Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 182947, 15 pages
doi:10.1155/2012/182947

Research Article

Cyclic Contractions on *G*-Metric Spaces

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Received 29 June 2012; Accepted 1 September 2012

Academic Editor: Irena Rachůnková

Conditions for existence and uniqueness of fixed points of two types of cyclic contractions defined on G-metric spaces are established and some illustrative examples are given. In addition, cyclic maps satisfying integral type contractive conditions are presented as applications.

1. Introduction

The extensive application potential of fixed point theory in various fields resulted in several generalizations of the metric spaces. Amongst them, one can mention quasimetric spaces, partial metric spaces, rectangular metric spaces, D-metric spaces, and G-metric spaces. Perhaps one of the most interesting generalizations is the G-metric space. Introduced by Mustafa and Sims [1] in 2006, the concept of G-metric space has drawn the attention of mathematicians and became a very popular subject especially from the point of view of fixed point theory [2–13].

Another attractive topic in fixed point theory is the concept of cyclic maps and best proximity points introduced by Kirk et al. [14] in 2003. Cyclic maps and in particular the fixed points of cyclic maps have been a subject of growing interest recently (see, e.g., [15–27]).

The purpose of this work is to combine these two notions and investigate cyclic maps on G-metric spaces. We concentrate on two types of cyclic contractions: cyclic type Banach contractions and cyclic weak ϕ-contractions.

Mustafa and Sims [1] introduced the concept of G-metric spaces as follows.

Definition 1.1 (see [1]). Let X be a nonempty set, $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:

(G1) $G(x, y, z) = 0$ if $x = y = z$,

(G2) $0 < G(x,x,y)$ for all $x,y \in X$ with $x \neq y$,

(G3) $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $y \neq z$,

(G4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$ (symmetry in all three variables),

(G5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$ (rectangle inequality).

Then the function G is called a generalized metric, or, more specifically, a G-metric on X, and the pair (X,G) is called a G-metric space.

Note that every G-metric on X induces a metric d_G on X defined by

$$d_G(x,y) = G(x,y,y) + G(y,x,x), \quad \forall x,y \in X. \tag{1.1}$$

To have better idea about the subject, we give the following examples of G-metrics.

Example 1.2. Let (X,d) be a metric space. The function $G : X \times X \times X \to [0,+\infty)$, defined by

$$G(x,y,z) = \max\{d(x,y), d(y,z), d(z,x)\}, \tag{1.2}$$

for all $x,y,z \in X$, is a G-metric on X.

Example 1.3. Let $X = [0,+\infty)$. The function $G : X \times X \times X \to [0,+\infty)$, defined by

$$G(x,y,z) = |x-y| + |y-z| + |z-x|, \tag{1.3}$$

for all $x,y,z \in X$, is a G-metric on X.

In their initial paper, Mustafa and Sims [1] defined also the basic topological concepts in G-metric spaces as follows.

Definition 1.4 (see [1]). Let (X,G) be a G-metric space, and let $\{x_n\}$ be a sequence of points of X. We say that $\{x_n\}$ is G-convergent to $x \in X$ if

$$\lim_{n,m \to +\infty} G(x, x_n, x_m) = 0, \tag{1.4}$$

that is, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x,x_n,x_m) < \varepsilon$, for all $n,m \geq N$. We call x the limit of the sequence and write $x_n \to x$ or $\lim_{n\to+\infty} x_n = x$.

Proposition 1.5 (see [1]). *Let (X,G) be a G-metric space. The following are equivalent:*

(1) *$\{x_n\}$ is G-convergent to x,*

(2) *$G(x_n,x_n,x) \to 0$ as $n \to +\infty$,*

(3) *$G(x_n,x,x) \to 0$ as $n \to +\infty$,*

(4) *$G(x_n,x_m,x) \to 0$ as $n,m \to +\infty$.*

Definition 1.6 (see [1]). Let (X,G) be a G-metric space. A sequence $\{x_n\}$ is called a G-Cauchy sequence if, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n,x_m,x_l) < \varepsilon$ for all $m,n,l \geq N$, that is, $G(x_n,x_m,x_l) \to 0$ as $n,m,l \to +\infty$.

Proposition 1.7 (see [1]). *Let (X, G) be a G-metric space. Then the following are equivalent:*

(1) *the sequence $\{x_n\}$ is G-Cauchy,*

(2) *for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $m, n \geq N$.*

Definition 1.8 (see [1]). A G-metric space (X, G) is called G-complete if every G-Cauchy sequence is G-convergent in (X, G).

