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Research Article

Weighted Pseudo Almost-Periodic Functions and Applications to Semilinear Evolution Equations

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We first give a solution to a key problem concerning the completeness of the space of weighted pseudo almost-periodic functions and then establish a new composition theorem with respect to these functions. Some important remarks with concrete examples are also presented. Moreover, we prove an existence theorem for the weighted pseudo almost-periodic mild solution to the semilinear evolution equation: $x'(t) = Ax(t) + f(t, x(t))$, $t \in \mathbb{R}$, where A is the infinitesimal generator of an exponentially stable C_0-semigroup. An application is also given to illustrate the abstract existence theorem.

1. Introduction

It is well known that periodicity and almost periodicity are natural and important phenomena in the real world. In 2006, Diagana [1] introduced the concept of weighted pseudo almost-periodic functions, which is a generalization of the classical almost-periodic functions of Bohr as well as the vector-valued almost-periodic functions of Bochner (cf., e.g., [1–4]). Recently, weighted pseudo almost-periodic functions are widely investigated and used in the study of differential equations. Many basic properties and applications to several classes of differential equations were established, see, for example, Blot et al. [5] and Diagana [6]. On the other hand, the properties of weighted pseudo almost-periodic functions are more complicated and changeable than the almost-periodic functions and the pseudo almost-periodic functions because the influence of the weight ρ is very strong sometimes. Very recently, we constructed examples to show that the decomposition of weighted pseudo almost-periodic functions is not generally unique [3]. Hence, the space of the weighted pseudo almost-periodic functions may not be a Banach space under the usual supremum norm. Actually, the completeness of the space of these functions is worthy to be studied deeply by new ideas. In this paper, we will present a solution to the fundamental problem. Then, we will study the corresponding composition problem of weighted pseudo almost-periodic functions, as well as the existence of weighted pseudo almost-periodic mild solution to the following semilinear evolution equation:

$$x'(t) = Ax(t) + f(t, x(t)), \quad t \in \mathbb{R}, \tag{1.1}$$

where A is the infinitesimal generator of an exponentially stable C_0-semigroup.

The paper is organized as follows. In Section 2, we prove a completeness theorem for the space of weighted pseudo almost-periodic functions by introducing a new norm on the space. Moreover, some remarks on weighted pseudo almost-periodic functions are given. In Section 3, we first establish a composition theorem for weighted pseudo almost-periodic functions and then give an existence theorem for the weighted pseudo almost-periodic mild solution to the evolution equation (1.1). An example is presented to illustrate the abstract existence theorem.

2. Completeness Theorem

Throughout this paper, we let $\mathbb{X}$, $\mathbb{Y}$ be Banach spaces and $BC(\mathbb{R}, \mathbb{X})$ the Banach space of all $\mathbb{X}$-valued bounded continuous functions equipped with the supremum norm. $L^1_{\mathrm{loc}}(\mathbb{R})$ denotes the space of all locally integrable functions on $\mathbb{R}$, and $\mathbb{U}$ stands for

$$\mathbb{U} := \left\{ \rho \in L^1_{\mathrm{loc}}(\mathbb{R}) : \rho(t) > 0, \text{ a.e. } t \in \mathbb{R} \right\}. \tag{2.1}$$

If $\rho \in \mathbb{U}$, then we set

$$\mu(T, \rho) := \int_{-T}^{T} \rho(t) dt, \quad T > 0. \tag{2.2}$$

Define the space $\mathbb{U}_\infty$ as

$$\mathbb{U}_\infty := \left\{ \rho \in \mathbb{U} : \lim_{T \to \infty} \mu(T, \rho) = \infty \right\}, \tag{2.3}$$

and define $\mathbb{U}_b$ as the set of all $\rho \in \mathbb{U}_\infty$ such that ρ is bounded with $\inf_{t \in \mathbb{R}} \rho(t) > 0$. Obviously,

$$\mathbb{U}_b \subset \mathbb{U}_\infty \subset \mathbb{U}. \tag{2.4}$$

In what follows we recall some definitions and notations needed in this paper.

Definition 2.1 (S. Bochner). A continuous function $f : \mathbb{R} \mapsto \mathbb{X}$ is called almost periodic if for each $\varepsilon > 0$ there exists an $l(\varepsilon) > 0$ such that every interval I of length $l(\varepsilon)$ contains a number τ with the property that

$$\|f(t+\tau) - f(t)\| < \varepsilon, \quad \forall t \in \mathbb{R}. \tag{2.5}$$

The set of all such functions will be denoted by $\mathrm{AP}(\mathbb{X})$ in this paper.

