0 + |\beta|$ *such that for any* $x(0) = \{x_{m,n}(0)\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in A(N,M)$, *the Mann iterative sequence with errors* $\{x(s)\}_{s\in\mathbb{N}_0} = \{x_{m,n}(s)\}_{(m,n,s)\in\mathbb{Z}_{\alpha,\beta}\times\mathbb{N}_0}$ *generated by the scheme:*

$$x_{m,n}(s+1) = \begin{cases} [1-\alpha(s)-\beta(s)]x_{m,n}(s) + \alpha(s) \\ \quad \times \left\{ L - b_{m,n}x_{m-\tau_0,n-\sigma_0}(s) + \sum_{j=m}^{\infty}\sum_{i=j}^{\infty}\frac{1}{a_{i,n}} \right. \\ \qquad \left. \times \sum_{t=n}^{\infty}\left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}}(s),\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}(s)\right) - c_{i,t}\right] \right\} \\ \quad + \beta(s)\gamma_{m,n}(s), \quad (m,n) \in \mathbb{Z}_{m_1,n_1},\ s \in \mathbb{N}_0, \\ [1-\alpha(s)-\beta(s)]_{x_{m_1,n_1}}(s) + \alpha(s) \\ \quad \times \left\{ L - b_{m_1,n_1}x_{m_1-\tau_0,n_1-\sigma_0}(s) + \sum_{j=m_1}^{\infty}\sum_{i=j}^{\infty}\frac{1}{a_{i,n_1}}\sum_{t=n_1}^{\infty} \right. \\ \qquad \left. \times \left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}}(s),\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}(s)\right) - c_{i,t}\right] \right\} \\ \quad + \beta(s)\gamma_{m_1,n_1}(s), \quad (m,n) \in \mathbb{Z}_{\alpha,\beta} \setminus \mathbb{Z}_{m_1,n_1},\ s \in \mathbb{N}_0, \end{cases} \tag{2.6}$$

converges to a bounded positive solution $x \in A(M,N)$ *of* (1.11) *and has the following error estimate:*

$$\|x(s+1) - x\| \leq [1 - (1-\theta)\alpha(s)]\|x(s) - x\| + 2M\beta(s), \quad s \in \mathbb{N}_0, \tag{2.7}$$

where $\{\gamma(s)\}_{s\in\mathbb{N}_0}$ *is an arbitrary sequence in* $A(M,N)$, $\{\alpha(s)\}_{s\in\mathbb{N}_0}$ *and* $\{\beta(s)\}_{s\in\mathbb{N}_0}$ *are any sequences in* $[0,1]$ *such that*

$$\sum_{s=0}^{\infty}\alpha(s) = +\infty, \tag{2.8}$$

$$\sum_{s=0}^{\infty}\beta(s) < +\infty \text{ or there exists a sequence } \{\xi(s)\}_{s\in\mathbb{N}_0} \subseteq [0,+\infty) \text{ satisfying}$$
$$\beta(s) = \xi(s)\alpha(s), \quad s \in \mathbb{N}_0, \qquad \lim_{s\to\infty}\xi(s) = 0; \tag{2.9}$$

(b) (1.11) *possesses uncountably many bounded positive solutions in* $A(M,N)$.

Proof. First of all we show that (a) holds. Set $L \in (N + b_1M, (1 - b_2)M)$. It follows from (2.1), (2.2), and (2.5) that there exist $\theta \in (0,1)$, $m_1 \geq m_0 + \tau_0 + |\alpha|$ and $n_1 \geq n_0 + \sigma_0 + |\beta|$ such that

$$\theta = b_1 + b_2 + \sum_{j=m_1}^{\infty}\sum_{i=j}^{\infty}\sup_{n\in\mathbb{N}_{n_1}}\left\{\frac{1}{|a_{i,n}|}\sum_{t=n}^{\infty}P_{i,t}\right\}, \tag{2.10}$$

$$-b_2 \leq b_{m,n} \leq b_1, \quad (m,n) \in \mathbb{N}_{m_1,n_1}, \tag{2.11}$$

$$\sum_{j=m_1}^{\infty}\sum_{i=j}^{\infty}\sup_{n\in\mathbb{N}_{n_1}}\left\{\frac{1}{|a_{i,n}|}\sum_{t=n}^{\infty}(Q_{i,t} + |c_{i,t}|)\right\} \leq \min\{(1-b_2)M - L, L - b_1M - N\}. \tag{2.12}$$

Define a mapping $T_L : A(N,M) \to l^{\infty}_{\alpha,\beta}$ by

$$T_Lx_{m,n} = \begin{cases} L - b_{m,n}x_{m-\tau_0,n-\sigma_0} + \sum_{j=m}^{\infty}\sum_{i=j}^{\infty}\frac{1}{a_{i,n}} \\ \quad \times\sum_{t=n}^{\infty}\left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right) - c_{i,t}\right], \quad (m,n) \in \mathbb{Z}_{m_1,n_1}, \\ T_Lx_{m_1,n_1}, \quad (m,n) \in \mathbb{Z}_{\alpha,\beta} \setminus \mathbb{Z}_{m_1,n_1} \end{cases} \tag{2.13}$$

for each $x = \{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in A(N,M)$. By employing (2.1)–(2.4) and (2.10)–(2.13), we infer that for $x = \{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}}, y = \{y_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in A(N,M)$ and $(m,n) \in \mathbb{Z}_{m_1,n_1}$

$$\begin{aligned} \left|T_Lx_{m,n} - T_Ly_{m,n}\right| &= \Bigg| b_{m,n}\left(x_{m-\tau_0,n-\sigma_0} - y_{m-\tau_0,n-\sigma_0}\right) \\ &\quad - \sum_{j=m}^{\infty}\sum_{i=j}^{\infty}\frac{1}{a_{i,n}}\sum_{t=n}^{\infty}\left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right)\right. \\ &\qquad \left.-f\left(i,t,y_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,y_{i-\tau_{k,i},t-\sigma_{k,t}}\right)\right]\Bigg| \\ &\leq |b_{m,n}|\left|x_{m-\tau_0,n-\sigma_0} - y_{m-\tau_0,n-\sigma_0}\right| \\ &\quad + \sum_{j=m}^{\infty}\sum_{i=j}^{\infty}\frac{1}{|a_{i,n}|}\sum_{t=n}^{\infty}\left|f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right)\right. \\ &\qquad \left.-f\left(i,t,y_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,y_{i-\tau_{k,i},t-\sigma_{k,t}}\right)\right| \\ &\leq (b_1+b_2)\left\|x-y\right\| + \sum_{j=m}^{\infty}\sum_{i=j}^{\infty}\frac{1}{|a_{i,n}|} \\ &\quad \times\sum_{t=n}^{\infty}P_{i,t}\max\left\{\left|x_{i-\tau_{l,i},t-\sigma_{l,t}} - y_{i-\tau_{l,i},t-\sigma_{l,t}}\right| : 1 \leq l \leq k\right\} \end{aligned}$$

$$
\begin{aligned}
&\le \left(b_1 + b_2 + \sum_{j=m_1}^{\infty}\sum_{i=j}^{\infty} \sup_{n\in\mathbb{N}_{n_1}} \left\{ \frac{1}{|a_{i,n}|} \sum_{t=n}^{\infty} P_{i,t} \right\} \right) \|x-y\| = \theta \|x-y\|,\\
T_L x_{m,n} &= L - b_{m,n} x_{m-\tau_0,n-\sigma_0} + \sum_{j=m}^{\infty}\sum_{i=j}^{\infty} \frac{1}{a_{i,n}} \sum_{t=n}^{\infty} \left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right) - c_{i,t} \right]\\
&\le L + b_2 M + \sum_{j=m}^{\infty}\sum_{i=j}^{\infty} \frac{1}{|a_{i,n}|} \sum_{t=n}^{\infty} \left[\left| f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right) \right| + |c_{i,t}| \right]\\
&\le L + b_2 M + \sum_{j=m_1}^{\infty}\sum_{i=j}^{\infty} \sup_{n\in\mathbb{N}_{n_1}} \left\{ \frac{1}{|a_{i,n}|} \sum_{t=n}^{\infty} (Q_{i,t} + |c_{i,t}|) \right\}\\
&\le L + b_2 M + \min\{(1-b_2)M - L, L - b_1 M - N\} \le M,\\
T_L x_{m,n} &= L - b_{m,n} x_{m-\tau_0,n-\sigma_0} + \sum_{j=m}^{\infty}\sum_{i=j}^{\infty} \frac{1}{a_{i,n}} \sum_{t=n}^{\infty} \left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right) - c_{i,t} \right]\\
&\ge L - b_1 M - \sum_{j=m}^{\infty}\sum_{i=j}^{\infty} \frac{1}{|a_{i,n}|} \sum_{t=n}^{\infty} \left[\left| f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right) \right| + |c_{i,t}| \right]\\
&\ge L - b_1 M - \sum_{j=m_1}^{\infty}\sum_{i=j}^{\infty} \sup_{n\in\mathbb{N}_{n_1}} \left\{ \frac{1}{|a_{i,n}|} \sum_{t=n}^{\infty} (Q_{i,t} + |c_{i,t}|) \right\}\\
&\ge L - b_1 M - \min\{(1-b_2)M - L, L - b_1 M - N\} \ge N,
\end{aligned}
\tag{2.14}
$$

which lead to

$$
T_L(A(N,M)) \subseteq A(N,M), \qquad \|T_L x - T_L y\| \le \theta \|x-y\|, \quad x,y \in A(N,M). \tag{2.15}
$$