Definition 1.9. Let (X, G) be a G-metric space. A mapping $F : X \times X \times X \to X$ is said to be continuous if for any three G-convergent sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converging to x, y, and z, respectively, $\{F(x_n, y_n, z_n)\}$ is G-convergent to $F(x, y, z)$.

Note that each G-metric on X generates a topology τ_G on X whose base is a family of open G-balls $\{B_G(x, \varepsilon), x \in X, \varepsilon > 0\}$, where $B_G(x, \varepsilon) = \{y \in X, G(x, y, y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$. A nonempty set $A \subset X$ is G-closed in the the G-metric space (X, G) if $\overline{A} = A$. Observe that

$$x \in \overline{A} \Longleftrightarrow B_G(x, \varepsilon) \cap A \neq \emptyset, \quad \forall \varepsilon > 0. \tag{1.5}$$

Finally, we have the following proposition.

Proposition 1.10. *Let (X, G) be a G-metric space and A be a nonempty subset of X. A is G-closed if for any G-convergent sequence $\{x_n\}$ in A with limit x, one has $x \in A$.*

2. Banach Contractive Cyclic Maps on G-Metric Spaces

Our first result is a fixed point theorem which is the Banach contraction mapping analog for cyclic maps on G-metric spaces.

Theorem 2.1. *Let (X, G) be a G-complete G-metric space and $\{A_j\}_{j=1}^m$ be a family of nonempty G-closed subsets of X. Let $Y = \cup_{j=1}^m A_j$ and $T : Y \to Y$ be a map satisfying*

$$T(A_j) \subseteq A_{j+1}, \quad j = 1, \ldots, m, \text{ where } A_{m+1} = A_1. \tag{2.1}$$

If there exists $k \in (0, 1)$ such that

$$G(Tx, Ty, Tz) \leq kG(x, y, z) \tag{2.2}$$

holds for all $x \in A_j$ and $y, z \in A_{j+1}$, $j = 1, \ldots, m$ then, T has a unique fixed point in $\cap_{j=1}^m A_j$.

Proof. We prove first the existence part. Take an arbitrary $x_0 \in Y$ and without loss of generality assume that $x_0 \in A_1$. Define the sequence $\{x_n\}$ as

$$x_n = Tx_{n-1}, \quad n = 1, 2, 3, \ldots. \tag{2.3}$$

Since T is cyclic, $x_0 \in A_1$, $x_1 = Tx_0 \in A_2$, $x_2 = Tx_1 \in A_3, \ldots,$ and so on. If $x_{n_0+1} = x_{n_0}$ for some $n_0 \in \mathbb{N}$, then obviously, the fixed point of T is x_{n_0}. Assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$.

Put $x = x_{n-1}$ and $y = z = x_n$ in (2.2). Then

$$\begin{aligned}0 &\le G(Tx_{n-1}, Tx_n, Tx_n) = G(x_n, x_{n+1}, x_{n+1}) \le kG(x_{n-1}, x_n, x_n)\\&\le k^2 G(x_{n-2}, x_{n-1}, x_{n-1})\\&\ \ \vdots\\&\le k^n G(x_0, x_1, x_1).\end{aligned} \tag{2.4}$$

Then, we have,

$$0 \le G(x_n, x_{n+1}, x_{n+1}) \le k^n G(x_0, x_1, x_1), \tag{2.5}$$

which upon letting $n \to \infty$ implies

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0. \tag{2.6}$$

On the other hand, by symmetry (G4) and the rectangle inequality (G5), we have

$$G(x, y, y) = G(y, y, x) \le G(y, x, x) + G(x, y, x) = 2G(y, x, x). \tag{2.7}$$

The inequality (2.7) with $x = x_n$ and $y = x_{n-1}$ becomes

$$G(x_n, x_{n-1}, x_{n-1}) \le 2G(x_{n-1}, x_n, x_n). \tag{2.8}$$

Letting $n \to \infty$ in (2.8), we get

$$\lim_{n\to\infty} G(x_n, x_{n-1}, x_{n-1}) = 0. \tag{2.9}$$

We show next that the sequence $\{x_n\}$ is a Cauchy sequence in the metric space (X, d_G) where d_G is given in (1.1). For $n \ge l$ we have