A continuous function $f : \mathbb{R} \times \mathbb{Y} \mapsto \mathbb{X}$ is said to be almost periodic if $f(t,x)$ is almost periodic in $t \in \mathbb{R}$ uniformly for all $x \in K$, where K is any bounded subset of $\mathbb{Y}$. Denote by $\mathrm{AP}(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ the set of all such functions.

The set of bounded continuous functions with vanishing mean value is denoted by $\mathrm{PAP}_0(\mathbb{R}, \mathbb{X})$, that is,

$$\mathrm{PAP}_0(\mathbb{R}, \mathbb{X}) := \left\{ \phi \in \mathrm{BC}(\mathbb{R}, \mathbb{X}) : \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \|\phi(\sigma)\| d\sigma = 0 \right\}. \tag{2.6}$$

Let $\rho \in \mathbb{U}_\infty$, and we define the set of continuous functions with vanishing mean value under weight ρ by

$$\mathrm{PAP}_0(\mathbb{R}, \mathbb{X}, \rho) := \left\{ \phi \in \mathrm{BC}(\mathbb{R}, \mathbb{X}) : \lim_{T \to \infty} \frac{1}{\mu(T,\rho)} \int_{-T}^{T} \|\phi(\sigma)\| \rho(\sigma) d\sigma = 0 \right\}, \tag{2.7}$$

and define

$$\begin{aligned} \mathrm{PAP}_0(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \rho) := \Bigg\{ \phi \in \mathrm{BC}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}) : \lim_{T \to \infty} \frac{1}{\mu(T,\rho)} \int_{-T}^{T} \|\phi(\sigma, u)\| \rho(\sigma) d\sigma = 0 \\ \text{uniformly for } u \text{ in any bounded subset of } \mathbb{Y} \Bigg\}. \end{aligned} \tag{2.8}$$

For simplicity of notation, we write

$$\mathrm{PAP}_0(\mathbb{X}) \text{ and } \mathrm{PAP}_0(\mathbb{X}) \tag{2.9}$$

instead of

$$\mathrm{PAP}_0(\mathbb{R}, \mathbb{X}) \text{ and } \mathrm{PAP}_0(\mathbb{R}, \mathbb{X}), \tag{2.10}$$

respectively.

The following definition is a slight modification of that given by Diagana [1].

Definition 2.2. A function $f \in \mathrm{BC}(\mathbb{R}, \mathbb{X})$ ($f \in \mathrm{BC}(\mathbb{R} \times \mathbb{X}, \mathbb{X})$) is called weighted pseudo almost periodic (or ρ-pseudo almost periodic) if it can be expressed as $f = g + \phi$, where

$g \in \mathrm{AP}(\mathbb{X})$ $(g \in \mathrm{AP}(\mathbb{R}\times\mathbb{X},\mathbb{X}))$ and $\phi \in \mathrm{PAP}_0(\mathbb{X},\rho)$ $(\phi \in \mathrm{PAP}_0(\mathbb{R}\times\mathbb{X},\mathbb{X},\rho))$. The set of weighted pseudo almost-periodic functions from $\mathbb{R}$ into $\mathbb{X}$ is denoted by $\mathrm{PAP}(\mathbb{X},\rho)$ $(\mathrm{PAP}(\mathbb{R}\times\mathbb{X},\mathbb{X},\rho))$ here.

Remark 2.3. (i) Clearly, $\mathrm{AP}(\mathbb{X})$ and $\mathrm{PAP}_0(\mathbb{X},\rho)$ are closed linear subspaces of $\mathrm{BC}(\mathbb{R},\mathbb{X})$ under the supremum norm.

(ii) There are a lot of weighted pseudo almost-periodic functions. For example, the following function:

$$f = \sin t + \sin\sqrt{2}t + \frac{1}{1+t^2} \tag{2.11}$$

is weighted pseudo almost periodic under weight $\rho = 1 + t^2$, see Example 3.6 for another example.

(iii) If one set $\rho(t) = 1$, then weighted pseudo almost-periodic functions are exactly the so-called pseudo almost-periodic functions. Furthermore, when the weight $\rho \in \mathbb{U}_b$, Diagana's work showed that the class of functions in $\mathrm{PAP}_0(\mathbb{X},\rho)$ coincides with $\mathrm{PAP}_0(\mathbb{X})$. So, the concept of weighted pseudo almost-periodic functions is a generalization of the concept of pseudo almost-periodic functions. From the following example given by us, one will see that this concept is a real generalization.

Example 2.4. Let

$$\rho(t) = \begin{cases} e^t, & t \le 0, \\ 1, & t > 0. \end{cases} \tag{2.12}$$

Then the bounded continuous function

$$f(t) = \sin t + \sin \pi t + \arctan t \tag{2.13}$$

is pseudo almost-periodic under this weight ρ but it is not the usual pseudo almost periodic function.

