Research Article

Multigrid Discretization and Iterative Algorithm for Mixed Variational Formulation of the Eigenvalue Problem of Electric Field

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This paper discusses highly finite element algorithms for the eigenvalue problem of electric field. Combining the mixed finite element method with the Rayleigh quotient iteration method, a new multi-grid discretization scheme and an adaptive algorithm are proposed and applied to the eigenvalue problem of electric field. Theoretical analysis and numerical results show that the computational schemes established in the paper have high efficiency.

1. Introduction

The finite element method for eigenvalue problem of electric field has become a hot topic in the field of mathematics and physics (see, e.g., [1–7]). This paper discusses high efficient mixed finite element calculation schemes for the eigenvalue problem of electric field.

Kikuchi [6] introduced the first type of mixed variational formulation for the eigenvalue problem of electric field. Based on this formulation, in [3] Buffa et al.analyzed the approximation of nodal finite element. Boffi et al. [1] discussed the second type of mixed variational formulation for the eigenvalue problem of electric field and analyzed approximations of edge element and nodal element. Yang et al. [7] studied a two-grid discretization scheme of finite element for the first type of mixed variational formulation.

Based on the work mentioned above, in this paper a new multi-grid discretization scheme and an adaptive algorithm are proposed for the first type of mixed variational formulation of eigenvalue problem and applied to the eigenvalue problem of electric field. The main features of the research in this paper are as follows.

(1) Our multi-grid discretization scheme and adaptive algorithm, which are the exten-sion of conforming finite element multi-grid discretization scheme (see scheme 3 in [8] and scheme 1 in [9]), are a combination of the mixed finite element method and the Rayleigh quotient iteration method (see the algorithm 27.3 in [10]). With our algorithm one solves an eigenvalue problem on a coarse grid just at the first step and always solves a linear algebraic system on finer and finer grids at each following step. We derive the error estimates for the algorithm and prove that the constants appeared in the error estimates are independent of the iteration degrees. Thus we prove the convergence of iterations.

(2) The eigenvalue problem of electric field is so complicated that it is very difficult to obtain local a posteriori error estimates for the eigenfunctions of mixed finite ele-ment. As yet, there is no achievement reported in this field. Our adaptive algorithm substitutes the weight method established by Costabel and Dauge (see [3, 11]) for local refinement, which uses $\theta \times |\lambda^{h_l} - \lambda^{h_{l-1}}| (\theta \in (0,1])$ as a posteriori error estimator of λ^{h_l} instead of estimating local a posteriori error for the eigenfunction. And the results are satisfying.

(3) We analyze the mixed finite element error for the eigenvalue problem of electric field (see Theorem 2.2 and Theorem 4.2). We refer to [12] to propose a new proof method which differs from the usual one in [13].

The rest of this paper is organized as follows. Some preliminaries of finite element approximations for eigenvalue problems which are needed in this paper are provided in the next section. In Section 3, for the first type of mixed variational formulation of eigenvalue problem, the finite element multi-grid discretization scheme and the adaptive algorithm are established and the validity of these schemes is proved theoretically. In Section 4, the multi-grid discretization scheme is applied to the eigenvalue problem of electric field. Finally, numerical experiments are presented in Section 5.

2. Preliminaries

Let V, W, and D be three real Hilbert spaces with inner products and norms $(\cdot,\cdot)_V$, $\|\cdot\|_V$, $(\cdot,\cdot)_W$, $\|\cdot\|_W$, $(\cdot,\cdot)_D$, and $\|\cdot\|_D$, respectively. Suppose that $V \hookrightarrow D$ (continuously imbedded), $a(\cdot,\cdot)$ is a symmetric, continuous, and V-elliptic bilinear form on $V \times V$, that is,

$$
\left| a(q,\psi) \right| \le M_1 \|q\|_V \|\psi\|_V, \quad \forall q, \psi \in V,
$$
$$
a(q,q) \ge \nu \|q\|_V^2, \quad \forall 0 \ne q \in V.
$$

(2.1)

$b(\cdot,\cdot)$ is a continuous bilinear form on $V \times W$, that is,

$$
\left| b(\psi,v) \right| \le M_2 \|\psi\|_V \|v\|_W, \quad \forall \psi \in V, \ v \in W.
$$

(2.2)

It is obvious that $a(\cdot,\cdot)$ is an inner product on V and $\|\cdot\|_a = \sqrt{a(\cdot,\cdot)}$ is an equivalent norm to $\|\cdot\|_V$.

In scientific and engineering computations, many eigenvalue problems have the following first type of mixed variational formulation: find $(\lambda, u, \sigma) \in R \times V \times W$, $(u, \sigma) \neq (0,0)$, such that

$$a(u, \psi) + b(\psi, \sigma) = \lambda(u, \psi)_D, \quad \forall \psi \in V, \tag{2.3}$$

$$b(u, v) = 0, \quad \forall v \in W. \tag{2.4}$$

In order to solve problem (2.3)-(2.4), one should construct finite element spaces $V_h \subset V$ and $W_h \subset W$. Restricting (2.3)-(2.4) on $V_h \times W_h$, we get the conforming mixed finite element approximation as follows: find $(\lambda_h, u_h, \sigma_h) \in R \times V_h \times W_h$, $(u_h, \sigma_h) \neq (0,0)$, such that

$$a(u_h, \psi) + b(\psi, \sigma_h) = \lambda_h(u_h, \psi)_D, \quad \forall \psi \in V_h, \tag{2.5}$$

$$b(u_h, v) = 0, \quad \forall v \in W_h. \tag{2.6}$$

Consider the associated source and approximate source problems. Given $f \in D$, find $(w, p) \in V \times W$ satisfying

$$a(w, \psi) + b(\psi, p) = (f, \psi)_D, \quad \forall \psi \in V,$$
$$b(w, v) = 0, \quad \forall v \in W. \tag{2.7}$$

Given $f \in D$, find $(w_h, p_h) \in V_h \times W_h$ satisfying

$$a(w_h, \psi) + b(\psi, p_h) = (f, \psi)_D, \quad \forall \psi \in V_h,$$
$$b(w_h, v) = 0, \quad \forall v \in W_h. \tag{2.8}$$

Note that the source term f is independent of the solution.

As for the mixed finite element method for boundary value problems, Brezzi and Fortinand so forth have established a systematic theory (see [14]). By Brezzi-Babuska Theorem, we have the following.

Lemma 2.1 (Brezzi-Babuska). *Suppose that*

(C1) *(2.1)-(2.2) hold;*

(C2) *inf-sup condition holds, namely, there exists a constant $\nu_1 > 0$, such that*

$$\sup_{\psi \in V, \psi \neq 0} \frac{b(\psi, v)}{\|\psi\|_V} \geq \nu_1 \|v\|_W, \quad \forall v \in W, \tag{2.9}$$

then there exists a unique solution (w, p) to the problem (2.7) *and*

$$\|w\|_a + \|p\|_W \leq C_r \|f\|_D, \tag{2.10}$$

where C_r just depends on ν, ν_1, and M_1. Moreover, suppose;

(C3) *discrete inf-sup condition holds, namely, there exists a constant $v_2 > 0$ independent of h, such that*

$$\sup_{\psi \in V_h, \psi \neq 0} \frac{b(\psi, v)}{\|\psi\|_V} \geq v_2 \|v\|_W, \quad \forall v \in W_h, \tag{2.11}$$

then there exists a unique solution (w_h, p_h) to the problem (2.8) and the following error estimate is valid:

$$\|w - w_h\|_a + \|p - p_h\|_W \leq C_e \left\{ \inf_{q \in V_h} \|w - q\|_a + \inf_{v \in W_h} \|p - v\|_W \right\}, \tag{2.12}$$

where C_e just depends on v, v_2 and M_1, M_2.

Suppose conditions (C1)–(C3) hold in Lemma 2.1. Then there exist unique solutions to the problem (2.7) and (2.8), respectively. Thus, we can define linear bounded operators as follows: $T : D \to V, \ S : D \to W : \forall f \in D$,

$$a(Tf, \psi) + b(\psi, Sf) = (f, \psi)_D, \quad \forall \psi \in V,$$
$$b(Tf, v) = 0, \quad \forall v \in W. \tag{2.13}$$

$T_h : D \to V_h \subset V, \ S_h : D \to W_h \subset W : \forall f \in D$,

$$a(T_h f, \psi) + b(\psi, S_h f) = (f, \psi)_D, \quad \forall \psi \in V_h;$$
$$b(T_h f, v) = 0, \quad \forall v \in W_h. \tag{2.14}$$

Obviously, (2.3)-(2.4) has the following equivalent operator form

$$\lambda T u = u,$$
$$\sigma = S(\lambda u), \tag{2.15}$$

and (2.5)-(2.6) has the following equivalent operator form

$$\lambda_h T_h u_h = u_h, \tag{2.16}$$

$$\sigma_h = S_h(\lambda_h u_h). \tag{2.17}$$

It is easy to verify that $T : D \to D, \ T_h : D \to D$ are self-adjoint operators in the sense of inner product $(\cdot, \cdot)_D$ (see [7]).

Assume $V \overset{c}{\hookrightarrow} D$ (compactly embedded), then it's easy to prove that $T : D \to D$ is completely continuous, $T : V \to V$ is completely continuous, and T_h is a finite rank operator. Combining (2.3)-(2.4), (2.5)-(2.6), and the V-ellipticity of $a(\cdot, \cdot)$, we deduce

$$\lambda = \frac{a(u, u)}{(u, u)_D} > 0, \quad \lambda_h = \frac{a(u_h, u_h)}{(u_h, u_h)_D} > 0. \tag{2.18}$$

Then from the spectral theory of self-adjoint and completely continuous operator, we know that the eigenvalues of (2.3)-(2.4) can be sorted as

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots \nearrow +\infty, \tag{2.19}$$

and the corresponding eigenfunctions are

$$(u_1, \sigma_1), (u_2, \sigma_2), \ldots, (u_k, \sigma_k), \ldots, \tag{2.20}$$

where $(u_i, u_j)_D = \delta_{ij}$.