$$\begin{aligned}d_G(x_n, x_l) &\le d_G(x_n, x_{n-1}) + d_G(x_{n-1}, x_{n-2}) + \cdots + d_G(x_{l+1}, x_l)\\&= G(x_n, x_{n-1}, x_{n-1}) + G(x_{n-1}, x_n, x_n)\\&\quad + G(x_{n-1}, x_{n-2}, x_{n-2}) + G(x_{n-2}, x_{n-1}, x_{n-1}) + \cdots\\&\quad + G(x_{l+1}, x_l, x_l) + G(x_l, x_{l+1}, x_{l+1})\\&= \sum_{i=l+1}^{n} [G(x_i, x_{i-1}, x_{i-1}) + G(x_{i-1}, x_i, x_i)],\end{aligned} \tag{2.10}$$

and making use of (2.4) and (2.8) we obtain

$$\begin{aligned} 0 \leq d_G(x_n, x_l) &\leq \sum_{i=l+1}^{n} 3k^{i-1} G(x_0, x_1, x_1) \\ &\leq 3G(x_0, x_1, x_1)\left[\sum_{i=0}^{n} k^{i-1} - \sum_{i=0}^{l} k^{i-1}\right]. \end{aligned} \tag{2.11}$$

Hence,

$$d_G(x_n, x_l) \longrightarrow 0 \quad \text{as } n, l \longrightarrow \infty. \tag{2.12}$$

That is, the sequence $\{x_n\}$ is Cauchy in (X, d_G). Since the space (X, G) is G-complete then (X, d_G) is complete (see Proposition 10 in [1]) and hence, $\{x_n\}$ converges to a number say, $u \in X$. Moreover, $\{x_n\}$ is G-Cauchy in (X, G) (see Proposition 9 in [1]) and it is easy to see that $u \in \cap_{j=1}^{m} A_j$. Indeed, if $x_0 \in A_1$, then the subsequence $\{x_{m(n-1)}\}_{n=1}^{\infty} \in A_1$, the subsequence $\{x_{m(n-1)+1}\}_{n=1}^{\infty} \in A_2$, and, continuing in this way, the subsequence $\{x_{mn-1}\}_{n=1}^{\infty} \in A_m$. All the m subsequences are G-convergent and hence, they all converge to the same limit u. In addition, the sets A_j are G-closed, thus, the limit $u \in \cap_{j=1}^{m} A_j$.

We show now that $u \in X$ is a fixed point of T, that is, $u = Tu$. Consider now (1.1) and (2.2) with $x = x_n$, $y = z = Tu$ and suppose that $u \neq Tu$ or $d_G(u, Tu) > 0$, then we have,

$$\begin{aligned} 0 \leq d_G(x_n, Tu) &= G(x_n, Tu, Tu) + G(Tu, x_n, x_n) \\ &= G(Tx_{n-1}, Tu, Tu) + G(Tu, Tx_{n-1}, Tx_{n-1}) \\ &\leq k[G(x_{n-1}, u, u) + G(u, x_{n-1}, x_{n-1})]. \end{aligned} \tag{2.13}$$

Passing to limit as $n \rightarrow \infty$, we end up with $0 \leq d_G(u, Tu) \leq 0$ which contradicts the assumption $d_G(u, Tu) > 0$. Hence, $u = Tu$, that is, $u \in$ is a fixed point of T.

To prove the uniqueness, we assume that $v \in X$ is another fixed point of T such that $v \neq u$. Both u and v lie in $\cap_{j=1}^{m} A_j$; thus, we can substitute $x = u$ and $y = z = v$ in (2.2). This yields $G(Tv, Tu, Tu) \leq kG(u, v, v)$ which is true only for $k = 1$ but $k \in (0, 1)$ by definition. Thus, the fixed point of T is unique. □

As a particular case of Theorem 2.1, we give the following corollary.

Corollary 2.2. *Let (X, G) be a G-complete G-metric space and $\{A_j\}_{j=1}^{m}$ be a family of nonempty G-closed subsets of X. Let $Y = \cup_{j=1}^{m} A_j$ and $T : Y \rightarrow Y$ be a map satisfying*

$$T(A_j) \subseteq A_{j+1}, \quad j = 1, \ldots, m, \text{ where } A_{m+1} = A_1. \tag{2.14}$$

If there exists $k \in (0, 1)$ such that

$$G(Tx, Ty, Ty) \leq kG(x, y, y) \tag{2.15}$$

holds for all $x \in A_j$ and $y \in A_{j+1}$, $j = 1, \ldots, m$ then, T has a unique fixed point in $\cap_{j=1}^{m} A_j$.