(iv) Suppose $f = g + \phi : \mathbb{R} \mapsto \mathbb{X}$ is a weighted pseudo almost-periodic function. From [3] we know that the decomposition of f is not generally unique, that is, the function space $\mathrm{PAP}(\mathbb{X},\rho)$ cannot be decomposed as

$$\mathrm{AP}(\mathbb{X}) \bigoplus \mathrm{PAP}_0(\mathbb{X},\rho). \tag{2.14}$$

Therefore, we do not know whether $\mathrm{PAP}(\mathbb{X},\rho)$ is a Banach space under the norm

$$\|f(t)\| = \sup_{t\in\mathbb{R}}\|g(t)\| + \sup_{t\in\mathbb{R}}\|\phi(t)\|, \tag{2.15}$$

although $\mathrm{AP}(\mathbb{X})$ and $\mathrm{PAP}_0(\mathbb{X},\rho)$ are closed linear subspaces of $\mathrm{BC}(\mathbb{R},\mathbb{X})$ under the supremum norm, and hence Banach spaces.

In order to obtain a completeness theorem for the space $\mathrm{PAP}(\mathbb{X},\rho)$ by overcoming the trouble showed in (iii) of the above remark, we use the "modular" idea and endow the weighted pseudo almost-periodic function in $\mathrm{PAP}(\mathbb{X},\rho)$ with a new norm as follows.

Let $\{g_i+\phi_i,\ i\in I\}$ be all the possible decomposition of $f\in\mathrm{PAP}(\mathbb{X},\rho)$. We define

$$\|f\|_\rho=\inf_{i\in I}\left(\|g_i\|+\|\phi_i\|\right)=\inf_{i\in I}\left(\sup_{t\in\mathbb{R}}\|g_i(t)\|+\sup_{t\in\mathbb{R}}\|\phi_i(t)\|\right). \tag{2.16}$$

Clearly, $\|\cdot\|_\rho$ is a norm on $\mathrm{PAP}(\mathbb{X},\rho)$. Moreover, we have the following result.

Theorem 2.5 (completeness theorem). $\mathrm{PAP}(\mathbb{X},æ)$ *is a Banach space under the norm* $\|\cdot\|_\rho$.

Proof. Suppose $\{f_n\}$ is a Cauchy sequence in $\mathrm{PAP}(\mathbb{X},\rho)$, relative to the norm $\|\cdot\|_\rho$. Then we can choose a subsequence $\{f_{n_i}\}$ with

$$\left\|f_{n_{i+1}}-f_{n_i}\right\|_\rho\le\frac{1}{2^{i+1}},\quad i=1,2,\ldots. \tag{2.17}$$

Set

$$h_0=f_{n_1},\qquad h_i=f_{n_{i+1}}-f_{n_i}\quad\text{for } i=1,2,\ldots. \tag{2.18}$$

Then by the definition of the norm $\|\cdot\|_\rho$, we can decompose each h_i as $h_i=g_i+\phi_i$, where $g_i\in\mathrm{AP}(\mathbb{R},\mathbb{X})$ and $\phi_i\in\mathrm{PAP}_0(\mathbb{X},\rho)$ with

$$\|h_i\|_\rho\ge\|g_i\|+\|\phi_i\|-\frac{1}{2^{i+1}},\quad i=0,1,2,\ldots. \tag{2.19}$$

Observing that

$$\begin{aligned}f_{n_i}&=f_{n_1}+\left(f_{n_2}-f_{n_1}\right)+\cdots+\left(f_{n_i}-f_{n_{i-1}}\right)\\&=h_0+h_1+\cdots+h_{i-1}\\&=g_0+g_1+\cdots+g_{i-1}+\phi_0+\phi_1+\cdots+\phi_{i-1}\\&=\sum_{k=0}^{i-1}g_k+\sum_{k=0}^{i-1}\phi_k,\quad i=0,1,2,\ldots,\end{aligned} \tag{2.20}$$

we claim that f_{n_i} converges to a weighted pseudo almost-periodic function. Actually, it follows from (2.17) and (2.19) that

$$\|g_i\|+\|\phi_i\|\le\|h_i\|_\rho+\frac{1}{2^{i+1}}\le\left\|f_{n_{i+1}}-f_{n_i}\right\|_\rho+\frac{1}{2^{i+1}}\le\frac{1}{2^i},\quad i=1,2,\ldots. \tag{2.21}$$