Consequently, (2.15) means that T_L is a contraction mapping in $A(N,M)$ and it has a unique fixed-point $x = \{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in A(N,M)$, which together with (2.13) gives that for $(m,n) \in \mathbb{Z}_{m_1,n_1}$

$$
x_{m,n} = L - b_{m,n} x_{m-\tau_0,n-\sigma_0} + \sum_{j=m}^{\infty}\sum_{i=j}^{\infty} \frac{1}{a_{i,n}} \sum_{t=n}^{\infty} \left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right) - c_{i,t} \right], \tag{2.16}
$$

which yields that for $(m,n) \in \mathbb{Z}_{m_1,n_1}$

$$
\begin{aligned}
\Delta_m (x_{m,n} + b_{m,n} x_{m-\tau_0,n-\sigma_0}) &= -\sum_{i=m}^{\infty} \frac{1}{a_{i,n}} \sum_{t=n}^{\infty} \left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right) - c_{i,t} \right],\\
\Delta_m^2 (x_{m,n} + b_{m,n} x_{m-\tau_0,n-\sigma_0}) &= \frac{1}{a_{m,n}} \sum_{t=n}^{\infty} \left[f\left(m,t,x_{m-\tau_{1,m},t-\sigma_{1,t}},\ldots,x_{m-\tau_{k,m},t-\sigma_{k,t}}\right) - c_{m,t} \right],\\
\Delta_n \left(a_{m,n} \Delta_m^2 (x_{m,n} + b_{m,n} x_{m-\tau_0,n-\sigma_0}) \right) &= -f\left(m,n,x_{m-\tau_{1,m},n-\sigma_{1,t}},\ldots,x_{m-\tau_{k,m},t-\sigma_{k,t}}\right) + c_{m,n},
\end{aligned}
\tag{2.17}
$$

that is, $x = \{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}}$ is a bounded positive solution of (1.11) in $A(N,M)$.

Using (2.6), (2.13), and (2.15), we infer that for any $s \in \mathbb{N}_0$ and $(m,n) \in \mathbb{Z}_{m_1,n_1}$

$$\begin{aligned}
|x_{m,n}(s+1) - x_{m,n}| &= \Bigg| \big[1-\alpha(s)-\beta(s)\big] x_{m,n}(s) + \alpha(s) \\
&\quad \times \Bigg\{ L - b_{m,n} x_{m-\tau_0,n-\sigma_0}(s) \\
&\qquad + \sum_{j=m}^{\infty} \sum_{i=j}^{\infty} \frac{1}{a_{i,n}} \sum_{t=n}^{\infty} \big[f\big(i,t, x_{i-\tau_{1,i},t-\sigma_{1,t}}(s), \ldots, x_{i-\tau_{k,i},t-\sigma_{k,t}}(s)\big) - c_{i,t} \big] \Bigg\} \\
&\quad + \beta(s)\gamma_{m,n}(s) - x_{m,n} \Bigg| \\
&\le \big[1-\alpha(s)-\beta(s)\big] |x_{m,n}(s) - x_{m,n}| + \alpha(s) |T_L x_{m,n}(s) - T_L x_{m,n}| \\
&\quad + \beta(s) \big| \gamma_{m,n}(s) - x_{m,n} \big| \\
&\le \big[1-\alpha(s)-\beta(s)\big] \|x(s) - x\| + \alpha(s)\theta \|x(s) - x\| + 2M\beta(s) \\
&\le [1-(1-\theta)\alpha(s)] \|x(s) - x\| + 2M\beta(s),
\end{aligned} \tag{2.18}$$

which yields that

$$\|x(s+1) - x\| \le [1-(1-\theta)\alpha(s)] \|x(s) - x\| + 2M\beta(s), \quad s \in \mathbb{N}_0. \tag{2.19}$$

That is, (2.7) holds. Consequently, Lemma 1.1 and (2.7)–(2.9) imply that $\lim_{s\to\infty} x(s) = x$.

Next we show that (b) holds. Let $L_1, L_2 \in (N+b_1M, (1-b_2)M)$ and let $L_1 \neq L_2$. As in the proof of (a), we infer that for each $i \in \{1,2\}$, there exist θ_i, m_{i+1}, n_{i+1} and T_{L_i} satisfying (2.10)–(2.13), where θ, m_1, n_1, L and T_L are replaced by θ_i, m_{i+1}, n_{i+1}, L_i, and T_{L_i}, respectively, and the mapping T_{L_i} has a fixed-point $x^i = \{x^i_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in A(N,M)$, which is a bounded positive solution of (1.11), that is,

$$\begin{aligned}
x^1_{m,n} &= L_1 - b_{m,n} x^1_{m-\tau_0,n-\sigma_0} \\
&\quad + \sum_{j=m}^{\infty} \sum_{i=j}^{\infty} \frac{1}{a_{i,n}} \sum_{t=n}^{\infty} \Big[f\Big(i,t, x^1_{i-\tau_{1,i},t-\sigma_{1,t}}, \ldots, x^1_{i-\tau_{k,i},t-\sigma_{k,t}}\Big) - c_{i,t} \Big], \quad (m,n) \in \mathbb{Z}_{m_2,n_2}, \\
x^2_{m,n} &= L_2 - b_{m,n} x^2_{m-\tau_0,n-\sigma_0} \\
&\quad + \sum_{j=m}^{\infty} \sum_{i=j}^{\infty} \frac{1}{a_{i,n}} \sum_{t=n}^{\infty} \Big[f\Big(i,t, x^2_{i-\tau_{1,i},t-\sigma_{1,t}}, \ldots, x^2_{i-\tau_{k,i},t-\sigma_{k,t}}\Big) - c_{i,t} \Big], \quad (m,n) \in \mathbb{Z}_{m_3,n_3}.
\end{aligned} \tag{2.20}$$

In order to show that the set of bounded positive solutions of (1.11) is uncountable, it is sufficient to prove that $x^1 \neq x^2$. It follows from (2.3), (2.10), (2.11), (2.20) that for $(m,n) \in \mathbb{Z}_{\max\{m_2,m_3\},\max\{n_2,n_3\}}$