3. Generalized Cyclic Weak ϕ-Contractions on G-Metric Spaces

The main goal of a number of studies regarding fixed points is to weaken the contractive conditions on the map under consideration. Inspired by this idea, in 1969, Boyd and Wong [28] defined the concept of Φ-contraction. Later, in 1997, Alber and Guerre-Delabriere [29] defined the weak ϕ-contractions on Hilbert spaces and proved fixed point theorem regarding such contractions. A map $T : X \to X$ on a metric space (X, d) is called a weak ϕ-contraction if there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)), \tag{3.1}$$

for all $x, y \in X$. These types of contractions have also been a subject of extensive research (see, e.g., [30–32]). In what follows, we discuss cyclic weak ϕ-contractions on G-metric spaces.

Consider the set Ψ of continuous functions $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ and $\phi(t) > 0$ for $t > 0$. We have the following fixed point theorem.

Theorem 3.1. *Let (X, G) be a G-complete G-metric space and $\{A_j\}_{j=1}^m$ be a family of nonempty G-closed subsets of X with $Y = \cup_{j=1}^m A_j$. Let $T : Y \to Y$ be a map satisfying*

$$T(A_j) \subseteq A_{j+1}, \quad j = 1, \ldots, m, \text{ where } A_{m+1} = A_1. \tag{3.2}$$

Suppose that there exists a function $\phi \in \Psi$ such that the map T satisfies

$$G(Tx, Ty, Tz) \leq M(x, y, z) - \phi(M(x, y, z)), \tag{3.3}$$

for all $x \in A_j$ and $y, z \in A_{j+1}$, $j = 1, \ldots, m$ where

$$M(x, y, z) = \max\{G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz)\}. \tag{3.4}$$

Then T has a unique fixed point in $\cap_{j=1}^m A_j$.

Proof. To prove the existence part, we construct a sequence of Picard iterations as usual. Take an arbitrary $x_0 \in A_1$ and define the sequence $\{x_n\}$ as

$$x_n = Tx_{n-1}, \quad n = 1, 2, 3, \ldots. \tag{3.5}$$

Since T is cyclic, $x_0 \in A_1$, $x_1 = Tx_0 \in A_2$, $x_2 = Tx_1 \in A_3, \ldots$, and so on. If $x_{n_0+1} = x_{n_0}$ for some $n_0 \in \mathbb{N}$, then, obviously, the fixed point of T is x_{n_0}. Assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$.

Let $x = x_n$ and $y = z = x_{n+1}$ in (3.3)

$$G(Tx_n, Tx_{n+1}, Tx_{n+1}) = G(x_{n+1}, x_{n+2}, x_{n+2}) \leq M(x_n, x_{n+1}, x_{n+1}) - \phi(M(x_n, x_{n+1}, x_{n+1})), \tag{3.6}$$

where

$$\begin{aligned} M(x_n, x_{n+1}, x_{n+1}) &= \max\{G(x_n, x_{n+1}, x_{n+1}), G(x_n, Tx_n, Tx_n), \\ &\qquad G(x_{n+1}, Tx_{n+1}, Tx_{n+1}), G(x_{n+1}, Tx_{n+1}, Tx_{n+1})\} \\ &= \max\{G(x_n, x_{n+1}, x_{n+1}), G(x_{n+1}, x_{n+2}, x_{n+2})\}. \end{aligned} \tag{3.7}$$

If $M(x_n, x_{n+1}, x_{n+1}) = G(x_{n+1}, x_{n+2}, x_{n+2})$, then (3.6) yields

$$G(x_{n+1}, x_{n+2}, x_{n+2}) \leq G(x_{n+1}, x_{n+2}, x_{n+2}) - \phi(G(x_{n+1}, x_{n+2}, x_{n+2})), \tag{3.8}$$

which implies $\phi(G(x_{n+1}, x_{n+2}, x_{n+2})) = 0$ and hence $G(x_{n+1}, x_{n+2}, x_{n+2}) = 0$. This contradicts the assumption $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Then we must have $M(x_n, x_{n+1}, x_{n+1}) = G(x_n, x_{n+1}, x_{n+1})$, in (3.6), so that

$$\begin{aligned} G(x_{n+1}, x_{n+2}, x_{n+2}) &\leq G(x_n, x_{n+1}, x_{n+1}) - \phi(G(x_n, x_{n+1}, x_{n+1})) \\ &\leq G(x_n, x_{n+1}, x_{n+1}). \end{aligned} \tag{3.9}$$