So

$$\|g_i\| \leq \frac{1}{2^i}, \quad i = 1, 2, \ldots. \tag{2.22}$$

Therefore

$$\left\| \sum_{k=0}^{\infty} g_k \right\| \leq \sum_{k=0}^{\infty} \|g_k\| \leq \|g_0\| + \sum_{k=1}^{\infty} \frac{1}{2^k} = \|g_0\| + 1. \tag{2.23}$$

This, together with the fact that $\mathrm{AP}(\mathbb{R}, \mathbb{X})$ is a Banach space and $\lim_{i \to \infty} \|g_i\| = 0$, implies that $g := \sum_{k=0}^{\infty} g_k$ exists and $g \in \mathrm{AP}(\mathbb{R}, \mathbb{X})$. The same arguments indicate that $\phi := \sum_{k=0}^{\infty} \phi_k$ exists and $\phi \in \mathrm{PAP}(\mathbb{X}, \rho)$. Thus, by (2.20) we have

$$\lim_{i \to \infty} f_{n_i} = \sum_{k=0}^{\infty} g_k + \sum_{k=0}^{\infty} \phi_k = g + \phi. \tag{2.24}$$

Therefore, f_{n_i} converges to $f = g + \phi$, which belongs to $\mathrm{PAP}(\mathbb{X}, \rho)$. In other words, $\{f_n\}$ is a Cauchy sequence having a convergent subsequence. Consequently, $\{f_n\}$ is a convergent sequence under the norm $\|\cdot\|_\rho$. This means that $\mathrm{PAP}(\mathbb{X}, \rho)$ is a Banach space under the norm $\|\cdot\|_\rho$. □

Remark 2.6. Clearly, the norm $\|\cdot\|_1$ coincides with the sup norm when f is a pseudo almost-periodic function, as a result of the uniqueness of decomposition of pseudo almost-periodic functions.

3. Composition Theorem and Existence Theorem

To obtain the existence of weighted pseudo almost periodic mild solutions to the semilinear evolution equation (1.1), we first establish a new composition theorem as follows.

Theorem 3.1 (composition theorem). *Let* $f \in \mathrm{PAP}(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \text{æ})$ *and* $f = g + \phi$ *with* $g \in \mathrm{AP}(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ *and* $\phi \in \mathrm{PAP}_0(\mathbb{R} \times \mathbb{X}, \text{æ})$. *Assume that* $f(t, x)$ *is uniformly continuous in every bounded subset* $K \subset \mathbb{X}$ *uniformly for* $t \in \mathbb{R}$. *If* $x \in \mathrm{PAP}(\mathbb{X}, \text{æ})$ *and* $x(t) = \alpha(t) + \beta(t)$, *with* $\alpha(t) \in \mathrm{AP}(\mathbb{X})$, *and* $\beta(t) \in \mathrm{PAP}_0(\mathbb{X}, \text{æ})$, *then*

$$g(\cdot, \alpha(\cdot)) \in \mathrm{AP}(\mathbb{X}), \qquad \mathrm{f}(\cdot, \mathrm{x}(\cdot)) - \mathrm{g}(\cdot, \text{ff}(\cdot)) \in PAP_0(\mathbb{X}, \text{æ}), \tag{3.1}$$

and furthermore $f(\cdot, x(\cdot)) \in \mathrm{PAP}(\mathbb{X}, \text{æ})$.

Proof. Write

$$f(t, x(t)) = g(t, \alpha(t)) + f(t, x(t)) - f(t, \alpha(t)) + \phi(t, \alpha(t)). \tag{3.2}$$

Then, we claim that

(a) $g(t,\alpha(t)) \in \mathrm{AP}(\mathbb{X})$,

(b) $F(t) := f(t,x(t)) - f(t,\alpha(t)) \in \mathrm{PAP}_0(\mathbb{X},\rho)$,

(c) $\phi(t,\alpha(t)) \in \mathrm{PAP}_0(\mathbb{X},\rho)$.

Actually, by the theorem of composition of almost-periodic functions, it is easy to see that (a) holds.

Next, let us prove (b).