The eigenvalues of (2.5)-(2.6) can be sorted as

$$0 < \lambda_{1,h} \le \lambda_{2,h} \le \cdots \le \lambda_{k,h} \le \cdots \le \lambda_{N_h,h}, \tag{2.21}$$

and the corresponding eigenfunctions are

$$(u_{1,h}, \sigma_{1,h}), (u_{2,h}, \sigma_{2,h}), \ldots, (u_{k,h}, \sigma_{k,h}), \ldots, (u_{N_h,h}, \sigma_{N_h,h}), \tag{2.22}$$

where $N_h = \dim V_h$, $(u_{i,h}, u_{j,h})_D = \delta_{ij}$.

From (2.5), by taking $u_h = u_{i,h}$, $\psi = u_{j,h}$, we have

$$a(u_{i,h}, u_{j,h}) + b(u_{j,h}, \sigma_h) = \lambda_{i,h}(u_{i,h}, u_{j,h})_D, \tag{2.23}$$

and from (2.6) we see that $b(u_{j,h}, \sigma_h) = 0$, then

$$a(u_{i,h}, u_{j,h}) = \lambda_{i,h}(u_{i,h}, u_{j,h})_D = \lambda_{i,h}\delta_{ij}, \tag{2.24}$$

thus $\{u_{i,h}/\|u_{i,h}\|_a\}$ is a completely normal eigenvector system on V_h in the sense of the inner product $a(\cdot, \cdot)$.

Denote $\lambda_k = 1/\mu_k$, $\lambda_{k,h} = 1/\mu_{k,h}$. In this paper, μ_k and $\mu_{k,h}$, λ_k and $\lambda_{k,h}$ are all called eigenvalues.

Let μ be the kth eigenvalue with algebraic multiplicity q, $\mu = \mu_k = \mu_{k+1} = \cdots = \mu_{k+q-1}$. $M(\mu)$ is the space spanned by all eigenfunctions $\{u_j\}_k^{k+q-1}$ corresponding to μ of T. $M_h(\mu)$ is the space spanned by all eigenfunctions $\{u_{j,h}\}_k^{k+q-1}$ corresponding to all eigenvalues of T_h that converge to μ. Let $\widehat{M}(\mu) = \{v : v \in M(\mu), \|v\|_a = 1\}$, $\widehat{M}_h(\mu) = \{v : v \in M_h(\mu), \|v\|_a = 1\}$.

We call $\lambda = 1/\mu$ as the kth eigenvalue, too. Denote $M(\lambda) = M(\mu)$, $M_h(\lambda) = M_h(\mu)$, and $\widehat{M}(\lambda) = \widehat{M}(\mu)$. Define

$$\left\|(T - T_h)|_{M(\lambda)}\right\|_D = \max_{u \in M(\lambda)} \frac{\left\|(T - T_h)u\right\|_D}{\|u\|_D},$$

$$\left\|(T - T_h)|_{M(\lambda)}\right\|_a = \max_{u \in M(\lambda)} \frac{\left\|(T - T_h)u\right\|_a}{\|u\|_a}.$$

(2.25)

The convergence and error estimates about the mixed element method of eigenvalue problem have been studied in [7, 12, 13, 15, 16]. From [12], we know that the following results are valid.

Theorem 2.2. *Suppose $V \overset{c}{\hookrightarrow} D$, $a(u,v)$ is symmetric, and the conditions of Lemma 2.1 hold; moreover, for any $f \in D$,*

$$\inf_{q \in V_h} \|Tf - q\|_V \longrightarrow 0 \quad (h \longrightarrow 0),$$

(2.26)

$$\inf_{v \in W_h} \|Sf - v\|_W \longrightarrow 0 \quad (h \longrightarrow 0).$$

(2.27)

Then $\|T_h - T\|_D \to 0$ ($h \to 0$).

Proof. From $V \overset{c}{\hookrightarrow} D$, we derive that $T : D \to D$ is a completely continuous operator. It is obvious that $T_h : D \to D$ is a finite rank operator. From (2.12), (2.26), and (2.27), we deduce

$$\|Tf - T_h f\|_a \leq C_e \left(\inf_{q \in V_h} \|Tf - q\|_a + \inf_{v \in W_h} \|Sf - v\|_W \right) \longrightarrow 0 \quad (h \longrightarrow 0).$$

(2.28)

It shows that $T_h : D \to V$ pointwisely converges to T. From (2.10) and (2.12) we derive that both $T : D \to V$ and $T_h : D \to V$ are linear bounded. Hence, from Banach-Steinhaus Theorem, we know that there exists a positive constant M independent of h, such that

$$\sup_h \|T_h\|_{L(D,V)} \leq M.$$

(2.29)

Thus, $\cup_{h>0}(T - T_h)B$ is a bounded set in V with respect to the unit ball B of D. From $V \overset{c}{\hookrightarrow} D$, we know that $\cup_{h>0}(T - T_h)B$ is a relatively compact set in D, which proves that $\{T_h\}$ is collectively compact. From (2.28), we know that $T : D \to D$, $T_h : D \to D$, and T_h pointwisely converge to T. From [7], $T : D \to D$, $T_h : D \to D$ are self-adjoint operators in the sense of inner product $(\cdot, \cdot)_D$. Then by Lemma 3.7 or Table 3.1 in [17], we get $\|T_h - T\|_D \to 0$ ($h \to 0$). The proof is completed. $\qquad\square$

Lemma 2.3. *Suppose that the conditions of Theorem 2.2 hold. Let $(\lambda_h, u_h, \sigma_h)$ be the kth eigenpair of (2.5)-(2.6) with $\|u_h\|_a = 1$, let λ be the kth eigenvalue of (2.3)-(2.4). Then $\lambda_h \to \lambda$ $(h \to 0)$, and there exists an eigenfunction (u, σ) corresponding to λ, such that*

$$|\lambda_h - \lambda| + \|u_h - u\|_D \le C_1 \big\|(T - T_h)|_{M(\lambda)}\big\|_D, \tag{2.30}$$

$$\|\sigma - \sigma_h\|_W \le \|S_h(\lambda u) - S(\lambda u)\|_W + C\big\|(T - T_h)|_{M(\lambda)}\big\|_D, \tag{2.31}$$

$$\|u - u_h\|_a \le C_2 \big\|(T_h - T)|_{M(\lambda)}\big\|_a, \tag{2.32}$$

let $u \in \widehat{M}(\lambda)$, then there exists $u_h \in M_h(\lambda)$ such that

$$\|u - u_h\|_a \le C_3 \big\|(T_h - T)|_{M(\lambda)}\big\|_a, \tag{2.33}$$

where u depends on h in general, and C_1, C_2, and C_3 are constants independent of h.

Proof. By Theorem 2.2, we know $\|T_h - T\|_D \to 0(h \to 0)$. Thus from Theorem 2.2 in [7], we see that the desired results are valid. The proof is completed. □

For $(u^*, \sigma^*) \in V \times W$, $u^* \ne 0$, define the Rayleigh quotient

$$\lambda^r = \frac{a(u^*, u^*) + 2b(u^*, \sigma^*)}{(u^*, u^*)_D}. \tag{2.34}$$

Lemma 2.4. *Suppose (λ, u, σ) is an eigenpair of (2.3)-(2.4), then for any $(u^*, \sigma^*) \in V \times W$, $u^* \ne 0$, the Rayleigh quotient λ^r satisfies*

$$\lambda^r - \lambda = \frac{a(u^* - u, u^* - u) + 2b(u^* - u, \sigma^* - \sigma)}{(u^*, u^*)_D} - \lambda \frac{(u^* - u, u^* - u)_D}{(u^*, u^*)_D}. \tag{2.35}$$

Proof. The proof is completed by using the same argument as that of Lemma 9.1 (see [7, 18]). □

Since $V \hookrightarrow D$ (continuously imbedded), there exists a constant C_4 independent of h such that

$$\|v\|_D \le C_4 \|v\|_a, \quad \forall v \in V. \tag{2.36}$$

Taking $(u^*, \sigma^*) = (u_h, \sigma_h)$ in (2.35) and using (2.4) and (2.6), we deduce the following.

Lemma 2.5. *Suppose* $(\lambda_h, u_h, \sigma_h)$ *is an approximation of* (λ, u, σ) *and* $\|u_h\|_a = 1$, *then*

$$\lambda_h - \lambda = \frac{a(u_h - u, u_h - u) + 2b(u_h - u, v - \sigma)}{(u_h, u_h)_D}$$

$$- \lambda \frac{(u_h - u, u_h - u)_D}{(u_h, u_h)_D}, \quad \forall v \in W_h, \tag{2.37}$$

$$|\lambda_h - \lambda| \le \left(\lambda_h + \lambda \lambda_h C_4^2\right)\|u_h - u\|_a^2$$

$$+ 2\lambda_h M_2 \|u_h - u\|_V \|\sigma - v\|_W, \quad \forall v \in W_h. \tag{2.38}$$

3. Mixed Finite Element Multigrid Discretization Scheme Based on the Rayleigh Quotient Iteration

In this section, we develop the work in [8], noticing that in [8], when $k > 1$, $\lambda_k^{h_1} - \lambda_k$, $\lambda_k^{h_{l-1}} - \lambda_{k,h_l}$ and $\lambda_k^{h_{l-1}} - \lambda_k$ sholud be modified to their absolute values, respectively, and establish the following mixed finite element multi-grid discretization scheme based on the Rayleigh quotient iteration, and give a rigorous theoretical analysis. Suppose the partition satisfies the following conditions.

Condition 1. $\{K^{h_i}\}$ is a family of regular meshes (see [19]) with the mesh diameter $\{h_i\}$ and $h_i = h_{i-1}^{t_i}$, $t_i \in (1, 3)$ is arbitrarily chosen, $i = 1, 2, \ldots$, and $\inf_i t_i > 1$.

Let $\{V_{h_i}\}_0^l$ and $\{W_{h_i}\}_0^l$ be finite element spaces on $\{K^{h_i}\}_0^l$.

Scheme 1. Multigrid Discretization.