Thus, the sequence $\{G(x_n, x_{n+1}, x_{n+1})\}$ is a nonnegative nonincreasing sequence which converges to $L \geq 0$. Letting $n \to \infty$ in (3.9) we get

$$L \leq L - \phi(L). \tag{3.10}$$

It follows that $\phi(L) = 0$; therefore, $L = 0$, that is,

$$\lim_{n \to \infty} G(x_n, x_{n+1}, x_{n+1}) = 0. \tag{3.11}$$

The equation (2.7) in the proof of Theorem 2.1 with $x = x_n$ and $y = x_{n-1}$ yields

$$G(x_n, x_{n-1}, x_{n-1}) \leq 2G(x_{n-1}, x_n, x_n), \tag{3.12}$$

and hence,

$$\lim_{n \to \infty} G(x_n, x_{n-1}, x_{n-1}) = 0. \tag{3.13}$$

We claim that $\{x_n\}$ is a G-Cauchy sequence in (X, G). Assume the contrary, that is, $\{x_n\}$ is not G-Cauchy. Then, according to Proposition 1.7 there exist $\varepsilon > 0$ and corresponding subsequences $\{n(k)\}$ and $\{l(k)\}$ of $\mathbb{N}$ satisfying $n(k) > l(k) > k$ for which

$$G\left(x_{l(k)}, x_{n(k)}, x_{n(k)}\right) \geq \varepsilon, \tag{3.14}$$

where $n(k)$ is chosen as the smallest integer satisfying (3.14), that is,

$$G\left(x_{l(k)}, x_{n(k)-1}, x_{n(k)-1}\right) < \varepsilon. \tag{3.15}$$

It is easy to see from (3.14) and (3.15) and the rectangle inequality (G5) that

$$\begin{aligned}\varepsilon \leq G(x_{l(k)}, x_{n(k)}, x_{n(k)}) &\leq G(x_{l(k)}, x_{n(k)-1}, x_{n(k)-1}) \\ &\quad + G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}) < \varepsilon + G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}).\end{aligned} \tag{3.16}$$

Taking limit as $k \to \infty$ in (3.16) and using (3.11) we obtain

$$\lim_{k\to\infty} G(x_{l(k)}, x_{n(k)}, x_{n(k)}) = \varepsilon. \tag{3.17}$$

Observe that for every $k \in \mathbb{N}$ there exists $s(k)$ satisfying $0 \leq s(k) \leq m$ such that

$$n(k) - l(k) + s(k) \equiv 1(m). \tag{3.18}$$

Therefore, for large enough values of k we have $r(k) = l(k) - s(k) > 0$ and $x_{r(k)}$ and $x_{n(k)}$ lie in the consecutive sets A_j and A_{j+1}, respectively, for some $0 \leq j \leq m$. We next substitute $x = x_{r(k)}$ and $y = z = x_{n(k)}$ in (3.3) to obtain

$$G(Tx_{r(k)}, Tx_{n(k)}, Tx_{n(k)}) \leq M(x_{r(k)}, x_{n(k)}, x_{n(k)}) - \phi(M(x_{r(k)}, x_{n(k)}, x_{n(k)})), \tag{3.19}$$

where

$$\begin{aligned}M(x_{r(k)}, x_{n(k)}, x_{n(k)}) = \max\{&G(x_{r(k)}, x_{n(k)}, x_{n(k)}), G(x_{r(k)}, x_{r(k)+1}, x_{r(k)+1}), \\ &G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1})\}.\end{aligned} \tag{3.20}$$

Employing rectangle inequality (G5) repeatedly we see that

$$\begin{aligned}G(x_{r(k)}, x_{n(k)}, x_{n(k)}) &\leq G(x_{r(k)}, x_{r(k)+1}, x_{r(k)+1}) + G(x_{r(k)+1}, x_{n(k)}, x_{n(k)}) \\ &\leq G(x_{r(k)}, x_{r(k)+1}, x_{r(k)+1}) + G(x_{r(k)+1}, x_{r(k)+2}, x_{r(k)+2}) \\ &\quad + G(x_{r(k)+2}, x_{n(k)}, x_{n(k)}) \\ &\leq \cdots \\ &\leq \sum_{i=r}^{l-1} [G(x_{i(k)}, x_{i(k)+1}, x_{i(k)+1})] + G(x_{l(k)}, x_{n(k)}, x_{n(k)}),\end{aligned} \tag{3.21}$$

or equivalently

$$\begin{aligned}0 &\leq G(x_{r(k)}, x_{n(k)}, x_{n(k)}) - G(x_{l(k)}, x_{n(k)}, x_{n(k)}) \\ &\leq \sum_{i=r}^{l-1} G(x_{i(k)}, x_{i(k)+1}, x_{i(k)+1}).\end{aligned} \tag{3.22}$$