Clearly, $F(t)$ is bounded and continuous. By the uniformly continuity of $f(t,x)$, we know that for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|\beta(t)\| := \|x(t) - \alpha(t)\| < \delta \tag{3.3}$$

implies that

$$\|f(t,x(t)) - f(t,\alpha(t))\| < \varepsilon. \tag{3.4}$$

Set

$$E_1 := \{t \in \mathbb{R} : \|\beta(t)\| < \delta\}, \qquad E_2 := \{t \in \mathbb{R} : \|\beta(t)\| \geq \delta\}. \tag{3.5}$$

Then E_1 and E_2 are measurable sets and $E_1 \cup E_2 = \mathbb{R}$. In addition, $\|F(t)\| < \varepsilon$ on the set E_1. Observe

$$\begin{aligned}
\frac{1}{\mu(T,\rho)}\int_{E_2\cap[-T,T]}\rho(s)ds &= \frac{1}{\delta\mu(T,\rho)}\int_{E_2\cap[-T,T]}\delta\rho(s)ds \\
&\leq \frac{1}{\delta\mu(T,\rho)}\int_{E_2\cap[-T,T]}\|\beta(s)\|\rho(s)ds \\
&\leq \frac{1}{\delta\mu(T,\rho)}\int_{-T}^{T}\|\beta(s)\|\rho(s)ds.
\end{aligned} \tag{3.6}$$

Using $\beta(t) \in \mathrm{PAP}_0(\mathbb{X},\rho)$, we get

$$\lim_{T\to\infty}\frac{1}{\mu(T,\rho)}\int_{E_2\cap[-T,T]}\rho(s)ds = 0. \tag{3.7}$$

Moreover, for any $T > 0$,

$$\begin{aligned}
&\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\|F(s)\|\rho(s)ds \\
&\quad= \frac{1}{\mu(T,\rho)}\int_{E_1\cap[-T,T]}\|F(s)\|\rho(s)ds + \frac{1}{\mu(T,\rho)}\int_{E_2\cap[-T,T]}\|F(s)\|\rho(s)ds
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\varepsilon}{\mu(T,\rho)} \int_{E_1\cap[-T,T]} \rho(s)ds + \frac{\sup_{t\in\mathbb{R}}\|F(t)\|}{\mu(T,\rho)} \int_{E_2\cap[-T,T]} \rho(s)ds \\
&\leq \varepsilon + \sup_{t\in\mathbb{R}}\|F(t)\| \frac{1}{\mu(T,\rho)} \int_{E_2\cap[-T,T]} \rho(s)ds.
\end{aligned} \tag{3.8}$$

Therefore, we have

$$\lim_{T\to\infty} \frac{1}{\mu(T,\rho)} \int_{-T}^{T} \|F(s)\|\rho(s)ds = 0. \tag{3.9}$$

That means that (b) is true.

Finally, we prove (c).

Since $\phi(t,\alpha(t))$ is continuous in $[-T,T]$, it is uniformly continuous in $[-T,T]$. Set $I = \alpha([-T,T])$. Then I is compact in $\mathbb{X}$. So one can find finite open balls B_k $(k = 1,2,\ldots,m)$ with center $x_k \in I$ and radius δ small enough such that $I \subset \bigcup_{k=1}^{m} B_k$ and

$$\left\|\phi(t,\alpha(t)) - \phi(t,x_k)\right\| < \frac{\varepsilon}{2}, \quad \alpha(t) \in B_k,\ t \in [-T,T]. \tag{3.10}$$

The set $O_k = \{t \in [-T,T] : \alpha(t) \in B_k\}$ is open in $[-T,T]$ and $[-T,T] = \bigcup_{k=1}^{m} O_k$. Let

$$V_1 = O_1, \qquad V_k = O_k \setminus \bigcup_{j=1}^{k-1} O_j \quad (2 \leq k \leq m). \tag{3.11}$$

Then $V_i \cap V_j = \emptyset$ when $i \neq j,\ 1 \leq i,\ j \leq m$, and

$$V_k \subset O_k, \qquad \bigcup_{k=1}^{m} V_k = [-T,T]. \tag{3.12}$$

On the other hand, by $\phi \in \mathrm{PAP}_0(\mathbb{R}\times\mathbb{X},\mathbb{X},\rho)$, we see that there exists $T_0 > 0$ such that

$$\frac{1}{\mu(T,\rho)} \int_{-T}^{T} \left\|\phi(s,x_k)\right\|\rho(s)ds < \frac{\varepsilon}{2m}, \quad \text{for } T \geq T_0,\ k \in \{1,2,\ldots,m\}. \tag{3.13}$$

Hence for $T \geq T_0$,

$$\begin{aligned}
&\frac{1}{\mu(T,\rho)} \int_{-T}^{T} \left\|\phi(s,\alpha(s))\right\|\rho(s)ds \\
&\quad = \frac{1}{\mu(T,\rho)} \sum_{k=1}^{m} \int_{V_k\cap[-T,T]} \left\|\phi(s,\alpha(s))\right\|\rho(s)ds
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{\mu(T,\rho)}\sum_{k=1}^{m}\int_{V_k\cap[-T,T]}\|\phi(s,\alpha(s))-\phi(s,x_k)\|\rho(s)ds\\
&\quad+\frac{1}{\mu(T,\rho)}\sum_{k=1}^{m}\int_{V_k\cap[-T,T]}\|\phi(s,x_k)\|\rho(s)ds\\
&\le \frac{1}{\mu(T,\rho)}\sum_{k=1}^{m}\int_{V_k\cap[-T,T]}\frac{\varepsilon}{2}\rho(s)ds+\sum_{k=1}^{m}\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\|\phi(s,x_k)\|\rho(s)ds\\
&\le \frac{\varepsilon}{2}+m\frac{\varepsilon}{2m}=\varepsilon.
\end{aligned}
\tag{3.14}
$$

Thus, (c) holds.