Step 1. Solve the eigenvalue problem (2.3)-(2.4) on $V_H \times W_H$: find $(\lambda_H, u_H, \sigma_H) \in R \times V_H \times W_H$, $\|u_H\|_a = 1$ such that

$$a(u_H, \psi) + b(\psi, \sigma_H) = \lambda_H (u_H, \psi)_D, \quad \forall \psi \in V_H,$$

$$b(u_H, v) = 0, \quad \forall v \in W_H. \tag{3.1}$$

Step 2. $u^{h_0} \Leftarrow u_H$, $\lambda^{h_0} \Leftarrow \lambda_H$, $i \Leftarrow 1$.

Step 3. Solve an equation on $V_{h_i} \times W_{h_i}$: find $(u', \sigma') \in V_{h_i} \times W_{h_i}$ such that

$$a(u', \psi) + b(\psi, \sigma') - \lambda^{h_{i-1}}(u', \psi)_D = \left(u^{h_{i-1}}, \psi\right)_D, \quad \forall \psi \in V_{h_i},$$

$$b(u', v) = 0, \quad \forall v \in W_{h_i}. \tag{3.2}$$

Set $u^{h_i} = u'/\|u'\|_a$, $\sigma^{h_i} = \sigma'/\|u'\|_a$.

Step 4. Compute the Rayleigh quotient

$$\lambda^{h_i} = \frac{a(u^{h_i}, u^{h_i})}{(u^{h_i}, u^{h_i})_D}. \tag{3.3}$$

Step 5. If $i = l$, then output $(\lambda^{h_l}, u^{h_l}, \sigma^{h_l})$, stop; else, $i \Leftarrow i + 1$, and return to Step 3.

Let $(\lambda_H, u_H, \sigma_H)$ be the kth eigenpair, and we use $(\lambda^{h_l}, u^{h_l}, \sigma^{h_l})$ as the kth approximation eigenpair of (2.3)-(2.4).

Next, we will discuss the validity of Scheme 1.

Lemma 3.1. *For any nonzero $u, v \in V$,*

$$\left\| \frac{u}{\|u\|_a} - \frac{v}{\|v\|_a} \right\|_a \le 2 \frac{\|u - v\|_a}{\|u\|_a}, \qquad \left\| \frac{u}{\|u\|_a} - \frac{v}{\|v\|_a} \right\|_a \le 2 \frac{\|u - v\|_a}{\|v\|_a}. \tag{3.4}$$

Proof. See [8]. □

Denote $\mathrm{dist}(u, V) = \inf_{v \in V} \|u - v\|_a$.

Consider the eigenvalue problem (2.16) on V_h.

Lemma 3.2. *Suppose that μ and μ_h are the kth eigenvalue of T and T_h, respectively, and (μ_0, u_0) is an approximate eigenpair, where μ_0 is not an eigenvalue of T_h, $u_0 \in V_h$, $\|u_0\|_a = 1$, $\mathrm{dist}(u_0, M_h(\mu)) \le 1/2$, $\max_{k \le j \le k+q-1} |(\mu_{j,h} - \mu_h)/(\mu_0 - \mu_{j,h})| \le 1/2$, $|\mu_0 - \mu_{j,h}| \ge (\rho/2)(j \ne k, k+1, \ldots, k+q-1)$, and $u^s \in V_h, u^h \in V_h$ satisfy*

$$(\mu_0 - T_h) u^s = u_0, \qquad u^h = \frac{u^s}{\|u^s\|_a}. \tag{3.5}$$

Then

$$\mathrm{dist}\left(u^h, \widehat{M}_h(\mu)\right) \le \frac{16}{\rho} |\mu_0 - \mu_h| \, \mathrm{dist}(u_0, M_h(\mu)), \tag{3.6}$$

where $\rho = \min_{\mu_j \ne \mu} |\mu_j - \mu|$ is the separation constant of the eigenvalue μ.

Proof. See [8] □

Since the convergence rate of $V_{h_{l-1}}$ and $W_{h_{l-1}}$ approximating eigenfunctions is lower than that of V_{h_l} and W_{h_l} approximating eigenfunctions, respectively, the approximation order of $(\lambda_{h_{l-1}}, u_{h_{l-1}})$ is lower than that of (λ_{h_l}, u_{h_l}). However, in general, the accuracy order of $(\lambda^{h_{l-1}}, u^{h_{l-1}})$ will not exceed that of $(\lambda_{h_{l-1}}, u_{h_{l-1}})$; therefore in the following Theorem 3.3 we assume that the accuracy order of $(\lambda^{h_{l-1}}, u^{h_{l-1}})$ is lower than that of (λ_{h_l}, u_{h_l}).

Theorem 3.3. *Suppose that $\|T_h - T\|_D \to 0 \, (h \to 0)$, H is small properly, and Condition 1 holds. Let $(\lambda^{h_l}, u^{h_l}, \sigma^{h_l})$ be the approximate eigenpair obtained by Scheme 1, and let $u^{h_{l-1}}$ approximate*

$\overline{u} \in \widehat{M}(\lambda)$, $\lambda^{h_{l-1}}$ approximate λ, and the accuracy order of $(\lambda^{h_{l-1}}, u^{h_{l-1}})$ be lower than that of (λ_{h_l}, u_{h_l}). Then there exists $u \in M(\lambda)$ such that

$$\left\| u^{h_l} - u \right\|_a \leq \frac{32}{\rho} C_5 C_6 \left(\left| \lambda^{h_{l-1}} - \lambda \right|^2 + \left| \lambda^{h_{l-1}} - \lambda \right| \left\| u^{h_{l-1}} - \overline{u} \right\|_D \right) \tag{3.7}$$
$$+ (C_2 \times q + 3) \left\| (T - T_{h_l})|_{M(\lambda)} \right\|_a,$$

$$\left| \lambda^{h_l} - \lambda \right| \leq \left(\lambda^{h_l} + \lambda \lambda^{h_l} C_4^2 \right) \left\| u^{h_l} - u \right\|_a^2 + 2\lambda^{h_l} M_2 \left\| u^{h_l} - u \right\|_V \inf_{v \in W_{h_l}} \|\sigma - v\|_W, \tag{3.8}$$

where C_5 and C_6 are determined, respectively, by (3.10) and (3.14) in the following proof.

Proof. Let $\mu_0 = 1/\lambda^{h_{l-1}}$, and $u_0 = \lambda^{h_{l-1}} T_{h_l} u^{h_{l-1}} / \|\lambda^{h_{l-1}} T_{h_l} u^{h_{l-1}}\|_a$. Since $\overline{u} \in M(\lambda)$, by calculation we deduce

$$\left\| \lambda^{h_{l-1}} T_{h_l} u^{h_{l-1}} - \overline{u} \right\|_a = \left\| \lambda^{h_{l-1}} T_{h_l} u^{h_{l-1}} - \lambda T \overline{u} \right\|_a$$
$$= \left| \lambda^{h_{l-1}} - \lambda \right| \left\| T_{h_l} u^{h_{l-1}} \right\|_a + \lambda \left\| T_{h_l} \left(u^{h_{l-1}} - \overline{u} \right) \right\|_a + \lambda \| (T_{h_l} - T) \overline{u} \|_a. \tag{3.9}$$

From Lemma 2.1, there exists a positive constant C_5 depending only on ν, ν_1, ν_2, M_1, and M_2 such that

$$\|T_{h_l} v\|_a \leq C_5 \|v\|_D, \quad \forall v \in D. \tag{3.10}$$

Then

$$\left\| \lambda^{h_{l-1}} T_{h_l} u^{h_{l-1}} - \overline{u} \right\|_a \leq C_5 \left(C_4 \left| \lambda^{h_{l-1}} - \lambda \right| + \lambda \left\| u^{h_{l-1}} - \overline{u} \right\|_D \right) + \lambda \left\| (T_{h_l} - T)|_{M(\lambda)} \right\|_a. \tag{3.11}$$

By Lemma 3.1, we derive

$$\text{dist}\left(u_0, \widehat{M}(\lambda) \right) \leq \|u_0 - \overline{u}\|_a \leq 2 \left\| \lambda^{h_{l-1}} T_{h_l} u^{h_{l-1}} - \overline{u} \right\|_a$$
$$\leq 2 C_5 \left(C_4 \left| \lambda^{h_{l-1}} - \lambda \right| + \lambda \left\| u^{h_{l-1}} - \overline{u} \right\|_D \right) + 2\lambda \left\| (T_{h_l} - T)|_{M(\lambda)} \right\|_a. \tag{3.12}$$

Using the triangle inequality and (2.33), we deduce

$$\text{dist}(u_0, M_{h_l}(\lambda)) \leq \text{dist}\left(u_0, \widehat{M}(\lambda) \right) + C_3 \left\| (T_{h_l} - T)|_{M(\lambda)} \right\|_a. \tag{3.13}$$

According to the hypotheses of the theorem, we know that $\lambda^{h_{l-1}} \to \lambda$ and $\lambda^{h_{l-1}} - \lambda$ are an infinitesimal of lower order comparing with $\lambda_{j,h_l} - \lambda$. Hence, there exists a positive constant C_6 independent of h_l ($l = 1, 2, \ldots$) such that for $j = k, k+1, \ldots, k+q-1$ we have

$$\left| \mu_0 - \mu_{j,h_l} \right| = \left| \frac{\lambda^{h_{l-1}} - \lambda + \lambda - \lambda_{j,h_l}}{\lambda_{j,h_l} \lambda^{h_{l-1}}} \right| \leq C_6 \left| \lambda^{h_{l-1}} - \lambda \right|. \tag{3.14}$$

Note that H is small enough and $h_l \ll h_{l-1}$; from (3.13) and (3.12), we obtain

$$\text{dist}(u_0, M_{h_l}(\lambda)) \leq \frac{1}{2}. \tag{3.15}$$

Noticing that $\lambda = \lambda_k = \lambda_{k+1} = \cdots = \lambda_{k+q-1}$, we have

$$\left| \mu_{j,h_l} - \mu_{h_l} \right| = \left| \frac{\lambda_{h_l} - \lambda_{j,h_l}}{\lambda_{h_l} \lambda_{j,h_l}} \right| = \left| \frac{\lambda_{h_l} - \lambda + \lambda_j - \lambda_{j,h_l}}{\lambda_{h_l} \lambda_{j,h_l}} \right|, \tag{3.16}$$

which together with (3.14), noting that $\lambda_{j,h_l} - \lambda$ is an infinitesimal of higher order comparing with $\lambda^{h_{l-1}} - \lambda$, yields

$$\max_{k \leq j \leq k+q-1} \left| \frac{\mu_{j,h_l} - \mu_{h_l}}{\mu_0 - \mu_{j,h_l}} \right| \leq \frac{1}{2}. \tag{3.17}$$