$$\begin{aligned}
\left|x^1_{m,n} - x^2_{m,n}\right| &= \Bigg| L_1 - L_2 - b_{m,n}\left(x^1_{m-\tau_0,n-\sigma_0} - x^2_{m-\tau_0,n-\sigma_0}\right) \\
&\quad + \sum_{j=m}^{\infty}\sum_{i=j}^{\infty}\frac{1}{a_{i,n}}\sum_{t=n}^{\infty}\Big[f\left(i,t,x^1_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x^1_{i-\tau_{k,i},t-\sigma_{k,t}}\right) \\
&\qquad - f\left(i,t,x^2_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x^2_{i-\tau_{k,i},t-\sigma_{k,t}}\right)\Big]\Bigg| \\
&\geq |L_1 - L_2| - |b_{m,n}|\left|x^1_{m-\tau_0,n-\sigma_0} - x^2_{m-\tau_0,n-\sigma_0}\right| \\
&\quad - \sum_{j=m}^{\infty}\sum_{i=j}^{\infty}\frac{1}{|a_{i,n}|}\sum_{t=n}^{\infty}\Big[\left|f\left(i,t,x^1_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x^1_{i-\tau_{k,i},t-\sigma_{k,t}}\right)\right. \\
&\qquad \left. - f\left(i,t,x^2_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x^2_{i-\tau_{k,i},t-\sigma_{k,t}}\right)\right|\Big] \\
&\geq |L_1 - L_2| - (b_1 + b_2)\left\|x^1 - x^2\right\| \\
&\quad - \sum_{j=m}^{\infty}\sum_{i=j}^{\infty}\frac{1}{|a_{i,n}|}\sum_{t=n}^{\infty}P_{i,t}\max\left\{\left|x^1_{i-\tau_{l,i},t-\sigma_{l,t}} - x^2_{i-\tau_{l,i},t-\sigma_{l,t}}\right| : 1 \leq l \leq k\right\} \\
&\geq |L_1 - L_2| - \left(b_1 + b_2 + \sum_{j=m}^{\infty}\sum_{i=j}^{\infty}\frac{1}{|a_{i,n}|}\sum_{t=n}^{\infty}P_{i,t}\right)\left\|x^1 - x^2\right\| \\
&\geq |L_1 - L_2| - \left(b_1 + b_2 + \sum_{j=\max\{m_2,m_3\}}^{\infty}\sum_{i=j}^{\infty}\sup_{n\in\mathbb{N}_{\max\{n_2,n_3\}}}\left\{\frac{1}{|a_{i,n}|}\sum_{t=n}^{\infty}P_{i,t}\right\}\right)\left\|x^1 - x^2\right\| \\
&\geq |L_1 - L_2| - \max\{\theta_1,\theta_2\}\left\|x^1 - x^2\right\|,
\end{aligned} \tag{2.21}$$

which implies that

$$\left\|x^1 - x^2\right\| \geq \frac{|L_1 - L_2|}{1 + \max\{\theta_1,\theta_2\}} > 0, \tag{2.22}$$

that is, $x^1 \neq x^2$. This completes the proof. □

Theorem 2.2. *Assume that there exist positive constants M and N, negative constants b_1 and b_2 and nonnegative sequences $\{P_{m,n}\}_{(m,n)\in\mathbb{N}_{m_0,n_0}}$ and $\{Q_{m,n}\}_{(m,n)\in\mathbb{N}_{m_0,n_0}}$ satisfying* (2.3)–(2.5) *and*

$$b_1 < -1, \qquad N(1 + b_2) > M(1 + b_1); \tag{2.23}$$

$$b_2 \leq b_{m,n} \leq b_1, \quad \textit{eventually}. \tag{2.24}$$

Then

(a) *for any* $L \in (M(1+b_1), N(1+b_2))$, *there exist* $\theta \in (0,1)$, $m_1 \geq m_0 + \tau_0 + |\alpha|$ *and* $n_1 \geq n_0 + \sigma_0 + |\beta|$ *such that for each* $x(0) = \{x_{m,n}(0)\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in A(N,M)$, *the Mann iterative sequence with errors* $\{x(s)\}_{s\in\mathbb{N}_0} = \{x_{m,n}(s)\}_{(m,n,s)\in\mathbb{Z}_{\alpha,\beta}\times\mathbb{N}_0}$ *generated by the scheme:*

$$x_{m,n}(s+1) = \begin{cases} [1-\alpha(s)-\beta(s)]x_{m,n}(s) + \alpha(s) \\ \quad \times \left\{ \dfrac{L}{b_{m+\tau_0,n+\sigma_0}} - \dfrac{x_{m+\tau_0,n+\sigma_0}(s)}{b_{m+\tau_0,n+\sigma_0}} + \dfrac{1}{b_{m+\tau_0,n+\sigma_0}} \displaystyle\sum_{j=m+\tau_0}^{\infty}\sum_{i=j}^{\infty} \frac{1}{a_{i,n+\sigma_0}} \right. \\ \qquad \left. \times \displaystyle\sum_{t=n+\sigma_0}^{\infty} \left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}}(s),\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}(s)\right) - c_{i,t} \right] \right\} \\ \quad + \beta(s)\gamma_{m,n}(s), \quad (m,n) \in \mathbb{Z}_{m_1,n_1},\ s \in \mathbb{N}_0, \\ [1-\alpha(s)-\beta(s)]x_{m_1,n_1}(s) + \alpha(s) \\ \quad \times \left\{ \dfrac{L}{b_{m_1+\tau_0,n_1+\sigma_0}} - \dfrac{x_{m_1+\tau_0,n_1+\sigma_0}(s)}{b_{m_1+\tau_0,n_1+\sigma_0}} + \dfrac{1}{b_{m_1+\tau_0,n_1+\sigma_0}} \displaystyle\sum_{j=m_1+\tau_0}^{\infty}\sum_{i=j}^{\infty} \frac{1}{a_{i,n_1+\sigma_0}} \right. \\ \qquad \left. \times \displaystyle\sum_{t=n_1+\sigma_0}^{\infty} \left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}}(s),\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}(s)\right) - c_{i,t} \right] \right\} \\ \quad + \beta(s)\gamma_{m_1,n_1}(s), \quad (m,n) \in \mathbb{Z}_{\alpha,\beta} \setminus \mathbb{Z}_{m_1,n_1},\ s \in \mathbb{N}_0 \end{cases} \tag{2.25}$$

converges to a bounded positive solution $x \in A(N,M)$ *of* (1.11) *and has the error estimate* (2.7), *where* $\{\gamma(s)\}_{s\in\mathbb{N}_0}$ *is an arbitrary sequence in* $A(N,M)$, $\{\alpha(s)\}_{s\in\mathbb{N}_0}$ *and* $\{\beta(s)\}_{s\in\mathbb{N}_0}$ *are any sequences in* $[0,1]$ *satisfying* (2.8) *and* (2.9);

(b) (1.11) *possesses uncountably many bounded positive solutions in* $A(M,N)$.

Proof. First of all we show (a). Taking $L \in (M(1+b_1), N(1+b_2))$, from (2.5), (2.23), and (2.24) we infer that there exist $\theta \in (0,1)$, $m_1 \geq m_0 + \tau_0 + |\alpha|$ and $n_1 \geq n_0 + \sigma_0 + |\beta|$ such that

$$\theta = -\frac{1}{b_1}\left(1 + \sum_{j=m_1}^{\infty}\sum_{i=j}^{\infty} \sup_{n\in\mathbb{N}_{n_1}} \left\{ \frac{1}{|a_{i,n}|} \sum_{t=n}^{\infty} P_{i,t} \right\}\right), \tag{2.26}$$

$$b_2 \leq b_{m,n} \leq b_1, \quad (m,n) \in \mathbb{N}_{m_1,n_1}, \tag{2.27}$$

$$\sum_{j=m_1}^{\infty}\sum_{i=j}^{\infty} \sup_{n\in\mathbb{N}_{n_1}} \left\{ \frac{1}{|a_{i,n}|} \sum_{t=n}^{\infty} (Q_{i,t} + |c_{i,t}|) \right\} \leq \min\left\{ L - M(1+b_1), b_1 N\left(1 + \frac{1}{b_2}\right) - \frac{b_1 L}{b_2} \right\}. \tag{2.28}$$

Define a mapping $T_L : A(N,M) \to l^{\infty}_{\alpha,\beta}$ by

$$T_L x_{m,n} = \begin{cases} \dfrac{L}{b_{m+\tau_0,n+\sigma_0}} - \dfrac{x_{m+\tau_0,n+\sigma_0}}{b_{m+\tau_0,n+\sigma_0}} + \dfrac{1}{b_{m+\tau_0,n+\sigma_0}} \displaystyle\sum_{j=m+\tau_0}^{\infty}\sum_{i=j}^{\infty} \frac{1}{a_{i,n+\sigma_0}} \\ \quad \times \displaystyle\sum_{t=n+\sigma_0}^{\infty} \left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right) - c_{i,t} \right], \quad (m,n) \in \mathbb{Z}_{m_1,n_1}, \\ T_L x_{m_1,n_1}, \quad (m,n) \in \mathbb{Z}_{\alpha,\beta} \setminus \mathbb{Z}_{m_1,n_1} \end{cases} \tag{2.29}$$

for each $x = \{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in A(N,M)$. It follows from (2.3), (2.4), (2.23), (2.24), and (2.26)–(2.29) that for $x = \{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}}$, $y = \{y_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in A(N,M)$ and $(m,n) \in \mathbb{Z}_{m_1,n_1}$:

$$\begin{aligned}
\left|T_L x_{m,n} - T_L y_{m,n}\right| &= \left| \frac{x_{m+\tau_0,n+\sigma_0} - y_{m+\tau_0,n+\sigma_0}}{b_{m+\tau_0,n+\sigma_0}} - \frac{1}{b_{m+\tau_0,n+\sigma_0}} \sum_{j=m+\tau_0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{a_{i,n+\sigma_0}} \right. \\
&\quad \times \sum_{t=n+\sigma_0}^{\infty} \left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right) \right. \\
&\quad \left. \left. - f\left(i,t,y_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,y_{i-\tau_{k,i},t-\sigma_{k,t}}\right) \right] \right| \\
&\le -\frac{\left|x_{m+\tau_0,n+\sigma_0} - y_{m+\tau_0,n+\sigma_0}\right|}{b_{m+\tau_0,n+\sigma_0}} - \frac{1}{b_{m+\tau_0,n+\sigma_0}} \sum_{j=m+\tau_0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{|a_{i,n+\sigma_0}|} \\
&\quad \times \sum_{t=n+\sigma_0}^{\infty} \left| f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right) \right. \\
&\quad \left. - f\left(i,t,y_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,y_{i-\tau_{k,i},t-\sigma_{k,t}}\right) \right| \\
&\le -\frac{\|x-y\|}{b_1} - \frac{1}{b_1} \sum_{j=m+\tau_0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{|a_{i,n+\sigma_0}|} \\
&\quad \times \sum_{t=n+\sigma_0}^{\infty} P_{i,t} \max\left\{ \left|x_{i-\tau_{l,i},t-\sigma_{l,t}} - y_{i-\tau_{l,i},t-\sigma_{l,t}}\right| : 1 \le l \le k \right\} \\
&\le -\frac{1}{b_1} \left(1 + \sum_{j=m_1}^{\infty} \sum_{i=j}^{\infty} \sup_{n\in\mathbb{N}_{n_1}} \left\{ \frac{1}{|a_{i,n}|} \sum_{t=n}^{\infty} P_{i,t} \right\} \right) \|x-y\| = \theta \|x-y\|, \\
T_L x_{m,n} &= \frac{L}{b_{m+\tau_0,n+\sigma_0}} - \frac{x_{m+\tau_0,n+\sigma_0}}{b_{m+\tau_0,n+\sigma_0}} + \frac{1}{b_{m+\tau_0,n+\sigma_0}} \sum_{j=m+\tau_0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{a_{i,n+\sigma_0}} \\
&\quad \times \sum_{t=n+\sigma_0}^{\infty} \left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right) - c_{i,t} \right] \\
&\le \frac{L}{b_1} - \frac{M}{b_1} - \frac{1}{b_1} \sum_{j=m+\tau_0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{|a_{i,n+\sigma_0}|} \\
&\quad \times \sum_{t=n+\sigma_0}^{\infty} \left[\left| f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right) \right| + |c_{i,t}| \right] \\
&\le \frac{L}{b_1} - \frac{M}{b_1} - \frac{1}{b_1} \sum_{j=m+\tau_0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{|a_{i,n+\sigma_0}|} \sum_{t=n+\sigma_0}^{\infty} \left(Q_{i,t} + |c_{i,t}| \right) \\
&\le \frac{L}{b_1} - \frac{M}{b_1} - \frac{1}{b_1} \sum_{j=m_1}^{\infty} \sum_{i=j}^{\infty} \sup_{n\in\mathbb{N}_{n_1}} \left\{ \frac{1}{|a_{i,n}|} \sum_{t=n}^{\infty} \left(Q_{i,t} + |c_{i,t}| \right) \right\} \\
&\le \frac{L}{b_1} - \frac{M}{b_1} - \frac{1}{b_1} \min\left\{ L - M(1+b_1), b_1 N\left(1 + \frac{1}{b_2}\right) - \frac{b_1 L}{b_2} \right\} \le M,
\end{aligned}$$

$$
\begin{aligned}
T_L x_{m,n} &= \frac{L}{b_{m+\tau_0,n+\sigma_0}} - \frac{x_{m+\tau_0,n+\sigma_0}}{b_{m+\tau_0,n+\sigma_0}} + \frac{1}{b_{m+\tau_0,n+\sigma_0}} \sum_{j=m+\tau_0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{a_{i,n+\sigma_0}} \\
&\quad \times \sum_{t=n+\sigma_0}^{\infty} \left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right) - c_{i,t} \right] \\
&\geq \frac{L}{b_2} - \frac{N}{b_2} + \frac{1}{b_1} \sum_{j=m+\tau_0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{|a_{i,n+\sigma_0}|} \\
&\quad \times \sum_{t=n+\sigma_0}^{\infty} \left[\left| f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right) \right| + |c_{i,t}| \right] \\
&\geq \frac{L}{b_2} - \frac{N}{b_2} + \frac{1}{b_1} \sum_{j=m+\tau_0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{|a_{i,n+\sigma_0}|} \sum_{t=n+\sigma_0}^{\infty} \left(Q_{i,t} + |c_{i,t}| \right) \\
&\geq \frac{L}{b_2} - \frac{N}{b_2} + \frac{1}{b_1} \sum_{j=m_1}^{\infty} \sum_{i=j}^{\infty} \sup_{n \in \mathbb{N}_{n_1}} \left\{ \frac{1}{|a_{i,n}|} \sum_{t=n}^{\infty} \left(Q_{i,t} + |c_{i,t}| \right) \right\} \\
&\geq \frac{L}{b_2} - \frac{N}{b_2} + \frac{1}{b_1} \min \left\{ L - M(1+b_1), b_1 N \left(1 + \frac{1}{b_2} \right) - \frac{b_1 L}{b_2} \right\} \geq N,
\end{aligned}
\tag{2.30}
$$

which imply that (2.15) holds. Consequently, (2.15) ensures that T_L is a contraction mapping in $A(N,M)$ and it has a unique fixed-point $x = \{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in A(N,M)$, which together with (2.29) gives that

$$
\begin{aligned}
x_{m,n} &= \frac{L}{b_{m+\tau_0,n+\sigma_0}} - \frac{x_{m+\tau_0,n+\sigma_0}}{b_{m+\tau_0,n+\sigma_0}} + \frac{1}{b_{m+\tau_0,n+\sigma_0}} \sum_{j=m+\tau_0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{a_{i,n+\sigma_0}} \\
&\quad \times \sum_{t=n+\sigma_0}^{\infty} \left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right) - c_{i,t} \right], \quad (m,n) \in \mathbb{Z}_{m_1,n_1},
\end{aligned}
\tag{2.31}
$$

which yields that for $(m,n) \in \mathbb{Z}_{m_1,n_1}$

$$
\begin{aligned}
\Delta_m \left(x_{m,n} + b_{m,n} x_{m-\tau_0,n-\sigma_0} \right) &= -\sum_{i=m}^{\infty} \frac{1}{a_{i,n}} \sum_{t=n}^{\infty} \left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right) - c_{i,t} \right], \\
\Delta_m^2 \left(x_{m,n} + b_{m,n} x_{m-\tau_0,n-\sigma_0} \right) &= \frac{1}{a_{m,n}} \sum_{t=n}^{\infty} \left[f\left(m,t,x_{m-\tau_{1,m},t-\sigma_{1,t}},\ldots,x_{m-\tau_{k,m},t-\sigma_{k,t}}\right) - c_{m,t} \right], \\
\Delta_n \left(a_{m,n} \Delta_m^2 \left(x_{m,n} + b_{m,n} x_{m-\tau_0,n-\sigma_0} \right) \right) &= -f\left(m,n,x_{m-\tau_{1,m},n-\sigma_{1,t}},\ldots,x_{m-\tau_{k,m},t-\sigma_{k,t}}\right) + c_{m,n},
\end{aligned}
\tag{2.32}
$$

which implies that $x = \{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}}$ is a bounded positive solution of (1.11) in $A(N,M)$.