Since $\mathrm{PAP}_0(\mathbb{X},\rho)$ is a linear space, we have $f(\cdot,x(\cdot))\in \mathrm{PAP}(\mathbb{X},\rho)$, by (a), (b), and (c). □

The following assumption on the weight $\rho\in\mathbb{U}_\infty$ will be used in our next investigation.

(H_0) For a $\omega>0$,

$$
P(\omega):=\lim_{T\to+\infty}\frac{1}{\mu(T,\rho)}\int_{-T}^{T}e^{-\omega(t+T)}\rho(t)dt=0,
\tag{3.15}
$$

$$
\int_{\sigma}^{T}e^{-\omega(t-\sigma)}\rho(t)dt\le M\rho(\sigma)
\tag{3.16}
$$

for some constant $M>0$ and T is sufficiently large.

Remark 3.2. (i) Since $\rho(t)\in\mathbb{U}_\infty$, condition (3.15) is satisfied when

$$
\sup_{T>0}\int_{-T}^{T}e^{-\omega(t+T)}\rho(t)dt<+\infty.
\tag{3.17}
$$

Condition (3.16) is satisfied when $\rho(t)$ is nonincreasing. Therefore, if the weight ρ is bounded and nonincreasing, then both (3.15) and (3.16) are satisfied.

(ii) Besides the weight mentioned in (i) above, there are lots of weights satisfying the assumption (H_0). By giving the following example, we present two unbounded weights satisfying the assumption (H_0).

Example 3.3. (1) Take $\rho(t)=e^{\tilde{\omega}t},\ \omega>\tilde{\omega}>0$. Then we have

$$
\begin{aligned}
0\le P(\omega)&=\lim_{T\to+\infty}\frac{1}{\mu(T,\rho)}\int_{-T}^{T}e^{-\omega(t+T)}\rho(t)dt\\
&=\lim_{T\to+\infty}\frac{e^{-\omega T}}{\mu(T,\rho)}\int_{-T}^{T}e^{-(\omega-\tilde{\omega})t}dt
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{T\to+\infty} -\frac{\widetilde{\omega}}{\omega-\widetilde{\omega}}\frac{e^{(\widetilde{\omega}-2\omega)T}-e^{-\widetilde{\omega}T}}{e^{\widetilde{\omega}T}-e^{-\widetilde{\omega}T}} \\
&\le \lim_{T\to+\infty} \frac{\widetilde{\omega}\left(e^{-2\widetilde{\omega}T}-e^{-2\omega T}\right)}{\omega-\widetilde{\omega}} \\
&= 0.
\end{aligned}
\tag{3.18}
$$

Therefore $P(\omega)$ satisfies condition (3.15). Furthermore,

$$
\int_{\sigma}^{T} e^{-\omega(t-\sigma)}\rho(t)dt = \frac{e^{\omega\sigma}\left(e^{-(\omega-\widetilde{\omega})\sigma}-e^{-(\omega-\widetilde{\omega})T}\right)}{\omega-\widetilde{\omega}} \le \frac{e^{\widetilde{\omega}\sigma}}{\omega-\widetilde{\omega}} := M\rho(\sigma),
\tag{3.19}
$$

where $M = 1/(\omega-\widetilde{\omega})$. So (3.16) is satisfied.

(2) Take $\rho(t) = 1+t^2$. Then for every $\omega > 0$, we obtain

$$
\begin{aligned}
0 \le P(\omega) &= \lim_{T\to+\infty}\frac{e^{-\omega T}}{\mu(T,\rho)}\int_{-T}^{T} e^{-\omega t}\left(1+t^2\right)dt \\
&= \lim_{T\to+\infty}\frac{e^{-\omega T}}{\mu(T,\rho)}\left[-\frac{\omega^2t^2+2\omega t+2+\omega^2}{\omega^3}e^{-\omega t}\right]\Bigg|_{-T}^{T} \\
&\le \lim_{T\to+\infty}\frac{\omega^2T^2-2\omega T+2+\omega^2}{2(T+T^3/3)\omega^3} \\
&= 0.
\end{aligned}
\tag{3.20}
$$