Since ρ is the separation constant, H is small enough, and $h_l \ll h_{l-1}$, there holds

$$\left| \mu_0 - \mu_{j,h_l} \right| \geq \frac{\rho}{2}, \quad j \neq k, \ k+1, \ldots, k+q-1. \tag{3.18}$$

For u' in Step 3 of Scheme 1, from the definition of T_h and S_h (taking $i = l$), we have

$$a\left(\lambda^{h_{l-1}} T_{h_l} u', \psi \right) + b\left(\psi, \lambda^{h_{l-1}} S_{h_l} u' \right) = \lambda^{h_{l-1}} (u', \psi)_D, \quad \forall \psi \in V_{h_l}, \tag{3.19}$$

$$b\left(\lambda^{h_{l-1}} T_{h_l} u', v \right) = 0, \quad \forall v \in W_{h_l}, \tag{3.20}$$

$$a\left(T_{h_l} u^{h_{l-1}}, \psi \right) + b\left(\psi, S_{h_l} u^{h_{l-1}} \right) = \left(u^{h_{l-1}}, \psi \right)_D, \quad \forall \psi \in V_{h_l}, \tag{3.21}$$

$$b\left(T_{h_l} u^{h_{l-1}}, v \right) = 0, \quad \forall v \in W_{h_l}. \tag{3.22}$$

Hence, when $i = l$, Step 3 of Scheme 1 is equivalent to the following: find $(u', \sigma') \in V_{h_l} \times W_{h_l}$ such that

$$a(u', \psi) + b(\psi, \sigma') - \lambda^{h_{l-1}} a\left(T_{h_l} u', \psi \right) - \lambda^{h_{l-1}} b\left(\psi, S_{h_l} u' \right)$$
$$= a\left(T_{h_l} u^{h_{l-1}}, \psi \right) + b\left(\psi, S_{h_l} u^{h_{l-1}} \right), \quad \forall \psi \in V_{h_l}, \tag{3.23}$$

$$b(u', v) = 0, \quad \forall v \in W_{h_l}. \tag{3.24}$$

And set $u^{h_l} = u'/\|u'\|_a$, $\sigma^{h_l} = \sigma'/\|u'\|_a$.

From (3.23), we obtain

$$a\left(u' - \lambda^{h_{l-1}} T_{h_l} u' - T_{h_l} u^{h_{l-1}}, \psi \right) + b\left(\psi, \sigma' - \lambda^{h_{l-1}} S_{h_l} u' - S_{h_l} u^{h_{l-1}} \right) = 0, \quad \forall \psi \in V_{h_l}. \tag{3.25}$$

Combining (3.24), (3.20) and (3.22), we get

$$b\left(u' - \lambda^{h_{l-1}}T_{h_l}u' - T_{h_l}u^{h_{l-1}}, v\right) = 0, \quad \forall v \in W_{h_l}. \tag{3.26}$$

By (3.26) and taking $\psi = u' - \lambda^{h_{l-1}}T_{h_l}u' - T_{h_l}u^{h_{l-1}}$ in (3.25), we obtain

$$a\left(u' - \lambda^{h_{l-1}}T_{h_l}u' - T_{h_l}u^{h_{l-1}}, u' - \lambda^{h_{l-1}}T_{h_l}u' - T_{h_l}u^{h_{l-1}}\right) = 0. \tag{3.27}$$

Thus

$$\left(\frac{1}{\lambda^{h_{l-1}}} - T_{h_l}\right)u' = \frac{1}{\lambda^{h_{l-1}}}T_{h_l}u^{h_{l-1}}, \quad u^{h_l} = \frac{u'}{\|u'\|_a}. \tag{3.28}$$

From (3.28) we know that the first term on the left-hand side of (3.25) is equal to 0, thus

$$b\left(\psi, \sigma' - \lambda^{h_{l-1}}S_{h_l}u' - S_{h_l}u^{h_{l-1}}\right) = 0, \quad \forall \psi \in V_{h_l}, \tag{3.29}$$

then, using discrete inf-sup condition we get

$$\sigma' = \lambda^{h_{l-1}}S_{h_l}u' + S_{h_l}u^{h_{l-1}}. \tag{3.30}$$

Thus Step 3 of Scheme 1 is equivalent to (3.28), (3.30), $u^{h_l} = u'/\|u'\|_a$, and $\sigma^{h_l} = \sigma'/\|u'\|_a$.

Noting that $(1/\lambda^{h_{l-1}})T_{h_l}u^{h_{l-1}} = \|(1/\lambda^{h_{l-1}})T_{h_l}u^{h_{l-1}}\|_a u_0$ differs from u_0 by only a constant and selecting $u^s = (\lambda^{h_{l-1}}u')/\|T_{h_l}u^{h_{l-1}}\|_a$, we have

$$\left(\frac{1}{\lambda^{h_{l-1}}} - T_{h_l}\right)u^s = u_0, \quad u^{h_l} = \frac{u^s}{\|u^s\|_a}. \tag{3.31}$$

By (3.15), (3.17), (3.18), and (3.31), we see that the conditions of Lemma 3.2 hold. Thus, substituting (3.13) and (3.14) into (3.6), we obtain

$$\mathrm{dist}\left(u^{h_l}, \widehat{M}_{h_l}(\lambda)\right) \leq \frac{16}{\rho}C_6\left(\lambda^{h_{l-1}} - \lambda\right)\left(\mathrm{dist}\left(u_0, \widehat{M}(\lambda)\right) + C_3\|(T - T_{h_l})|_{M(\lambda)}\|_a\right). \tag{3.32}$$

Let the eigenfunctions $\{u_{j,h_l}\}_k^{k+q-1}$ be an orthonormal system of $M_{h_l}(\lambda)$ in the sense of $a(\cdot,\cdot)$, then

$$\mathrm{dist}\left(u^{h_l}, M_{h_l}(\lambda)\right) = \left\|u^{h_l} - \sum_{j=k}^{k+q-1} a\left(u^{h_l}, u_{j,h_l}\right)u_{j,h_l}\right\|_a. \tag{3.33}$$

Let

$$u^* = \sum_{j=k}^{k+q-1} a\left(u^{h_l}, u_{j,h_l}\right) u_{j,h_l}, \tag{3.34}$$

noting $\|u^{h_l} - u^*\|_a \leq \text{dist}(u^{h_l}, \widehat{M}_{h_l}(\lambda))$, from (3.32) we deduce

$$\left\|u^{h_l} - u^*\right\|_a \leq \frac{16}{\rho} C_6\left(\lambda^{h_{l-1}} - \lambda\right)\left(\text{dist}\left(u_0, \widehat{M}(\lambda)\right) + C_3 \left\|(T - T_{h_l})|_{M(\lambda)}\right\|_a\right). \tag{3.35}$$

By Lemma 2.3, there exist $\{u_j^0\}_k^{k+q-1} \subset \widehat{M}(\lambda)$ such that $u_{j,h_l} - u_j^0$ satisfy (2.32). Let

$$u = \sum_{j=k}^{k+q-1} a\left(u^{h_l}, u_{j,h_l}\right) u_j^0, \tag{3.36}$$

then $u \in M(\lambda)$. Using (2.32), we deduce

$$\|u^* - u\|_a = \left\|\sum_{j=k}^{k+q-1} a\left(u^{h_l}, u_{j,h_l}\right)\left(u_{j,h_l} - u_j^0\right)\right\|_a \leq C_2 q \left\|(T_{h_l} - T)|_{M(\lambda)}\right\|_a. \tag{3.37}$$

Combining (3.35) with the above inequality, we have

$$\left\|u^{h_l} - u\right\|_a \leq \frac{16}{\rho} C_6\left(\lambda^{h_{l-1}} - \lambda\right) \text{dist}\left(u_0, \widehat{M}(\lambda)\right) \tag{3.38}$$
$$+ (C_2 q + 1)\left\|(T - T_{h_l})|_{M(\lambda)}\right\|_a.$$

Substituting (3.12) into (3.38), we get (3.7).
From Step 3 of Scheme 1, we know that $b(u^{h_l}, \sigma^{h_l}) = 0$, thus

$$\lambda^{h_l} = \frac{a(u^{h_l}, u^{h_l})}{(u^{h_l}, u^{h_l})_D} = \frac{a(u^{h_l}, u^{h_l}) + 2b(u^{h_l}, \sigma^{h_l})}{(u^{h_l}, u^{h_l})_D}. \tag{3.39}$$

Select $\lambda^r = \lambda^{h_l}$, $u^* = u^{h_l}$, and $\sigma^* = \sigma^{h_l}$. From Lemma 2.4, we get

$$\lambda^{h_l} - \lambda = \frac{a(u^{h_l} - u, u^{h_l} - u) + 2b(u^{h_l} - u, \sigma^{h_l} - \sigma)}{(u^{h_l}, u^{h_l})_D} - \lambda \frac{(u^{h_l} - u, u^{h_l} - u)_D}{(u^{h_l}, u^{h_l})_D}. \tag{3.40}$$

Noting that $\forall v \in W_{h_l}$, $b(u^{h_l} - u, v) = 0$, and $(u^{h_l}, u^{h_l})_D = (1/\lambda^{h_l})a(u^{h_l}, u^{h_l}) = 1/\lambda^{h_l}$, we have

$$
\begin{aligned}
\lambda^{h_l} - \lambda = \lambda^{h_l} a\left(u^{h_l} - u, u^{h_l} - u\right) + 2\lambda^{h_l} b\left(u^{h_l} - u, v - \sigma\right) \\
- \lambda\lambda^{h_l}\left(u^{h_l} - u, u^{h_l} - u\right)_D, \quad \forall v \in W_{h_l}.
\end{aligned}
\tag{3.41}
$$

Note $\|u^{h_l} - u\|_D \leq C_4 \|u^{h_l} - u\|_a$. From (3.41) we obtain (3.8). $\square$

For $l = 1$, Scheme 1 is actually the two-grid discretization scheme established in [7]. By Theorem 3.3 we get the following conclusion directly.