It follows from (2.15), (2.25) and (2.29) that for any $s \in \mathbb{N}_0$ and $(m,n) \in \mathbb{Z}_{m_1,n_1}$

$$\begin{aligned}
|x_{m,n}(s+1) - x_{m,n}| = \Bigg| & \left[1-\alpha(s)-\beta(s)\right] x_{m,n}(s) + \alpha(s) \\
& \times \left\{ \frac{L}{b_{m+\tau_0,n+\sigma_0}} - \frac{x_{m+\tau_0,n+\sigma_0}(s)}{b_{m+\tau_0,n+\sigma_0}} + \frac{1}{b_{m+\tau_0,n+\sigma_0}} \right. \\
& \left. \times \sum_{j=m+\tau_0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{a_{i,n+\sigma_0}} \sum_{t=n+\sigma_0}^{\infty} \left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}}(s),\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}(s)\right) - c_{i,t} \right] \right\} \\
& + \beta(s)\gamma_{m,n}(s) - x_{m,n} \Bigg| \\
\le\ & \left[1-\alpha(s)-\beta(s)\right] |x_{m,n}(s) - x_{m,n}| + \alpha(s) |T_L x_{m,n}(s) - T_L x_{m,n}| \\
& + \beta(s) \left| \gamma_{m,n}(s) - x_{m,n} \right| \\
\le\ & \left[1-\alpha(s)-\beta(s)\right] \|x(s) - x\| + \alpha(s)\theta \|x(s) - x\| + 2M\beta(s) \\
\le\ & [1-(1-\theta)\alpha(s)] \|x(s) - x\| + 2M\beta(s),
\end{aligned} \tag{2.33}$$

which yields (2.7). Thus Lemma 1.1 and (2.7)–(2.9) ensure that $\lim_{s\to\infty} x(s) = x$.

Next we show that (b) holds. Let $L_1, L_2 \in (M(1+b_1),\ N(1+b_2))$ let and $L_1 \neq L_2$. As in the proof of (a), we infer that for each $i \in \{1,2\}$, there exist $\theta_i,\ m_{i+1},\ n_{i+1}$ and T_{L_i} satisfying (2.26)–(2.29), where $\theta,\ m_1,\ n_1,\ L$ and T_L are replaced by $\theta_i,\ m_{i+1},\ n_{i+1},\ L_i$ and T_{L_i}, respectively, and the mapping T_{L_i} has a fixed-point $x^i = \{x^i_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in A(N,M)$, which is a bounded positive solution of (1.11), that is:

$$\begin{aligned}
x^1_{m,n} = & \frac{L_1}{b_{m+\tau_0,n+\sigma_0}} - \frac{x^1_{m+\tau_0,n+\sigma_0}}{b_{m+\tau_0,n+\sigma_0}} + \frac{1}{b_{m+\tau_0,n+\sigma_0}} \sum_{j=m+\tau_0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{a_{i,n+\sigma_0}} \\
& \times \sum_{t=n+\sigma_0}^{\infty} \left[f\left(i,t,x^1_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x^1_{i-\tau_{k,i},t-\sigma_{k,t}}\right) - c_{i,t} \right], \quad (m,n) \in \mathbb{Z}_{m_2,n_2},
\end{aligned} \tag{2.34}$$

$$\begin{aligned}
x^2_{m,n} = & \frac{L_2}{b_{m+\tau_0,n+\sigma_0}} - \frac{x^2_{m+\tau_0,n+\sigma_0}}{b_{m+\tau_0,n+\sigma_0}} + \frac{1}{b_{m+\tau_0,n+\sigma_0}} \sum_{j=m+\tau_0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{a_{i,n+\sigma_0}} \\
& \times \sum_{t=n+\sigma_0}^{\infty} \left[f\left(i,t,x^2_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x^2_{i-\tau_{k,i},t-\sigma_{k,t}}\right) - c_{i,t} \right], \quad (m,n) \in \mathbb{Z}_{m_3,n_3}.
\end{aligned} \tag{2.35}$$

In order to show that the set of bounded positive solutions of (1.11) is uncountable, it is sufficient to prove that $x^1 \neq x^2$. It follows from (2.3), (2.26), (2.27), (2.34), and (2.35) that for $(m,n) \in \mathbb{Z}_{\max\{m_2,m_3\},\max\{n_2,n_3\}}$

$$\begin{aligned}
\left|x^1_{m,n} - x^2_{m,n}\right| &= \left| \frac{L_1 - L_2}{b_{m+\tau_0,n+\sigma_0}} - \frac{x^1_{m+\tau_0,n+\sigma_0} - x^2_{m+\tau_0,n+\sigma_0}}{b_{m+\tau_0,n+\sigma_0}} + \frac{1}{b_{m+\tau_0,n+\sigma_0}} \sum_{j=m+\tau_0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{a_{i,n+\sigma_0}} \right. \\
&\quad \left. \times \sum_{t=n+\sigma_0}^{\infty} \left[f\left(i,t,x^1_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x^1_{i-\tau_{k,i},t-\sigma_{k,t}}\right) - f\left(i,t,x^2_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x^2_{i-\tau_{k,i},t-\sigma_{k,t}}\right)\right] \right| \\
&\geq -\frac{|L_1 - L_2|}{b_{m+\tau_0,n+\sigma_0}} + \frac{\left|x^1_{m+\tau_0,n+\sigma_0} - x^2_{m+\tau_0,n+\sigma_0}\right|}{b_{m+\tau_0,n+\sigma_0}} + \frac{1}{b_{m+\tau_0,n+\sigma_0}} \sum_{j=m+\tau_0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{|a_{i,n+\sigma_0}|} \\
&\quad \times \sum_{t=n+\sigma_0}^{\infty} \left| f\left(i,t,x^1_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x^1_{i-\tau_{k,i},t-\sigma_{k,t}}\right) - f\left(i,t,x^2_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x^2_{i-\tau_{k,i},t-\sigma_{k,t}}\right)\right| \\
&\geq -\frac{|L_1 - L_2|}{b_2} + \frac{\left\|x^1 - x^2\right\|}{b_1} + \frac{1}{b_1} \sum_{j=m+\tau_0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{|a_{i,n+\sigma_0}|} \\
&\quad \times \sum_{t=n+\sigma_0}^{\infty} P_{i,t} \max\left\{ \left| x^1_{i-\tau_{1,i},t-\sigma_{1,t}} - x^2_{i-\tau_{1,i},t-\sigma_{1,t}} \right| : 1 \leq l \leq k \right\} \\
&\geq -\frac{|L_1 - L_2|}{b_2} + \frac{1}{b_1} \left(1 + \sum_{j=m+\tau_0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{|a_{i,n+\sigma_0}|} \sum_{t=n+\sigma_0}^{\infty} P_{i,t} \right) \left\|x^1 - x^2\right\| \\
&\geq -\frac{|L_1 - L_2|}{b_2} + \frac{1}{b_1} \left(1 + \sum_{j=\max\{m_2,m_3\}}^{\infty} \sum_{i=j}^{\infty} \sup_{n \in \mathbb{N}_{\max\{n_2,n_3\}}} \left\{ \frac{1}{|a_{i,n}|} \sum_{t=n}^{\infty} P_{i,t} \right\} \right) \left\|x^1 - x^2\right\| \\
&\geq -\frac{|L_1 - L_2|}{b_2} - \max\{\theta_1,\theta_2\} \left\|x^1 - x^2\right\|,
\end{aligned} \tag{2.36}$$

which implies that

$$\left\|x^1 - x^2\right\| \geq -\frac{|L_1 - L_2|}{b_2(1 + \max\{\theta_1,\theta_2\})} > 0, \tag{2.37}$$

that is, $x^1 \neq x^2$. This completes the proof. □

Theorem 2.3. *Assume that there exist positive constants M and N, nonnegative constants b_1 and b_2, and nonnegative sequences $\{P_{m,n}\}_{(m,n)\in\mathbb{N}_{m_0,n_0}}$ and $\{Q_{m,n}\}_{(m,n)\in\mathbb{N}_{m_0,n_0}}$ satisfying* (2.3)–(2.5), (2.24) *and*

$$1 < b_2, \qquad b_1 < b_2^2, \qquad Mb_1\left(b_2^2 - b_1\right) > Nb_2\left(b_1^2 - b_2\right). \tag{2.38}$$

Then

(a) *for any* $L \in (b_1N + b_1M/b_2\ ,\ b_2M + b_2N/b_1\ )$, *there exist* $\theta \in (0,1)$, $m_1 \geq m_0 + \tau_0 + |\alpha|$ *and* $n_1 \geq n_0 + \sigma_0 + |\beta|$ *such that for each* $x(0) = \{x_{m,n}(0)\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in A(N,M)$, *the Mann iterative sequence with errors* $\{x(s)\}_{s\in\mathbb{N}_0} = \{x_{m,n}(s)\}_{(m,n,s)\in\mathbb{Z}_{\alpha,\beta}\times\mathbb{N}_0}$ *generated by* (2.25) *converges to a bounded positive solution* $x \in A(N,M)$ *of* (1.11) *and has the error estimate* (2.7), *where* $\{\gamma(s)\}_{s\geq 0}$ *is an arbitrary sequence in* $A(N,M)$, $\{\alpha(s)\}_{s\geq 0}$ *and* $\{\beta(s)\}_{s\geq 0}$ *are any sequences in* $[0,1]$ *satisfying* (2.8) *and* (2.9);

(b) (1.11) *possesses uncountably many bounded positive solutions in* $A(M,N)$.