Moreover,

$$
\begin{aligned}
\int_{\sigma}^{T} e^{-\omega(t-\sigma)}\rho(t)dt &= \int_{\sigma}^{T} e^{-\omega(t-\sigma)}\left(1+t^2\right)dt \\
&= e^{\omega\sigma}\left[-\frac{\omega^2t^2+2\omega t+2+\omega^2}{\omega^3}e^{-\omega t}\right]\Bigg|_{\sigma}^{T} \\
&\le \frac{\omega^2\sigma^2+2\omega\sigma+2+\omega^2}{\omega^3} \\
&\le \frac{(\omega^2+1)\sigma^2+2+2\omega^2}{\omega^3} \\
&\le \frac{2(\omega^2+1)}{\omega^3}\left(\sigma^2+1\right) \\
&:= M\rho(\sigma),
\end{aligned}
\tag{3.21}
$$

where $M = 2(\omega^2+1)/\omega^3$ is a constant.

Definition 3.4. A mild solution to (1.1) is a continuous function $x(t) : \mathbb{R} \mapsto \mathbb{X}$ satisfying

$$x(t) = T(t-a)x(a) + \int_a^t T(t-s)f(s,x(s))ds \tag{3.22}$$

for all $t \geq a$ and all $a \in \mathbb{R}$.

Theorem 3.5. *Let A generate an exponentially stable C_0- semigroup $\{T(t)\}_{t\geq 0}$, that is,*

$$\|T(t)\| \leq Ke^{-\omega t}, \quad \forall t \geq 0, \tag{3.23}$$

for some positive constants K and ω. Let the weight ρ satisfy (H_0) *for the ω in* (3.23)*. Assume that*

(H_1) *$f = g + \phi : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}$ is weighted pseudo almost periodic, with $g \in \mathrm{AP}(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ and $\phi \in \mathrm{PAP}_0(\mathbb{R} \times \mathbb{X}, \mathbb{X}, æ)$.*

(H_2) *for every $t \in \mathbb{R}$, $x, y \in \mathbb{X}$,*

$$\|f(t,x) - f(t,y)\| \leq L_f\|x-y\|, \qquad \|g(t,x) - g(t,y)\| \leq L_g\|x-y\|, \tag{3.24}$$

for some constants L_f and L_g with $(2L_g + L_f)K/\omega < 1$.

Then (1.1) *admits a unique weighted pseudo almost-periodic mild solution.*

Proof. Define a nonlinear operator $\digamma$ on $\mathrm{PAP}(\mathbb{X}, \rho)$ by

$$(\digamma x)(t) = \int_{-\infty}^t T(t-s)f(s,x(s))ds, \quad t \in \mathbb{R}. \tag{3.25}$$

Fix $x(t) \in \mathrm{PAP}(\mathbb{X}, \rho)$. Then

$$x(t) = \alpha(t) + \beta(t) \tag{3.26}$$

with

$$\alpha(t) \in \mathrm{AP}(\mathbb{X}), \qquad \beta(t) \in \mathrm{PAP}_0(\mathbb{X}, \rho). \tag{3.27}$$

By Theorem 3.1, we know that there exist two functions

$$\widehat{g}(t) = g(t, \alpha(t)) \in \mathrm{AP}(\mathbb{X}), \tag{3.28}$$

$$\widehat{\varphi}(t) = f(t, x(t)) - f(t, \alpha(t)) + \varphi(t, \alpha(t)) \in \mathrm{PAP}_0(\mathbb{X}, \rho) \tag{3.29}$$

such that

$$f(\cdot, x(\cdot)) = \widehat{g}(\cdot) + \widehat{\varphi}(\cdot). \tag{3.30}$$

Therefore $(\mathcal{F}x)(t)$ can be expressed as $(\mathcal{F}x)(t) = G(t) + \Phi(t)$, where

$$G(t) := \int_{-\infty}^{t} T(t-s)\widehat{g}(s)ds, \qquad \Phi(t) := \int_{-\infty}^{t} T(t-s)\widehat{\varphi}(s)ds. \tag{3.31}$$

From (3.28) and the fact that $\{T(t)\}$ is exponentially stable, it follows that $G(t) \in \mathrm{AP}(\mathbb{X})$.

Next, we show that $\Phi(t) \in \mathrm{PAP}_0(\mathbb{X}, \rho)$.