Theorem 3.4. *Suppose $\|T_h - T\|_D \to 0 (h \to 0)$. Let $h_0 = H$ be small properly. Let $(\lambda^{h_1}, u^{h_1}, \sigma^{h_1})$ be an approximate eigenpair obtained by Scheme 1 ($l = 1$). Then there exists $u \in M(\lambda)$ such that*

$$
\begin{aligned}
\left\|u^{h_1} - u\right\|_a \leq \frac{32}{\rho} C_5 C_6 \left(\left|\lambda^{h_0} - \lambda\right|^2 + \left|\lambda^{h_0} - \lambda\right| C_1 \left\|(T - T_{h_0})|_{M(\lambda)}\right\|_D\right) \\
+ (C_2 \times q + 3)\left\|(T - T_{h_1})|_{M(\lambda)}\right\|_a,
\end{aligned}
\tag{3.42}
$$

$$
\left|\lambda^{h_1} - \lambda\right| \leq \left(\lambda^{h_1} + \lambda\lambda^{h_1} C_4^2\right)\left\|u^{h_1} - u\right\|_a^2 + 2\lambda^{h_1} M_2 \left\|u^{h_1} - u\right\|_V \inf_{v \in W_{h_1}} \|\sigma - v\|_W.
\tag{3.43}
$$

Proof. Consider Scheme 1. Here $l = 1$, $u^{h_0} = u_H$. By Lemma 2.3, we know that u^{h_0} approximates $\bar{u} \in \widehat{M}(\lambda)$, and the accuracy order of (λ^{h_0}, u^{h_0}) is lower than (λ_{h_1}, u_{h_1}). Hence, for $l = 1$, the conditions of Theorem 3.3 hold. Select $\bar{u}$ in the proof of Theorem 3.3 such that $u^{h_0} - \bar{u}$ satisfies Lemma 2.3. Then from (2.30), we obtain

$$
\left\|u^{h_0} - \bar{u}\right\|_D \leq C_1 \left\|(T - T_{h_0})|_{M(\lambda)}\right\|_D,
\tag{3.44}
$$

substituting it into (3.7), we obtain (3.42). From (3.8), we deduce (3.43). $\square$

Theorem 3.4 is actually Theorem 3.3 in [7], but we analyze in detail the constants appeared in the error estimates.

From Theorem 3.4 and Theorem 3.3, we know that $\lambda^{h_l} \to \lambda$ ($l \to \infty$) and the convergence rate is high. Thus, we use $\theta \times |\lambda^{h_l} - \lambda^{h_{l-1}}|$ as a posteriori error indicator of $\lambda^{h_l} - \lambda$ (details can be seen in Remark 4.5). Then we establish the following adaptive algorithm.

Scheme 2 (Adaptive Algorithm). Give an error tolerance ε and choose the parameter $0 < \theta \leq 1$, H, t_1, and $h_1 = H^{t_1}$.

Step 1. Solve (2.3)-(2.4) on $V_H \times W_H$: find $(\lambda_H, u_H, \sigma_H) \in R \times V_H \times W_H$, $\|u_H\|_a = 1$ such that

$$
\begin{aligned}
a(u_H, \psi) + b(\psi, \sigma_H) = \lambda_H(u_H, \psi)_D, \quad \forall \psi \in V_H, \\
b(u_H, v) = 0, \quad \forall v \in W_H.
\end{aligned}
\tag{3.45}
$$

Step 2. $u^{h_0} \Leftarrow u_H$, $\lambda^{h_0} \Leftarrow \lambda_H$, $l \Leftarrow 1$.

Step 3. Solve an equation on $V_{h_l} \times W_{h_l}$: find $(u', \sigma') \in V_{h_l} \times W_{h_l}$ such that

$$a(u', \psi) + b(\psi, \sigma') - \lambda^{h_{l-1}}(u', \psi)_D = \left(u^{h_{l-1}}, \psi\right)_{D'} \quad \forall \psi \in V_{h_l},$$

$$b(u', v) = 0, \quad \forall v \in W_{h_l}. \tag{3.46}$$

Let $u^{h_l} = u'/\|u'\|_a$, $\sigma^{h_l} = \sigma'/\|u'\|_a$.

Step 4. Compute the Rayleigh quotient

$$\lambda^{h_l} = \frac{a\left(u^{h_l}, u^{h_l}\right)}{\left(u^{h_l}, u^{h_l}\right)_D}. \tag{3.47}$$

Step 5. If $\theta \times |\lambda^{h_l} - \lambda^{h_{l-1}}| > \varepsilon$, then select t_{l+1}, $h_{l+1} = h_l^{t_{l+1}}$, $l \Leftarrow l + 1$ and return to Step 3; else output $(\lambda^{h_l}, u^{h_l}, \sigma^{h_l})$, stop.

Remark 3.5. In Scheme 2, we use $\theta \times |\lambda^{h_l} - \lambda^{h_{l-1}}|$ as a posteriori error indicator which is global. In order to cope with difficulties caused by local singularity of a complicated problem in calculation, so far, most algorithms designed a local a posteriori error indicator to establish adaptive algorithm with local mesh refinement (e.g., see [20–22]). However, because the eigenvalue problem of electric field is so complicated, that it is very difficult to obtain a local a posteriori error indicator of eigenfunction. Fortunately, the influence of local singularity can be avoided by using the weight method which is established by Costabel and Dauge to discrete the eigenvalue problem of electric field. And the performance of the weigh method is very good (see [3, 11]). Hence, without local mesh refinement, by using the weight method mentioned above our algorithm can also guarantee its high efficiency.

4. The Eigenvalue Problem of Electric Field

Consider the eigenvalue problem of electric field:

$$c^2 \mathbf{curlcurl}\, \vec{u} = \omega^2 \vec{u}, \quad \text{in } \Omega,$$

$$\text{div}\, \vec{u} = 0, \quad \text{in } \Omega, \tag{4.1}$$

$$\vec{u} \times \vec{\gamma} = 0, \quad \text{on } \in \partial\Omega,$$

where Ω is a polyhedron in $\mathbf{R}^3$, $\vec{\gamma}$ is the unit outward normal to $\partial\Omega$.

Physically $\vec{u}$ denotes the electric field, ω denotes the time frequency, and c is the speed of the light velocity. Usually, let $\lambda = \omega^2/c^2$ named eigenvalue.

Let

$$H(\mathbf{curl}, \Omega) = \left\{\vec{q} \in L_2(\Omega)^3 : \mathbf{curl}\vec{q} \in L_2(\Omega)^3\right\},$$

$$H_0(\mathbf{curl}, \Omega) = \left\{\vec{q} \in H(\mathbf{curl}, \Omega) : \vec{q} \times \vec{\gamma}|_{\partial\Omega} = 0\right\}. \tag{4.2}$$

When Ω is a convex polyhedron, we define the following function space:

$$\chi = \left\{ \vec{q} \in H_0(\mathbf{curl}, \Omega) : \operatorname{div} \vec{q} \in L^2(\Omega) \right\}. \tag{4.3}$$

Denote

$$(\vec{q}, \vec{\varphi})_0 = \int_\Omega \vec{q} \cdot \vec{\varphi} \mathrm{d}x, \quad \|\vec{q}\|_0 = (\vec{q}, \vec{q})_0^{1/2}.$$

$$(\vec{q}, \vec{\varphi})_\chi = (\mathbf{curl}\vec{q}, \mathbf{curl}\vec{\varphi})_0 + (\operatorname{div} \vec{q}, \operatorname{div} \vec{\varphi})_0, \quad \|\vec{q}\|_\chi = (\vec{q}, \vec{q})_\chi^{1/2}, \tag{4.4}$$

From [23, 24], we know that $\chi \subset H^1(\Omega)^3$; $(\vec{q}, \vec{\varphi})_\chi$ is a coercive bilinear form on χ, and $\|\vec{q}\|_\chi$ is a norm.

When Ω is a nonconvex polyhedron, the problem is relatively complicated. Let E denote a set of edges of reentrant dihedral angles on $\partial\Omega$, and $d = d(x)$ denote the distance to the set E: $d(x) = \operatorname{dist}(x, \cup_{e \in E} \bar{e})$. We introduce a weight function ω_r which is a nonnegative smooth function of x. It can be represented by d^r in reentrant edge and angular domain. We shall write $\omega_r \simeq d^r$. Define the weighted functional spaces:

$$L_r^2(\Omega) = \left\{ v \in L_{\mathrm{loc}}^2(\Omega) : \omega_r v \in L_2(\Omega) \right\},$$

$$\chi_r = \left\{ \vec{q} \in H_0(\mathbf{curl}, \Omega) : \operatorname{div} \vec{q} \in L_r^2(\Omega) \right\}. \tag{4.5}$$

Denote

$$(\vec{q}, \vec{\varphi})_{L_r^2} = \int_\Omega \omega_r^2 \vec{q} \cdot \vec{\varphi} \, \mathrm{d}x, \qquad \|\vec{q}\|_{L_r^2} = (\vec{q}, \vec{q})_{L_r^2}^{1/2}.$$

$$(\vec{q}, \vec{\varphi})_{\chi_r} = (\mathbf{curl}\, \vec{q}, \mathbf{curl}\, \vec{\varphi})_0 + (\operatorname{div} \vec{q}, \operatorname{div} \vec{\varphi})_{L_r^2}, \qquad \|\vec{q}\|_{\chi_r} = (\vec{q}, \vec{q})_{\chi_r}^{1/2}. \tag{4.6}$$

Let σ_Δ^D be the following smallest singular exponent in the Laplace problem with homogenous Dirichlet boundary condition:

$$\left\{ \phi \in H_0^1(\Omega) : \Delta\phi \in L_2(\Omega) \right\} \subset \cap_{s < \sigma_\Delta^D} H^s(\Omega),$$

$$\left\{ \phi \in H_0^1(\Omega) : \Delta\phi \in L_2(\Omega) \right\} \not\subset H^{\sigma_\Delta^D}(\Omega). \tag{4.7}$$

From the regularity estimate, we know $\sigma_\Delta^D \in ((3/2), 2)$. Let $r_{\min} = 2 - \sigma_\Delta^D$.