Proof. Set $L \in (b_1N + b_1M/b_2), b_2M + b_2N/b_1)$. It follows from (2.5), (2.24), and (2.38) that there exist $\theta \in (0,1)$, $m_1 \geq m_0 + \tau_0 + |\alpha|$ and $n_1 \geq n_0 + \sigma_0 + |\beta|$ satisfying (2.27):

$$\theta = \frac{1}{b_2}\left(1 + \sum_{j=m_1}^{\infty}\sum_{i=j}^{\infty}\sup_{n\in\mathbb{N}_{n_1}}\left\{\frac{1}{|a_{i,n}|}\sum_{t=n}^{\infty}P_{i,t}\right\}\right), \tag{2.39}$$

$$\sum_{j=m_1}^{\infty}\sum_{i=j}^{\infty}\sup_{n\in\mathbb{N}_{n_1}}\left\{\frac{1}{|a_{i,n}|}\sum_{t=n}^{\infty}(Q_{i,t} + |c_{i,t}|)\right\} \leq \min\left\{b_2M - L + \frac{b_2N}{b_1}, \frac{b_2L}{b_1} - M - b_2N\right\}. \tag{2.40}$$

Let the mapping $T_L : A(N,M) \to l^{\infty}_{\alpha,\beta}$ be defined by (2.29). It follows from (2.3), (2.4), (2.24), (2.27), (2.29), and (2.38)–(2.40) that for $x = \{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}}$, $y = \{y_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in A(N,M)$ and $(m,n) \in \mathbb{Z}_{m_1,n_1}$

$$\begin{aligned}
|T_Lx_{m,n} - T_Ly_{m,n}| &= \left|\frac{x_{m+\tau_0,n+\sigma_0} - y_{m+\tau_0,n+\sigma_0}}{b_{m+\tau_0,n+\sigma_0}} - \frac{1}{b_{m+\tau_0,n+\sigma_0}}\sum_{j=m+\tau_0}^{\infty}\sum_{i=j}^{\infty}\frac{1}{a_{i,n+\sigma_0}}\right.\\
&\quad\times\sum_{t=n+\sigma_0}^{\infty}\left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right)\right.\\
&\qquad\left.\left.-f\left(i,t,y_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,y_{i-\tau_{k,i},t-\sigma_{k,t}}\right)\right]\right|\\
&\leq \frac{\left|x_{m+\tau_0,n+\sigma_0} - y_{m+\tau_0,n+\sigma_0}\right|}{b_{m+\tau_0,n+\sigma_0}} + \frac{1}{b_{m+\tau_0,n+\sigma_0}}\sum_{j=m+\tau_0}^{\infty}\sum_{i=j}^{\infty}\frac{1}{|a_{i,n+\sigma_0}|}\\
&\quad\times\sum_{t=n+\sigma_0}^{\infty}\left|f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right)\right.\\
&\qquad\left.-f\left(i,t,y_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,y_{i-\tau_{k,i},t-\sigma_{k,t}}\right)\right|\\
&\leq \frac{\|x-y\|}{b_2} + \frac{1}{b_2}\sum_{j=m+\tau_0}^{\infty}\sum_{i=j}^{\infty}\frac{1}{|a_{i,n+\sigma_0}|}\\
&\quad\times\sum_{t=n+\sigma_0}^{\infty}P_{i,t}\max\left\{\left|x_{i-\tau_{l,i},t-\sigma_{l,t}} - y_{i-\tau_{l,i},t-\sigma_{l,t}}\right| : 1\leq l\leq k\right\}\\
&\leq \frac{1}{b_2}\left(1 + \sum_{j=m_1}^{\infty}\sum_{i=j}^{\infty}\sup_{n\in\mathbb{N}_{n_1}}\left\{\frac{1}{|a_{i,n}|}\sum_{t=n}^{\infty}P_{i,t}\right\}\right)\|x-y\| = \theta\|x-y\|,
\end{aligned}$$

$$
\begin{aligned}
T_L x_{m,n} &= \frac{L}{b_{m+\tau_0,n+\sigma_0}} - \frac{x_{m+\tau_0,n+\sigma_0}}{b_{m+\tau_0,n+\sigma_0}} + \frac{1}{b_{m+\tau_0,n+\sigma_0}} \sum_{j=m+\tau_{0,m}}^{\infty} \sum_{i=j}^{\infty} \frac{1}{a_{i,n+\sigma_0}} \\
&\quad \times \sum_{t=n+\sigma_0}^{\infty} \left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right) - c_{i,t} \right] \\
&\le \frac{L}{b_2} - \frac{N}{b_1} + \frac{1}{b_2} \sum_{j=m+\tau_0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{|a_{i,n+\sigma_0}|} \\
&\quad \times \sum_{t=n+\sigma_0}^{\infty} \left[\left| f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right) \right| + |c_{i,t}| \right] \\
&\le \frac{L}{b_2} - \frac{N}{b_1} + \frac{1}{b_2} \sum_{j=m_1}^{\infty} \sum_{i=j}^{\infty} \sup_{n\in\mathbb{N}_{n_1}} \left\{ \frac{1}{|a_{i,n}|} \sum_{t=n}^{\infty} (Q_{i,t} + |c_{i,t}|) \right\} \\
&\le \frac{L}{b_2} - \frac{N}{b_1} + \frac{1}{b_2} \min\left\{ b_2 M - L + \frac{b_2 N}{b_1}, \frac{b_2 L}{b_1} - M - b_2 N \right\} \le M, \\
T_L x_{m,n} &= \frac{L}{b_{m+\tau_0,n+\sigma_0}} - \frac{x_{m+\tau_0,n+\sigma_0}}{b_{m+\tau_0,n+\sigma_0}} + \frac{1}{b_{m+\tau_0,n+\sigma_0}} \sum_{j=m+\tau_{0,m}}^{\infty} \sum_{i=j}^{\infty} \frac{1}{a_{i,n+\sigma_0}} \\
&\quad \times \sum_{t=n+\sigma_0}^{\infty} \left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right) - c_{i,t} \right] \\
&\ge \frac{L}{b_1} - \frac{M}{b_2} - \frac{1}{b_2} \sum_{j=m+\tau_0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{|a_{i,n+\sigma_0}|} \\
&\quad \times \sum_{t=n+\sigma_0}^{\infty} \left[\left| f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right) \right| + |c_{i,t}| \right] \\
&\ge \frac{L}{b_1} - \frac{M}{b_2} - \frac{1}{b_2} \sum_{j=m_1}^{\infty} \sum_{i=j}^{\infty} \sup_{n\in\mathbb{N}_{n_1}} \left\{ \frac{1}{a_{i,n}} \sum_{t=n}^{\infty} (Q_{i,t} + |c_{i,t}|) \right\} \\
&\ge \frac{L}{b_1} - \frac{M}{b_2} - \frac{1}{b_2} \min\left\{ b_2 M - L + \frac{b_2 N}{b_1}, \frac{b_2 L}{b_1} - M - b_2 N \right\} \ge N,
\end{aligned}
\tag{2.41}
$$

which imply that (2.15) holds. Consequently (2.15) ensures that T_L is a contraction mapping, and hence it has a unique fixed-point $x = \{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in A(N,M)$, which gives that

$$
\begin{aligned}
x_{m,n} &= \frac{L}{b_{m+\tau_0,n+\sigma_0}} - \frac{x_{m+\tau_0,n+\sigma_0}}{b_{m+\tau_0,n+\sigma_0}} + \frac{1}{b_{m+\tau_0,n+\sigma_0}} \sum_{j=m+\tau_0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{a_{i,n+\sigma_0}} \\
&\quad \times \sum_{t=n+\sigma_0}^{\infty} \left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right) - c_{i,t} \right], \ (m,n) \in \mathbb{Z}_{m_1,n_1}.
\end{aligned}
\tag{2.42}
$$

As in the proof of Theorem 2.2, it is easy to verify that $x = \{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}}$ is a bounded positive solution of (1.11) in $A(N,M)$; (2.7) holds and $\lim_{s\to\infty} x(s) = x$.