Note that the sum on the right-hand side of (3.22) consists of finite $s-1 \leq m$ number of terms, and due to (3.11) each term of this sum tends to 0 as $k \to \infty$. Therefore,

$$\lim_{k\to\infty} G(x_{r(k)}, x_{n(k)}, x_{n(k)}) = \lim_{k\to\infty} G(x_{l(k)}, x_{n(k)}, x_{n(k)}) = \varepsilon. \tag{3.23}$$

Using rectangle inequality (G5) again, we have

$$\begin{aligned}
0 &\leq G(x_{r(k)+1}, x_{n(k)+1}, x_{n(k)+1}) \\
&\leq G(x_{r(k)+1}, x_{n(k)}, x_{n(k)}) + G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}) \\
&\leq G(x_{r(k)+1}, x_{r(k)}, x_{r(k)}) + G(x_{r(k)}, x_{n(k)}, x_{n(k)}) + G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}),
\end{aligned} \tag{3.24}$$

from which we deduce

$$\lim_{k\to\infty} G(x_{r(k)+1}, x_{n(k)+1}, x_{n(k)+1}) = \varepsilon \tag{3.25}$$

upon letting $k \to \infty$ and using (3.23). Now, passing to limit as $k \to \infty$ in (3.19) and using (3.11), (3.23), and (3.25) we get

$$\varepsilon \leq \max\{\varepsilon, 0, 0\} - \phi(\max\{\varepsilon, 0, 0\}) = \varepsilon - \phi(\varepsilon), \tag{3.26}$$

and hence $\phi(\varepsilon) = 0$. We conclude that $\varepsilon = 0$ which contradicts the assumption that $\{x_n\}$ is not G-Cauchy. Thus, the sequence $\{x_n\}$ is G-Cauchy and since (X, G) is G-complete; it is G-convergent to a limit, say $w \in X$. It can be easily seen that $w \in \cap_{j=1}^{m} A_j$. Since $x_0 \in A_1$, then the subsequence $\{x_{m(n-1)}\}_{n=1}^{\infty} \in A_1$, the subsequence $\{x_{m(n-1)+1}\}_{n=1}^{\infty} \in A_2$, and, continuing in this way, the subsequence $\{x_{mn-1}\}_{n=1}^{\infty} \in A_m$. All the m subsequences are G-convergent in the G-closed sets A_j and hence, they all converge to the same limit $w \in \cap_{j=1}^{m} A_j$.

To show that the limit of the Picard sequence is the fixed point of T, that is, $w = Tw$ we employ (3.3) with $x = x_n$, $y = z = w$. This leads to

$$G(Tx_n, Tw, Tw) = M(x_n, w, w) - \phi(M(x_n, w, w)), \tag{3.27}$$

where

$$M(x_n, w, w) = \max\{G(x_n, w, w), G(x_n, x_{n+1}, x_{n+1}), G(w, Tw, Tw)\}. \tag{3.28}$$

Passing to limit as $n \to \infty$, we get

$$G(w, Tw, Tw) = G(w, Tw, Tw) - \phi(G(w, Tw, Tw)). \tag{3.29}$$

Thus, $\phi(G(w, Tw, Tw)) = 0$ and hence, $G(w, Tw, Tw) = 0$, that is, $w = Tw$.

Finally, we prove that the fixed point is unique. Assume that $v \in X$ is another fixed point of T such that $v \neq w$. Then, since both v and w belong to $\cap_{j=1}^{m} A_j$, we set $x = v$ and $y = z = w$ in (3.3) which yields

$$G(Tv, Tw, Tw) \leq M(v, w, w) - \phi(M(v, w, w)), \tag{3.30}$$

where,

$$M(v, w, w) = \max\{G(v, w, w), G(v, Tv, Tv), G(w, Tw, Tw)\} = G(v, w, w). \tag{3.31}$$

Then (3.30) becomes

$$G(v, w, w) \leq G(v, w, w) - \phi(G(v, w, w)), \tag{3.32}$$

and clearly, $G(v, w, w) = 0$, so we conclude that $v = w$, that is, the fixed point of T is unique. □

For particular choices of the function ϕ we obtain the following corollaries.

Corollary 3.2. *Let (X, G) be a G-complete G-metric space and $\{A_j\}_{j=1}^{m}$ be a family of nonempty G-closed subsets of X with $Y = \cup_{j=1}^{m} A_j$. Let $T : Y \to Y$ be a map satisfying*

$$T(A_j) \subseteq A_{j+1}, \quad j = 1, \ldots, m, \text{ where } A_{m+1} = A_1. \tag{3.33}$$

Suppose that there exists $k \in (0, 1)$ such that the map T satisfies

$$G(Tx, Ty, Tz) \leq kM(x, y, z), \tag{3.34}$$

for all $x \in A_j$ and $y, z \in A_{j+1}$, $j = 1, \ldots, m$ where

$$M(x, y, z) = \max\{G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz)\}. \tag{3.35}$$

Then T has a unique fixed point in $\cap_{j=1}^{m} A_j$.