From [11, 25], we see that for all $r \in (r_{\min}, 1)$, the seminorm $\|\vec{q}\|_{\chi_r}$ is a norm on χ_r, and $\chi_r \cap H^1(\Omega)^3$ is dense in χ_r.

In the following discussion, we will use $\chi_r, L_r^2(\Omega)$ for both nonconvex and convex domains. We select $r \in (r_{\min}, 1)$ for non-convex domain and $\chi_r = \chi, L_r^2(\Omega) = L^2(\Omega)$ for convex domain.

By introducing the Lagrange multiplier σ, [6] changed (4.1) into the mixed variational formulation: find $(\lambda, \vec{u}, \sigma) \in \mathbf{R}^+ \times \chi_r \times L_r^2(\Omega)$ such that

$$
\left(\vec{u}, \vec{\psi}\right)_{\chi_r} + \left(\operatorname{div} \vec{\psi}, \sigma\right)_{L_r^2} = \lambda \left(\vec{u}, \vec{\psi}\right)_0, \quad \forall \vec{\psi} \in \chi_r,
$$
$$
\left(\operatorname{div} \vec{u}, v\right)_{L_r^2} = 0, \quad \forall v \in L_r^2(\Omega).
$$

(4.8)

Let K^h be a regular simplex partition (tetrahedral partition) of Ω with the mesh diameter h. Define the finite element space $V_h \times W_h \subset \chi_r \times L_r^2(\Omega)$.

Restricting (4.8) on the above-mentioned finite space, we obtain the discrete mixed variational form: find $(\lambda_h, \vec{u}_h, \sigma_h) \in \mathbf{R}^+ \times V_h \times W_h$ such that

$$
\left(\vec{u}_h, \vec{\psi}\right)_{\chi_r} + \left(\operatorname{div} \vec{\psi}, \sigma_h\right)_{L_r^2} = \lambda_h \left(\vec{u}_h, \vec{\psi}\right)_0, \quad \forall \vec{\psi} \in V_h,
$$
$$
\left(\operatorname{div} \vec{u}_h, v\right)_{L_r^2} = 0, \quad \forall v \in W_h.
$$

(4.9)

Set

$$
V = \chi_r, \qquad \|\cdot\|_V = \|\cdot\|_a = \|\cdot\|_{\chi_r},
$$
$$
W = L_r^2(\Omega), \qquad \|\cdot\|_W = \|\cdot\|_{L_r^2},
$$
$$
D = L_2(\Omega)^3, \qquad \|\cdot\|_D = \|\cdot\|_0,
$$
$$
a\left(\vec{q}, \vec{\psi}\right) = \left(\vec{q}, \vec{\psi}\right)_{\chi_r} b\left(\vec{\psi}, v\right) = \left(\operatorname{div} \vec{\psi}, v\right)_{L_r^2}.
$$

(4.10)

Then (4.8) and (4.9) can be written as (2.3)-(2.4) and (2.5)-(2.6), respectively (it is needed to add $\vec{}$ for the vector function, e.g., u, ψ should be written in the forms of $\vec{u}, \vec{\psi}$).

We apply Schemes 1 and 2 to the eigenvalue problem of electric field (4.8). Adding the symbol $\vec{}$ for the vector function, we get a multi-grid discretization scheme and adaptive algorithm for mixed finite element of the eigenvalue problem of electric field which are still called Schemes 1 and 2.

It is easy to know that $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ are continuous bilinear forms on $V \times V$ and $V \times W$, respectively. $V \xhookrightarrow{c} D$. (It is true obviously when Ω is convex; when Ω is non-convex, see [25].)

Consider the source problem associated with (4.8): find $(\vec{w}, p) \in \chi_r \times L_r^2(\Omega)$ such that

$$
\left(\vec{w}, \vec{\psi}\right)_{\chi_r} + \left(\operatorname{div} \vec{\psi}, p\right)_{L_r^2} = \left(\vec{f}, \vec{\psi}\right)_0, \quad \forall \vec{\psi} \in \chi_r,
$$
$$
\left(\operatorname{div} \vec{w}, v\right)_{L_r^2} = 0, \quad \forall v \in L_r^2(\Omega).
$$

(4.11)

For (4.11) and its conforming finite element approximation, condition (C1) of Lemma 2.1 holds obviously; [3] has proved that condition (C2) holds; assume that the discrete inf-sup condition (C3) holds, then conditions of Lemma 2.1 hold. Thus we can define operators T, S, T_h, and S_h. (4.8) and (4.9) can be written as (2.15) and (2.16)-(2.17), respectively.

The following Lemma 4.1 is cited from [3, 25].

Lemma 4.1. *Equation* (4.1) *is equivalent to* (4.8), *and the solutions of* (4.8) $\sigma = S(\lambda \vec{u}) = 0$ *and* $\vec{u} \in \chi_r$, $\operatorname{div} \vec{u} = 0$.

Note that Lemma 4.1 shows that for the eigenvalue problem of electric field, the second term on the right-hand side in (3.8) is equal to 0.

Theorem 4.2. *Suppose that the discrete inf-sup condition (C3) holds. Then* $\|T - T_h\|_D \to 0(h \to 0)$. *Let* $(\lambda_h, \vec{u}_h, \sigma_h)$ *be the kth eigenpair of* (4.9) *with* $\|\vec{u}_h\|_{\chi_r} = 1$, *and let* λ *be the kth eigenvalue of* (4.8). *Then* $\lambda_h \to \lambda(h \to 0)$, *and there exists an eigenfunction* $\vec{u}$ *corresponding to* λ *with* $\|\vec{u}\|_{\chi_r} = 1$, *such that*

$$|\lambda_h - \lambda| \leq \left(\lambda_h + \lambda \lambda_h C_4^2 \right) C_2^2 \left\| (T_h - T)|_{M(\lambda)} \right\|_{\chi_r}^2, \tag{4.12}$$

$$\|\vec{u} - \vec{u}_h\|_{\chi_r} \leq C_2 \left\| (T_h - T)|_{M(\lambda)} \right\|_{\chi_r}, \tag{4.13}$$

let $\vec{u} \in \widehat{M}(\lambda)$, *then there exists* $\vec{u}_h \in M_h(\lambda)$ *such that*

$$\|\vec{u} - \vec{u}_h\|_{\chi_r} \leq C_3 \left\| (T_h - T)|_{M(\lambda)} \right\|_{\chi_r}, \tag{4.14}$$

where C_2, C_3, *and* C_4 *are constants independent of h.*

Proof. From the preceding discussion and hypotheses of the theorem, we know that $V \overset{c}{\hookrightarrow} D$, $a(u,v)$ is symmetric, and the conditions of Lemma 2.1 hold. Besides, since K^h is a regular partition, when Ω is a convex polyhedron, $\chi_r \subset H^1(\Omega)^3 (\chi_r = \chi)$; when Ω is a non-convex polyhedron, $\chi_r \cap H^1(\Omega)^3$ is dense in χ_r. Since $C^\infty(\overline{\Omega})^3$ is dense in $H^1(\Omega)^3$, thus, no matter Ω is convex or non-convex, $C^\infty(\overline{\Omega})^3$ is dense in χ_r. For any given $\vec{f} \in D$, we have $T\vec{f} \in \chi_r$. Thus for any $\varepsilon > 0$, according to the density, we know that there exists $\vec{w} \in C^\infty(\overline{\Omega})^3$ such that

$$\left\| T\vec{f} - \vec{w} \right\|_{\chi_r} \leq \frac{\varepsilon}{2}. \tag{4.15}$$

Selecting $h_0 > 0$ being small properly, when $0 < h \leq h_0$, we have

$$\|\vec{w} - I_h \vec{w}\|_{\chi_r} \leq Ch|\vec{w}|_2 \leq \frac{\varepsilon}{2}, \tag{4.16}$$

where $I_h : C^\infty(\overline{\Omega})^3 \to V_h$ is an interpolation operator. Thus

$$\inf_{\vec{q} \in V_h} \left\| T\vec{f} - \vec{q} \right\|_{\chi_r} \leq \left\| T\vec{f} - I_h \vec{w} \right\|_{\chi_r}$$

$$\leq \left\| T\vec{f} - \vec{w} \right\|_{\chi_r} + \|\vec{w} - I_h \vec{w}\|_{\chi_r} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \tag{4.17}$$

Namely, $\inf_{\vec{q} \in V_h} \| T\vec{f} - \vec{q} \|_{\chi_r} \to 0(h \to 0)$. Hence (2.26) is true. Analogously, using the density of $C^\infty(\overline{\Omega})$ in $L_r^2(\Omega)$, we deduce that $\inf_{v \in W_h} \| S\vec{f} - v \|_{L_r^2} \to 0(h \to 0)$, namely, (2.27), is true.