Next we show that (b) holds. Let $L_1, L_2 \in (b_1N + b_1M/b_2\ , b_2M + b_2N/b_1)$ and $L_1 \neq L_2$. As in the proof of (a), we infer that for each $i \in \{1,2\}$, there exist $\theta_i, m_{i+1}, n_{i+1}$ and T_{L_i} satisfying (2.27), (2.29), (2.39), and (2.40), where $\theta,\ m_1,\ n_1,\ L$ and T_L are replaced by $\theta_i,\ m_{i+1},\ n_{i+1},\ L_i$, and T_{L_i}, respectively, and the mapping T_{L_i} has a fixed-point $x^i = \{x^i_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in A(N,M)$, which is a bounded positive solution of (1.11) and satisfies (2.34) and (2.35). In order to show that the set of bounded positive solutions of (1.11) is uncountable, it is sufficient to prove that $x^1 \neq x^2$. It follows from (2.3), (2.27), (2.34), (2.35), and (2.39) that for $(m,n) \in \mathbb{Z}_{\max\{m_2,m_3\},\max\{n_2,n_3\}}$

$$
\begin{aligned}
\left|x^1_{m,n} - x^2_{m,n}\right| &= \left| \frac{L_1 - L_2}{b_{m+\tau_0,n+\sigma_0}} - \frac{x^1_{m+\tau_0,n+\sigma_0} - x^2_{m+\tau_0,n+\sigma_0}}{b_{m+\tau_0,n+\sigma_0}} + \frac{1}{b_{m+\tau_0,n+\sigma_0}} \sum_{j=m+\tau_0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{a_{i,n+\sigma_0}} \right. \\
&\quad \times \sum_{t=n+\sigma_0}^{\infty} \left[f\left(i,t,x^1_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x^1_{i-\tau_{k,i},t-\sigma_{k,t}}\right) \right. \\
&\qquad \left. \left. - f\left(i,t,x^2_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x^2_{i-\tau_{k,i},t-\sigma_{k,t}}\right) \right] \right| \\
&\geq \frac{|L_1 - L_2|}{b_{m+\tau_0,n+\sigma_0}} - \frac{\left|x^1_{m+\tau_0,n+\sigma_0} - x^2_{m+\tau_0,n+\sigma_0}\right|}{b_{m+\tau_0,n+\sigma_0}} - \frac{1}{b_{m+\tau_0,n+\sigma_0}} \sum_{j=m+\tau_0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{|a_{i,n+\sigma_0}|} \\
&\quad \times \sum_{t=n+\sigma_0}^{\infty} \left| f\left(i,t,x^1_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x^1_{i-\tau_{k,i},t-\sigma_{k,t}}\right) - f\left(i,t,x^2_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x^2_{i-\tau_{k,i},t-\sigma_{k,t}}\right) \right| \\
&\geq \frac{|L_1 - L_2|}{b_1} - \frac{\left\|x^1 - x^2\right\|}{b_2} - \frac{1}{b_2} \sum_{j=m+\tau_0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{|a_{i,n+\sigma_0}|} \\
&\quad \times \sum_{t=n+\sigma_0}^{\infty} P_{i,t} \max\left\{ \left| x^1_{i-\tau_{1,i},t-\sigma_{1,t}} - x^2_{i-\tau_{1,i},t-\sigma_{1,t}} \right| : 1 \leq l \leq k \right\} \\
&\geq \frac{|L_1 - L_2|}{b_1} - \frac{1}{b_2} \left(1 + \sum_{j=m+\tau_0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{|a_{i,n+\sigma_0}|} \sum_{t=n+\sigma_0}^{\infty} P_{i,t} \right) \left\|x^1 - x^2\right\| \\
&\geq \frac{|L_1 - L_2|}{b_1} - \frac{1}{b_2} \left(1 + \sum_{j=\max\{m_2,m_3\}}^{\infty} \sum_{i=j}^{\infty} \sup_{n\in\mathbb{N}_{\max\{n_2,n_3\}}} \left\{ \frac{1}{|a_{i,n}|} \sum_{t=n}^{\infty} P_{i,t} \right\} \right) \left\|x^1 - x^2\right\| \\
&\geq \frac{|L_1 - L_2|}{b_1} - \max\{\theta_1,\theta_2\} \left\|x^1 - x^2\right\|,
\end{aligned}
\tag{2.43}
$$

which implies that

$$
\left\|x^1 - x^2\right\| \geq \frac{|L_1 - L_2|}{b_1(1 + \max\{\theta_1,\theta_2\})} > 0, \tag{2.44}
$$

that is, $x^1 \neq x^2$. This completes the proof. □

3. Examples

Now we illustrate the results presented in Section 2 with the following three examples. Note that none of the known results can be applied to the examples.

Example 3.1. Consider the third-order nonlinear partial difference equation with delays:

$$\begin{aligned}&\Delta_n\left((-1)^{mn}m^4n^3\Delta_m^2\left(x_{m,n}+\frac{(-1)^{m+n}}{3}x_{m-\tau_0,n-\sigma_0}\right)\right)+\frac{\sqrt{n}}{m^2(n^2+1)}x^3_{m^2,n(n+1)/2}\\&-\frac{\cos\left(m^3n^5-\ln m\right)}{(m+1)^2n^2}x^2_{m^3-2m,n^2-n}=\frac{(-1)^n\sin\left(m^2-2n\right)}{\sqrt{m^5n^4+1}},\quad m\geq 1,\ n\geq 1,\end{aligned}\tag{3.1}$$

where $\tau_0,\sigma_0\in\mathbb{N}$ are fixed. Let $m_0=n_0=1$, $k=2$, $b_1=b_2=1/3$, $\alpha=\min\{1-\tau_0,-1\}$, $\beta=1-\sigma_0$, M and let N be two positive constants with $M>3N$ and

$$\begin{aligned}&a_{m,n}=(-1)^{mn}m^4n^3,\qquad b_{m,n}=\frac{(-1)^{m+n}}{3},\qquad c_{m,n}=\frac{(-1)^{m+n}\sin\left(m^2-2n\right)}{\sqrt{m^5n^4+1}},\\&f(m,n,u,v)=\frac{\sqrt{n}}{m^2(n^2+1)}u^3-\frac{\cos\left(m^3n^5-\ln m\right)}{(m+1)^2n^2}v^2,\\&\tau_{1,m}=m(1-m),\qquad \tau_{2,m}=m\left(3-m^2\right),\qquad \sigma_{1,n}=\frac{n(1-n)}{2},\qquad \sigma_{2,n}=n(2-n),\\&P_{m,n}=\frac{3M^2\sqrt{n}}{m^2(n^2+1)}+\frac{2M}{(m+1)^2n^2},\qquad Q_{m,n}=\frac{M^3\sqrt{n}}{m^2(n^2+1)}+\frac{M^2}{(m+1)^2n^2},\\&(m,n,u,v)\in\mathbb{N}_{m_0,n_0}\times\mathbb{R}^2.\end{aligned}\tag{3.2}$$

It is easy to verify that (2.1)–(2.4) hold. Note that

$$\begin{aligned}&\sum_{j=m_0}^{\infty}\sum_{i=j}^{\infty}\sup_{n\geq n_0}\left\{\frac{1}{|a_{i,n}|}\sum_{t=n}^{\infty}\max\{P_{i,t},Q_{i,t},|c_{i,t}|\}\right\}\\&=\sum_{j=m_0}^{\infty}\sum_{i=j}^{\infty}\sup_{n\geq n_0}\left\{\frac{1}{i^4n^3}\sum_{t=n}^{\infty}\max\left\{\frac{3M^2\sqrt{t}}{i^2(t^2+1)}+\frac{2M}{(i+1)^2t^2},\frac{M^3\sqrt{t}}{i^2(t^2+1)}+\frac{M^2}{(i+1)^2t^2},\frac{\left|\sin\left(i^2-2t\right)\right|}{\sqrt{i^5t^4+1}}\right\}\right\}\\&<\left(1+2M+4M^2+M^3\right)\left(\sum_{t=n_0}^{\infty}\frac{1}{\sqrt{t^3}}\right)\sum_{j=m_0}^{\infty}\sum_{i=j}^{\infty}\frac{1}{i^4}<+\infty.\end{aligned}\tag{3.3}$$

Hence the conditions of Theorem 2.1 are fulfilled. It follows from Theorem 2.1 that (3.1) possesses uncountably many bounded positive solutions in $A(N,M)$. On the other hand, for any $L\in(N+(1/3)M,(2/3)M)$, there exist $\theta\in(0,1)$ and $m_1\geq m_0+\tau_0+|\alpha|$, $n_1\geq n_0+\sigma_0+|\beta|$ such that the Mann iterative sequence with errors $\{x(s)\}_{s\geq 0}$ generated by (2.6) converges to a bounded positive solution $x\in A(N,M)$ of (3.1) and has the error estimate (2.7), where $\{\gamma(s)\}_{s\geq 0}$ is an arbitrary sequence in $A(N,M)$, $\{\alpha(s)\}_{s\geq 0}$ and $\{\beta(s)\}_{s\geq 0}$ are any sequences in $[0,1]$ satisfying (2.8) and (2.9).