Proof. The proof is obvious by choosing the function ϕ in Theorem 3.1 as $\phi(t) = (1 - k)t$. □

Corollary 3.3. *Let (X, G) be a G-complete G-metric space and $\{A_j\}_{j=1}^{m}$ be a family of nonempty G-closed subsets of X with $Y = \cup_{j=1}^{m} A_j$. Let $T : Y \to Y$ be a map satisfying*

$$T(A_j) \subseteq A_{j+1}, \quad j = 1, \ldots, m, \text{ where } A_{m+1} = A_1. \tag{3.36}$$

Suppose that there exist constants a, b, c, and d with $0 < a + b + c + d < 1$ such that the map T satisfies

$$G(Tx, Ty, Tz) \leq aG(x, y, z) + bG(x, Tx, Tx) + cG(y, Ty, Ty) + dG(z, Tz, Tz), \tag{3.37}$$

for all $x \in A_j$ and $y, z \in A_{j+1}$, $j = 1, \ldots, m$. Then, T has a unique fixed point in $\cap_{j=1}^{m} A_j$.

Proof. Clearly we have,

$$aG(x,y,z)+bG(x,Tx,Tx)+cG(y,Ty,Ty)+dG(z,Tz,Tz)\le(a+b+c+d)M(x,y,z),\tag{3.38}$$

where

$$M(x,y,z)=\max\{G(x,y,z),G(x,Tx,Tx),G(y,Ty,Ty),G(z,Tz,Tz)\}.\tag{3.39}$$

By Corollary 3.2, the map T has a unique fixed point. □

4. An Example and Applications

To illustrate the cyclic weak ϕ-contractions on G-metric spaces we give the following example.

Example 4.1. Let $X=[-1,1]$ and let $T:X\to X$ be given as $Tx=-x/3$. Let $A=[-1,0]$ and $B=[0,1]$. Define the function $G:X\times X\times X\to[0,\infty)$ as

$$G(x,y,z)=\left|x^3-y^3\right|+\left|y^3-z^3\right|+\left|z^3-x^3\right|.\tag{4.1}$$

Clearly, the function G is a G-metric on X. Define also $\phi:[0,\infty)\to[0,\infty)$ as $\phi(t)=2t/3$. Obviously, the map T has a unique fixed point $x=0\in A\cap B$.

It can be easily shown that the map T satisfies the condition (3.3). Indeed, note that

$$\begin{aligned}G(Tx,Ty,Tz)&=\frac{1}{27}\left[\left|x^3-y^3\right|+\left|y^3-z^3\right|+\left|z^3-x^3\right|\right],\\M(x,y,z)&=\max\left\{\left|x^3-y^3\right|+\left|y^3-z^3\right|+\left|z^3-x^3\right|,\frac{56}{27}\left|x^3\right|,\frac{56}{27}\left|y^3\right|,\frac{56}{27}\left|z^3\right|\right\}.\end{aligned}\tag{4.2}$$

Then,

$$M(x,y,z)-\phi(M(x,y,z))=\frac{1}{3}M(x,y,z),\tag{4.3}$$

and clearly,

$$\begin{aligned}G(Tx,Ty,Tz)&=\frac{1}{27}\left[\left|x^3-y^3\right|+\left|y^3-z^3\right|+\left|z^3-x^3\right|\right]\\&<\frac{1}{3}\left[\left|x^3-y^3\right|+\left|y^3-z^3\right|+\left|z^3-x^3\right|\right]\le\frac{1}{3}M(x,y,z).\end{aligned}\tag{4.4}$$

Hence, T has a unique fixed point by Theorem 3.1.

Cyclic maps satisfying integral type contractive conditions are amongst common applications of fixed point theorems. In this context, we consider the following applications.