Hence, from Theorem 2.2 and Lemma 2.3, we know that $\|T - T_h\|_D \to 0$ $(h \to 0)$, $\lambda_h \to \lambda$ $(h \to 0)$, (4.13), and (4.14) hold. From (2.38), (4.13), and Lemma 4.1, we get (4.12). The proof is completed. $\square$

Denote

$$\varepsilon_\lambda(h) = \sup_{\vec{u} \in \widehat{M}(\lambda)} \inf_{\vec{\varphi} \in V_h} \left\| \vec{u} - \vec{\varphi} \right\|_{X_r}. \tag{4.18}$$

From Lemma 2.1, noting $\sigma = 0$, we deduce

$$
\begin{aligned}
\left\| (T_h - T)|_{M(\lambda)} \right\|_{X_r} &= \sup_{\vec{u} \in \widehat{M}(\lambda)} \left\| (T_h - T)u \right\|_{X_r} \\
&\leq \sup_{\vec{u} \in \widehat{M}(\lambda)} C_e \inf_{\vec{\varphi} \in V_h} \left\| T\vec{u} - \vec{\varphi} \right\|_{X_r} \leq \lambda^{-1} C_e \varepsilon_\lambda(h) \equiv C_7 \varepsilon_\lambda(h).
\end{aligned}
\tag{4.19}
$$

Theorem 4.3. *Assume that the discrete inf-sup condition (C3) holds, $h_0 = H$ is small properly, Condition 1 holds and $\sup_i t_i < 3$. Let (λ^{h_l}, u^{h_l}) be an approximate eigenpair obtained by Scheme 1, then there exists $\vec{u} \in M(\lambda)$ such that*

$$\left\| \vec{u}^{h_l} - \vec{u} \right\|_{X_r} \leq 2(C_2 \times q + 3)C_7 \varepsilon_\lambda(h_l), \tag{4.20}$$

$$\left| \lambda^{h_l} - \lambda \right| \leq 4\left(\lambda^{h_l} + \lambda\lambda^{h_l} C_4^2 \right)(C_2 \times q + 3)^2 C_7^2 \varepsilon_\lambda(h_l)^2. \tag{4.21}$$

Proof. We use induction to complete the proof. Note that the conditions of Theorem 2.2 hold. For $l = 1$, Scheme 1 is actually two-grid discretization scheme. Substituting (4.19) into (4.12) and (4.13), we derive

$$\|\vec{u}_H - \vec{u}\|_{X_r} \leq C_2 C_7 \varepsilon_\lambda(H),$$

$$|\lambda - \lambda_H| \leq \left(\lambda_H + \lambda\lambda_H C_4^2 \right) C_2^2 C_7^2 \varepsilon_\lambda(H)^2. \tag{4.22}$$

Combining (3.7) with $l = 1$ and the above two inequalities, we know that there exists $\vec{u} \in M(\lambda)$ such that

$$
\begin{aligned}
\left\| u^{h_1} - \vec{u} \right\|_{X_r} &\leq \frac{32}{\rho} C_5 C_6 \left(\left| \lambda^{h_0} - \lambda \right|^2 + \left| \lambda^{h_0} - \lambda \right| \left\| u^{h_0} - \vec{u} \right\|_D \right) \\
&\quad + (C_2 \times q + 3) \left\| (T - T_{h_1})|_{M(\lambda)} \right\|_{X_r}
\end{aligned}
$$

$$\leq \frac{32}{\rho} C_5 C_6 \left(\left(\lambda_H + \lambda \lambda_H C_4^2 \right)^2 C_2^4 C_7^4 \varepsilon_\lambda(H)^4 \right.$$

$$+ \left(\lambda_H + \lambda \lambda_H C_4^2 \right) C_2^2 C_7^2 \varepsilon_\lambda(H)^2 C_2 C_4 C_7 \varepsilon_\lambda(H) \right)$$

$$+ (C_2 \times q + 3) C_7 \varepsilon_\lambda(h_1)$$

$$\leq 2(C_2 \times q + 3) C_7 \varepsilon_\lambda(h_1).$$

$$(4.23)$$

Since $\sigma = 0$, substituting (4.23) into (3.43), we deduce

$$\left| \lambda^{h_1} - \lambda \right| \leq \left(\lambda^{h_1} + \lambda \lambda^{h_1} C_4^2 \right) \left\| \vec{u}^{h_1} - \vec{u} \right\|_{X_r}^2$$

$$\leq \left(\lambda^{h_1} + \lambda \lambda^{h_1} C_4^2 \right) 4 (C_2 \times q + 3)^2 C_7^2 \varepsilon_\lambda(h_1)^2.$$

$$(4.24)$$

The above two inequalities show that Theorem 4.2 is true for $l = 1$.

Suppose that the theorem is true for $l - 1$, then by Theorem 3.3, we get

$$\left\| \vec{u}^{h_l} - \vec{u} \right\|_{X_r} \leq \frac{32}{\rho} C_5 C_6 \left(\left(\lambda^{h_{l-1}} + \lambda \lambda^{h_{l-1}} C_4^2 \right)^2 4^2 (C_2 \times q + 3)^4 C_2^4 C_7^4 \varepsilon_\lambda(h_{l-1})^4 \right.$$

$$+ \left(\lambda^{h_{l-1}} + \lambda \lambda^{h_{l-1}} C_4^2 \right) 4 (C_2 \times q + 3)^2 C_7^2 \varepsilon_\lambda(h_{l-1})^2 2 (C_2 \times q + 3) C_7 \varepsilon_\lambda(h_{l-1}) \right)$$

$$+ (C_2 \times q + 3) C_7 \varepsilon_\lambda(h_l)$$

$$\leq 2(C_2 \times q + 3) C_7 \varepsilon_\lambda(h_l),$$

$$(4.25)$$

That is, (4.20) is valid.

From (4.20) and (3.8), we obtain (4.21). The proof is completed.   □

Assume that K^h is a regular simplex partition (tetrahedral partition) of Ω with the mesh diameter h. Let V_h and W_h be the P_{k+1}-P_k finite element spaces as follows:

$$V_h = \left\{ \vec{q} \in C^0 \left(\overline{\Omega} \right)^3 : \vec{q} \times \vec{\gamma}|_{\partial \Omega} = 0, \vec{q}|_\kappa \in P_{k+1}(\kappa)^3 \cdot \forall \kappa \in K^h \right\},$$

$$W_h = \left\{ v \in C^0 \left(\overline{\Omega} \right) : v|_\kappa \in P_k(\kappa), \forall \kappa \in K^h, v|_{E_h} = 0 \right\}.$$

$$(4.26)$$

Here we set $E_h = \cup_{\kappa \in K^h, \partial \kappa \cap E \neq \phi} \kappa$. $v|_{E_h} = 0$ means that v is equal to 0 on the tetrahedron where reentrant edge and angular point are adjacent. Considering finite element approximation of (4.11), for the 3-DP_2-iso-P_1 Taylor-Hood finite element, Ciarlet and Girault [26] have discussed that the discrete inf-sup condition (C3) holds when Ω is a convex domain; for the P_{k+1}-P_k element, Ciarlet and Hechme have proved that the discrete inf-sup condition (C3) holds when Ω is a polyhedron (see Section 2.2 in [4] and pp. 509 in [3]).

From the above mentioned, we know that the P_{k+1}-P_k element approximation of (4.11) satisfies the conditions of Theorem 4.2.

Let σ_Δ^N be the smallest singular exponent in the Laplace problem with homogenous Neumman boundary condition, then $\sigma_\Delta^N \in (3/2, 2)$. Denote $\tau = \min(r - r_{\min}, \sigma_\Delta^N - 1)$.

Corollary 4.4. *Assume that $h_0 = H$ is small properly, $t_i \in (1, 3]$ $(i = 1, 2, ., l)$, and $h_i = h_{i-1}^{t_i}$ $(i = 1, 2, \ldots, l)$. Let (λ^{h_l}, u^{h_l}) be an approximate eigenpair of the P_{k+1}-P_k element obtained by Scheme 1. Then when Ω is a convex domain, there exists $\vec{u} \in M(\lambda)$ such that*

$$\left\| \vec{u}^{h_l} - \vec{u} \right\|_{X_r} \leq 2(C_2 \times q + 3)C_7 C' h_l, \tag{4.27}$$

$$\left| \lambda^{h_l} - \lambda \right| \leq 4\left(\lambda^{h_l} + \lambda\lambda^{h_l} C_4^2 \right)(C_2 \times q + 3)^2 C_7^2 C'^2 h_l^2. \tag{4.28}$$

When Ω is a non-convex domain, there exists $\vec{u} \in M(\lambda)$ such that

$$\left\| \vec{u}^{h_l} - \vec{u} \right\|_{X_r} \leq 2(C_2 \times q + 3)C_7 C'' h_l^\mu, \quad \forall \mu \in (0, \tau), \tag{4.29}$$

$$\left| \lambda^{h_l} - \lambda \right| \leq 4\left(\lambda^{h_l} + \lambda\lambda^{h_l} C_4^2 \right)(C_2 \times q + 3)^2 C_7^2 C''^2 h_l^{2\mu}, \quad \forall \mu \in (0, \tau), \tag{4.30}$$

where C' and C'' are determined by (4.31) and (4.32) in the proof, respectively.

Proof. The hypotheses of Corollary 4.4 imply that the conditions of Theorem 4.3 hold. When Ω is convex, for any $\vec{u} \in M(\lambda)$ we have $\vec{u} = T(\lambda\vec{u}) \in H^2(\Omega)$ (see (44) in [1]). Thus there exists C' independent of h_l $(l = 1, 2, \ldots)$ such that

$$\varepsilon_\lambda(h_l) \leq C' h_l. \tag{4.31}$$

Substituting the above inequality into (4.20) and (4.21), we get (4.27) and (4.28), respectively.

When Ω is a non-convex domain, for any $\vec{u} \in M(\lambda)$, by (36) in [3] we know that there exists C'' independent of h_l $(l = 1, 2, \ldots)$, such that

$$\varepsilon_\lambda(h_l) \leq C'' h_l^\mu. \tag{4.32}$$

Substituting the above inequality into (4.20) and (4.21), we derive (4.29) and (4.30), respectively. □

Remark 4.5. From Corollary 4.4, we see that the constants in the error estimates are not only independent of the mesh diameter but also independent of the iterative degrees. Thus, when $l \to \infty$, we have $\lambda^{h_l} \to \lambda$. Suppose that the precision order of (4.28) is optimal which cannot be improved any more, then

$$\lambda^{h_l} - \lambda^{h_{l+1}} = \lambda^{h_l} - \lambda + \lambda - \lambda^{h_{l+1}} = O\left(h_l^{2r} \right),$$

$$\lambda^{h_{l-1}} - \lambda^{h_l} = \lambda^{h_{l-1}} - \lambda + \lambda - \lambda^{h_l} = O\left(h_{l-1}^{2r} \right), \tag{4.33}$$
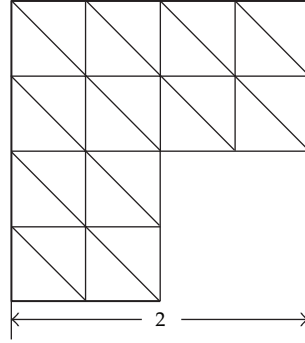
Figure 1

where for convex domain $r = 1$, while for non-convex domain $r = \mu \in (0, \tau)$ but approximates τ arbitrarily. Therefore we have that $\lambda^{h_l} - \lambda^{h_{l+1}} \leq \lambda^{h_{l-1}} - \lambda^{h_l}$. Then

$$
\begin{aligned}
\left|\lambda^{h_l} - \lambda\right| &\leq \left|\lambda^{h_l} - \lambda^{h_{l+1}}\right| + \left|\lambda^{h_{l+1}} - \lambda\right| \\
&\leq \left|\lambda^{h_{l-1}} - \lambda^{h_l}\right| + \vartheta\left(h_{l+1}^{2r}\right),
\end{aligned}
\tag{4.34}
$$

thus we can use $\eta(\lambda^{h_l}) = \theta \times |\lambda^{h_{l-1}} - \lambda^{h_l}|$ as a posteriori error indicator of $|\lambda^{h_l} - \lambda|$, where $\theta \in (0, 1]$.