Example 3.2. Consider the third-order nonlinear partial difference equation with delays:

$$\Delta_n\left((-1)^n m^3 \ln^2(m+n)\Delta_m^2\left(x_{m,n}-\frac{4n+(-1)^m n}{n+1}x_{m-\tau_0,n-\sigma_0}\right)\right) + \frac{x_{m-2,n-3}^2 x_{3m^2-2,n^5-1}^3}{m^3n^2} = \frac{\cos\left(nm^3-\sqrt{m}\right)}{\sqrt{n^3+1}}, \quad m\geq 1,\ n\geq 1, \tag{3.4}$$

where $\tau_0, \sigma_0 \in \mathbb{N}$ are fixed. Let $m_0 = n_0 = 1$, $k = 2$, $b_1 = -2$, $b_2 = -5$, $\alpha = \min\{1-\tau_0, -1\}$, $\beta = \min\{1-\sigma_0, -2\}$, M and let N be two positive constants with $M > 4N$ and

$$a_{m,n} = (-1)^n m^3 \ln^2(m+n), \qquad b_{m,n} = -\frac{4n+(-1)^m n}{n+1}, \qquad c_{m,n} = \frac{\cos\left(nm^3-\sqrt{m}\right)}{\sqrt{n^3+1}},$$

$$f(m,n,u,v) = \frac{u^2v^3}{m^3n^2}, \qquad \tau_{1,m} = 2, \qquad \tau_{2,m} = -3m^2+m+2, \qquad \sigma_{1,n} = 3, \qquad \sigma_{2,n} = -n^5+n+1,$$

$$P_{m,n} = \frac{5M^4}{m^3n^2}, \qquad Q_{m,n} = \frac{M^5}{m^3n^2}, \quad (m,n,u,v) \in \mathbb{N}_{m_0,n_0}\times\mathbb{R}^2. \tag{3.5}$$

It is clear that (2.3), (2.4), (2.23), and (2.24) hold. Observe that

$$\begin{aligned}
&\sum_{j=m_0}^{\infty}\sum_{i=j}^{\infty}\sup_{n\geq n_0}\left\{\frac{1}{|a_{i,n}|}\sum_{t=n}^{\infty}\max\{P_{i,t}, Q_{i,t}, |c_{i,t}|\}\right\} \\
&= \sum_{j=m_0}^{\infty}\sum_{i=j}^{\infty}\sup_{n\geq n_0}\left\{\frac{1}{i^3\ln^2(i+n)}\sum_{t=n}^{\infty}\max\left\{\frac{5M^4}{i^3t^2}, \frac{M^5}{i^3t^2}, \frac{\left|\cos\left(ti^3-\sqrt{i}\right)\right|}{\sqrt{t^3+1}}\right\}\right\} \\
&< \frac{1+5M^4+M^5}{\ln^2 2}\left(\sum_{t=n_0}^{\infty}\frac{1}{\sqrt{t^3}}\right)\sum_{j=m_0}^{\infty}\sum_{i=j}^{\infty}\frac{1}{i^3} < +\infty.
\end{aligned} \tag{3.6}$$

That is, the conditions of Theorem 2.2 are fulfilled. Thus Theorem 2.2 ensures that (3.4) has uncountably many bounded positive solutions in $A(N,M)$. On the other hand, for any $L \in (-M,-4N)$, there exist $\theta \in (0,1)$ and $m_1 \geq m_0+\tau_0+|\alpha|$, $n_1 \geq n_0+\sigma_0+|\beta|$ such that the Mann iterative sequence with errors $\{x(s)\}_{s\geq 0}$ generated by (2.25) converges to a bounded positive solution $x \in A(N,M)$ of (3.4) and has the error estimate (2.7), where $\{\gamma(s)\}_{s\geq 0}$ is an arbitrary sequence in $A(N,M)$,$\{\alpha(s)\}_{s\geq 0}$ and $\{\beta(s)\}_{s\geq 0}$ are any sequences in $[0,1]$ satisfying (2.8) and (2.9).

Example 3.3. Consider the third-order nonlinear partial difference equation with delays:

$$\Delta_n\left((-1)^n\left(m^5n\right)\Delta_m^2\left(x_{m,n}+\frac{2mn+3}{mn+1}x_{m-\tau_0,n-\sigma_0}\right)\right)+\frac{\left((-1)^m-1/n\right)x_{m-4,n-3}^4}{m^3n^2+x_{m^2-2m,n^3-n}^2}$$
$$=\frac{(-1)^{n+m}\left(m^2-3n^3\right)}{m^5(n^7+1)},\quad m\geq 1,\ n\geq 1, \tag{3.7}$$

where $\tau_0,\sigma_0\in\mathbb{N}$ are fixed. Let $m_0=n_0=1$, $k=2$, $b_1=3$, $b_2=2$, $\alpha=\min\{1-\tau_0,-3\}$, $\beta=\min\{1-\sigma_0,-2\}$, M and N be two positive constants with $M>(14/3)N$ and

$$a_{m,n}=(-1)^n\left(m^5n\right),\qquad b_{m,n}=\frac{2mn+3}{mn+1},\qquad c_{m,n}=\frac{(-1)^{n+m}\left(m^2-3n^3\right)}{m^5(n^7+1)},$$
$$f(m,n,u,v)=\frac{\left((-1)^m-1/n\right)u^4}{m^3n^2+v^2},\qquad \tau_{1,m}=4,\qquad \tau_{2,m}=m(3-m),\qquad \sigma_{1,n}=3,\qquad \sigma_{2,n}=-n^3+2n,$$
$$P_{m,n}=\frac{2M^3\left(2m^3n^2+2M^2+1\right)}{\left(m^3n^2+N^2\right)^2},\qquad Q_{m,n}=\frac{2M^4}{m^3n^2+N^2},\qquad (m,n,u,v)\in\mathbb{N}_{m_0,n_0}\times\mathbb{R}^2. \tag{3.8}$$

Clearly (2.3), (2.4), (2.24), and (2.38) hold. Notice that

$$\sum_{j=m_0}^{\infty}\sum_{i=j}^{\infty}\sup_{n\geq n_0}\left\{\frac{1}{|a_{i,n}|}\sum_{t=n}^{\infty}\max\{P_{i,t},Q_{i,t},|c_{i,t}|\}\right\}$$
$$=\sum_{j=m_0}^{\infty}\sum_{i=j}^{\infty}\sup_{n\geq n_0}\left\{\frac{1}{i^5n}\sum_{t=n}^{\infty}\max\left\{\frac{2M^3\left(2i^3t^2+2M^2+1\right)}{\left(i^3t^2+N^2\right)^2},\frac{2M^4}{i^3t^2+N^2},\frac{\left|i^2-3t^3\right|}{i^5(t^7+1)}\right\}\right\} \tag{3.9}$$
$$<\max\left\{4,2M^4,4M^3\left(1+M^2\right)\right\}\left(\sum_{t=n_0}^{\infty}\frac{1}{t^2}\right)\sum_{j=m_0}^{\infty}\sum_{i=j}^{\infty}\frac{1}{i^5}<+\infty.$$

Hence the conditions of Theorem 2.3 are fulfilled. Consequently Theorem 2.3 implies that (3.7) possesses uncountably many bounded positive solutions in $A(N,M)$. On the other hand, for any $L\in(3N+(3/2)M,2M+(2/3)N)$, there exist $\theta\in(0,1)$ and $m_1\geq m_0+\tau_0+|\alpha|$, $n_1\geq n_0+\sigma_0+|\beta|$ such that the Mann iterative sequence with errors $\{x(s)\}_{s\geq 0}$ generated by (2.25) converges to a bounded positive solution $x\in A(N,M)$ of (3.7) and has the error estimate (2.7), where $\{\gamma(s)\}_{s\geq 0}$ is an arbitrary sequence in $A(N,M)$, $\{\alpha(s)\}_{s\geq 0}$ and $\{\beta(s)\}_{s\geq 0}$ are any sequences in $[0,1]$ satisfying (2.8) and (2.9).

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