Corollary 4.2. *Let (X,G) be a G-complete G-metric space and $\{A_j\}_{j=1}^m$ be a family of nonempty G-closed subsets of X with $Y=\cup_{j=1}^m A_j$. Let $T:Y\to Y$ be a map satisfying*

$$T(A_j)\subseteq A_{j+1},\quad j=1,\dots,m,\ \text{where}\ A_{m+1}=A_1. \tag{4.5}$$

Suppose also that

$$\int_0^{G(Tx,Ty,Tz)} ds \le \int_0^{M(x,y,z)} ds - \phi\left(\int_0^{M(x,y,z)} ds\right), \tag{4.6}$$

where $\phi\in\Psi$ and

$$M(x,y,z)=\max\{G(x,y,z),G(x,Tx,Tx),G(y,Ty,Ty),G(z,Tz,Tz)\}, \tag{4.7}$$

for all $x\in A_j$ and $y,z\in A_{j+1}$, $j=1,\dots,m$. Then T has a unique fixed point in $\cap_{i=1}^m A_i$.

Corollary 4.3. *Let (X,G) be a G-complete G-metric space and $\{A_j\}_{j=1}^m$ be a family of nonempty G-closed subsets of X with $Y=\cup_{j=1}^m A_j$. Let $T:Y\to Y$ be a map satisfying*

$$T(A_j)\subseteq A_{j+1},\quad j=1,\dots,m,\ \text{where}\ A_{m+1}=A_1. \tag{4.8}$$

Suppose also that

$$\int_0^{G(Tx,Ty,Tz)} ds \le k\int_0^{M(x,y,z)} ds, \tag{4.9}$$

where $k\in(0,1)$ and

$$M(x,y,z)=\max\{G(x,y,z),G(x,Tx,Tx),G(y,Ty,Ty),G(z,Tz,Tz)\}, \tag{4.10}$$

for all $x\in A_j$ and $y,z\in A_{j+1}$, $j=1,\dots,m$. Then T has a unique fixed point in $\cap_{i=1}^m A_i$.

Very recently, Jachymski proved the equivalence of auxiliary functions (see Lemma, in [33]). Inspired by the results from the remarkable paper of Jachymski, we state the following theorem.

Theorem 4.4 (see [33]). *Let T be a self-map on a G-complete G-metric space (X,G) and $\{A_j\}_{j=1}^m$ be a family of nonempty G-closed subsets of X with $Y=\cup_{j=1}^m A_j$. Let $T:Y\to Y$ be a map satisfying*

$$T(A_j)\subseteq A_{j+1},\quad j=1,\dots,m,\ \text{where}\ A_{m+1}=A_1. \tag{4.11}$$

Assume that

$$M(x,y,z)=\max\{G(x,y,z),G(x,Tx,Tx),G(y,Ty,Ty),G(z,Tz,Tz)\}. \tag{4.12}$$

Then the following statements are equivalent:

(i) *there exist functions* $\psi,\eta\in\Psi$ *such that*

$$\psi(G(Tx,Ty,Tz))\leq\psi(M(x,y,z))-\eta(M(x,y,z)), \tag{4.13}$$

for any $x\in A_i,\ y,z\in A_{i+1},\ i=1,2,\ldots,m$,

(ii) *there exists a function* $\beta:[0,\infty)\to[0,1]$ *such that for any bounded sequence* $\{t_n\}$ *of positive reals,* $\beta(t_n)\to 1$ *implies* $t_n\to 0$ *and*

$$G(Tx,Ty,Tz)\leq\beta(M(x,y,z))\psi(M(x,y,z)), \tag{4.14}$$

for any $x\in A_i,\ y,z\in A_{i+1},\ i=1,2,\ldots,m$,

(iii) *there exists a continuous function* $\eta:[0,\infty)\to[0,\infty)$ *such that* $\eta^{-1}(\{0\})=0$ *and*

$$G(Tx,Ty,Tz)\leq M(x,y,z)-\eta(M(x,y,z)) \tag{4.15}$$

for any $x\in A_i,\ y,z\in A_{i+1},\ i=1,2,\ldots,m$,

(iv) *there exists function* $\psi\in\Psi$ *and a nondecreasing, right continuous function* $\varphi:[0,\infty)\to[0,\infty)$ *with* $\varphi(t)<t$ *and for all* $t>0$ *with*

$$\psi(G(Tx,Ty,Tz))\leq\varphi(\psi(M(x,y,z))), \tag{4.16}$$

for any $x\in A_i,\ y,z\in A_{i+1},\ i=1,2,\ldots,m$,

(v) *there exists a continuous and nondecreasing function* $\varphi:[0,\infty)\to[0,\infty)$ *such that* $\varphi(t)<t$ *and for all* $t>0$ *with*

$$\psi(G(Tx,Ty,Tz))\leq\varphi(M(x,y,z)), \tag{4.17}$$

for any $x\in A_i,\ y,z\in A_{i+1},\ i=1,2,\ldots,m$.

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