5. Numerical Experiments.

Consider the eigenvalue problem (4.1) of electric field, where $\overline{\Omega} = [0, \pi] \times [0, \pi]$ is a square domain or $\overline{\Omega} = [-1, 0] \times [-1, 0] \cup [-1, 1] \times [0, 1]$ is an L-shaped domain. For the square domain, the first five exact eigenvalues are $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = 2$, and $\lambda_4 = \lambda_5 = 4$; for the L-shaped domain, the first five eigenvalues are $\lambda_1 \approx 1.47562182408$, $\lambda_2 \approx 3.53403136678$, $\lambda_3 = \lambda_4 = \pi^2 \approx 9.86960440109$, and $\lambda_5 \approx 11.3894793979$, and the first eigenfunction has a strong singularity (see, e.g., [3]).

We adopt a uniform isosceles right triangulation for Ω (the edge in each element is along three fixed directions, see Figure 1) to produce the meshes K^{h_l} with mesh diameter h_l.

The definition of P_2-P_1 mixed finite element spaces is given by

$$
\begin{aligned}
V_{h_l} &= \left\{ \vec{q} \in C^0\left(\overline{\Omega}\right)^2 : \vec{q} \times \vec{\gamma}|_{\partial\Omega} = 0, \vec{q}|_\kappa \in P_2(\kappa)^2, \forall \kappa \in K^{h_l} \right\}, \\
W_{h_l} &= \left\{ v \in C^0\left(\overline{\Omega}\right) : v|_\kappa \in P_1(\kappa), \forall \kappa \in K^{h_l}, v|_{E_{h_l}} = 0 \right\}.
\end{aligned}
\tag{5.1}
$$

We make use of Matlab to compute the first five approximate eigenvalues by using Scheme 1 with P_2-P_1 element. The numerical results are listed in Tables 1, 2, and 3.

Table 1: The results on the square by Scheme 1 (P_2-P_1 element) for the eigenvalue problem of electric field ($r = 0$).

k	H	l	h_l	λ_{k,h_l}	$\lambda_k^{h_l}$	$\eta(\lambda_{k,h_l})/e(\lambda_{k,h_l})$	$\eta(\lambda_k^{h_l})/e(\lambda_k^{h_l})$
1	$\sqrt{2}/8$	1	$\sqrt{2}/32$	1.0000001280750	1.00000012807483	248.41	248.41
1	$\sqrt{2}/8$	2	$\sqrt{2}/64$	1.0000000080355	1.00000000803501	14.94	14.94
1	$\sqrt{2}/8$	3	$\sqrt{2}/128$	—	1.00000000050340	—	14.96
3	$\sqrt{2}/8$	1	$\sqrt{2}/32$	2.0000017958515	2.00000179585163	245.62	245.62
3	$\sqrt{2}/8$	2	$\sqrt{2}/64$	2.0000001126165	2.00000011261761	14.95	14.95
3	$\sqrt{2}/8$	3	$\sqrt{2}/128$	—	2.00000000704840	—	14.98
4	$\sqrt{2}/8$	1	$\sqrt{2}/32$	4.0000081820714	4.00000818207150	242.08	242.08
4	$\sqrt{2}/8$	2	$\sqrt{2}/64$	4.0000005140035	4.00000051400385	14.92	14.92
4	$\sqrt{2}/8$	3	$\sqrt{2}/128$	—	4.00000003219500	—	14.97

Table 2: The results on the L-shaped domain by Scheme 1 (P_2-P_1 element) for the eigenvalue problem of electric field ($r = 0.5$).

k	H	l	h_l	λ_{k,h_l}	$\lambda_k^{h_l}$	$\eta(\lambda_{k,h_l})/e(\lambda_{k,h_l})$	$\eta(\lambda_k^{h_l})/e(\lambda_k^{h_l})$
1	$\sqrt{2}/10$	1	$\sqrt{2}/40$	2.619901684020	2.619902307450	0.16	0.16
1	$\sqrt{2}/10$	2	$\sqrt{2}/80$	2.545994662814	2.545994677427	0.69	0.69
1	$\sqrt{2}/10$	3	$\sqrt{2}/160$	—	2.468072524066	—	0.78
2	$\sqrt{2}/10$	1	$\sqrt{2}/40$	3.540738971244	3.540738974231	5.75	5.75
2	$\sqrt{2}/10$	2	$\sqrt{2}/80$	3.536561194304	3.536561194305	1.65	1.65
2	$\sqrt{2}/10$	3	$\sqrt{2}/160$	—	3.534975905948	—	1.67
3	$\sqrt{2}/10$	1	$\sqrt{2}/40$	9.869612668412	9.869612668412	244.51	244.51
3	$\sqrt{2}/10$	2	$\sqrt{2}/80$	9.869604920374	9.869604920377	14.92	14.92
3	$\sqrt{2}/10$	3	$\sqrt{2}/160$	—	9.869604433619	—	14.92
5	$\sqrt{2}/10$	1	$\sqrt{2}/40$	11.392491049450	11.392491049225	7.11	7.11
5	$\sqrt{2}/10$	2	$\sqrt{2}/80$	11.390607786437	11.390607786437	1.67	1.67
5	$\sqrt{2}/10$	3	$\sqrt{2}/160$	—	11.389899921315	—	1.68

Table 3: The results on the L-shaped domain by Scheme 1 (P_2-P_1 element) for the eigenvalue problem of electric field ($r = 0.95$).

k	H	l	h_l	λ_{k,h_l}	$\lambda_k^{h_l}$	$\eta(\lambda_{k,h_l})/e(\lambda_{k,h_l})$	$\eta(\lambda_k^{h_l})/e(\lambda_k^{h_l})$
1	$\sqrt{2}/10$	1	$\sqrt{2}/40$	1.550099678021	1.550100277590	1.88	1.88
1	$\sqrt{2}/10$	2	$\sqrt{2}/80$	1.512784318422	1.512784324775	1.00	1.00
1	$\sqrt{2}/10$	3	$\sqrt{2}/160$	—	1.492972425344	—	1.14
2	$\sqrt{2}/10$	1	$\sqrt{2}/40$	3.534598663496	3.534598663588	14.65	14.65
2	$\sqrt{2}/10$	2	$\sqrt{2}/80$	3.534154160497	3.534154160495	3.62	3.62
2	$\sqrt{2}/10$	3	$\sqrt{2}/160$	—	3.534055109225	—	4.17
3	$\sqrt{2}/10$	1	$\sqrt{2}/40$	9.869612641692	9.869612641692	243.23	243.23
3	$\sqrt{2}/10$	2	$\sqrt{2}/80$	9.869604919289	9.869604919288	14.90	14.90
3	$\sqrt{2}/10$	3	$\sqrt{2}/160$	—	9.869604433575	—	14.92
5	$\sqrt{2}/10$	1	$\sqrt{2}/40$	11.389671749018	11.389671749041	28.83	28.83
5	$\sqrt{2}/10$	2	$\sqrt{2}/80$	11.389519405478	11.389519405477	3.81	3.81
5	$\sqrt{2}/10$	3	$\sqrt{2}/160$	—	11.389487122250	—	4.18

In Tables 1–3, $\lambda_{1,h_l}, \lambda_{2,h_l}, \ldots, \lambda_{5,h_l}$ denote the first five eigenvalues obtained by using the mixed element on K^{h_l} directly; $\lambda_1^{h_l}, \lambda_2^{h_l}, \ldots, \lambda_5^{h_l}$ denote the first five eigenvalues obtained by Scheme 1:

$$e(\lambda_{k,h_l}) = |\lambda_{k,h_l} - \lambda_k|, \qquad e\left(\lambda_k^{h_l}\right) = \left|\lambda_k^{h_l} - \lambda_k\right|,$$

$$\eta(\lambda_{k,h_l}) = \theta|\lambda_{k,h_l} - \lambda_{k,h_{l-1}}|, \quad \eta\left(\lambda_k^{h_l}\right) = \theta\left|\lambda_k^{h_l} - \lambda_k^{h_{l-1}}\right|, \quad (\theta = 1).$$

(5.2)

From Tables 1–3, we can see that (1) $\lambda_k^{h_l}$ and λ_{k,h_l} have the same accuracy. (2) It fails to find λ_{k,h_l} by direct computation by using the mixed element on K^{h_l} with $h_l = \sqrt{2}/128$ in the case of the square domain and $h_l = \sqrt{2}/160$ in the case of the L-shaped domain (here Matlab shows that Sparse lu with 4 outputs (UMFPACK) failed), but Scheme 1 still works. (3) $\eta(\lambda_k^{h_l}) = |\lambda_k^{h_l} - \lambda_k^{h_{l-1}}|$ is an efficient and reliable error indicator of $\lambda_k^{h_l}$.

It can be seen from Tables 1–3 that, in the calculation of error indicators, θ should be selected as follows: in the case of the square domain, $\theta = 1/15$; in the case of the L-shaped domain, θ is equal to $1, 3/5, 1/15$, and $3/5$, respectively when $r = 0.5$, and θ is equal to $1, 1/4, 1/15$ and $1/4$ respectively, when $r = 0.95$.

Remark 5.1. Taking Table 1, for example, we illustrate how to select t_i next. In Table 1, $h_0 = \sqrt{2}/8$, $h_1 = \sqrt{2}/32$, $h_2 = \sqrt{2}/64$, and $h_3 = \sqrt{2}/128$. According to $h_i = h_{i-1}^{t_i}$, by calculation, we get that $t_1 \approx 1.80$, $t_2 \approx 1.22$, and $t_3 \approx 1.18$.

Acknowledgments

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