By (3.29), we see that $\widehat{\varphi}(t)$ is bounded on $\mathbb{R}$, that is, $\sup_{t\in\mathbb{R}}\|\widehat{\varphi}(t)\| < +\infty$, and $\widehat{\varphi}$ is continuous. Hence, for any $T > 0$, we have

$$\begin{aligned}
\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\|\Phi(s)\|\rho(s)ds &= \frac{1}{\mu(T,\rho)}\int_{-T}^{T}\left\|\int_{-\infty}^{s} T(t-\sigma)\widehat{\varphi}(\sigma)d\sigma\right\|\rho(s)ds \\
&\leq \frac{1}{\mu(T,\rho)}\int_{-T}^{T}\int_{-\infty}^{s}\|T(s-\sigma)\|\left\|\widehat{\varphi}(\sigma)\right\|d\sigma\rho(s)ds \\
&\leq \frac{1}{\mu(T,\rho)}\int_{-T}^{T}\int_{-\infty}^{s} Ke^{-\omega(s-\sigma)}\left\|\widehat{\varphi}(\sigma)\right\|d\sigma\rho(s)ds \\
&\leq \frac{1}{\mu(T,\rho)}\int_{-T}^{T}\left(\int_{-\infty}^{-T}+\int_{-T}^{s}\right)Ke^{-\omega(s-\sigma)}\left\|\widehat{\varphi}(\sigma)\right\|d\sigma\rho(s)ds \\
&:= I + J,
\end{aligned} \tag{3.32}$$

where

$$\begin{aligned}
I &= \frac{1}{\mu(T,\rho)}\int_{-T}^{T}\int_{-\infty}^{-T} Ke^{-\omega(s-\sigma)}\left\|\widehat{\varphi}(\sigma)\right\|d\sigma\rho(s)ds, \\
J &= \frac{1}{\mu(T,\rho)}\int_{-T}^{T}\int_{-T}^{s} Ke^{-\omega(s-\sigma)}\left\|\widehat{\varphi}(\sigma)\right\|d\sigma\rho(s)ds.
\end{aligned} \tag{3.33}$$

Clearly,

$$\begin{aligned}
I &\leq \frac{1}{\mu(T,\rho)}\sup_{t\in\mathbb{R}}\left\|\widehat{\varphi}(\sigma)\right\|\int_{-T}^{T}\int_{-\infty}^{-T} Ke^{-\omega(s-\sigma)}\rho(s)d\sigma\, ds, \\
&\leq \frac{K}{\omega\mu(T,\rho)}\sup_{t\in\mathbb{R}}\left\|\widehat{\varphi}(\sigma)\right\|\int_{-T}^{T} e^{-\omega(s+T)}\rho(s)ds.
\end{aligned} \tag{3.34}$$

This, together with the assumption (H_0), implies that

$$I \longrightarrow 0 \quad \text{as } T \longrightarrow +\infty. \tag{3.35}$$

Moreover, by Fubini's theorem and assumption (H_0), we get

$$\begin{aligned} J &= \frac{K}{\mu(T,\rho)} \int_{-T}^{T} \|\widehat{\varphi}(\sigma)\| \int_{\sigma}^{T} e^{-\omega(s-\sigma)} \rho(s)\, ds\, d\sigma \\ &\leq \frac{MK}{\mu(T,\rho)} \int_{-T}^{T} \|\widehat{\varphi}(\sigma)\| \rho(\sigma) d\sigma. \end{aligned} \tag{3.36}$$

Thus, (3.29) implies that

$$J \longrightarrow 0 \quad \text{as } T \longrightarrow +\infty. \tag{3.37}$$

So, $\Phi(t) \in \mathrm{PAP}_0(\mathbb{X},\rho)$.

Consequently, $(\mathcal{F}x)(t) = G(t) + \Phi(t)$ is a weighted pseudo almost-periodic function on $\mathbb{R}$. This means that $\mathcal{F}$ maps $\mathrm{PAP}(\mathbb{X},\rho)$ into $\mathrm{PAP}(\mathbb{X},\rho)$.

Now, let $x, y \in \mathrm{PAP}(\mathbb{X},\rho)$ with

$$x = \alpha_x + \beta_x, \qquad y = \alpha_y + \beta_y, \tag{3.38}$$

where

$$\alpha_x, \alpha_y \in \mathrm{AP}(\mathbb{X}), \qquad \beta_x, \beta_y \in \mathrm{PAP}_0(\mathbb{X},\rho) \tag{3.39}$$

are periodic and vanishing mean value components of weighted pseudo almost-periodic functions x, y. Then by Theorem 3.1, we have

$$\begin{gathered} g(\cdot, \alpha_x(\cdot)), g(\cdot, \alpha_y(\cdot)) \in \mathrm{AP}(\mathbb{X}), \\ f(\cdot, x(\cdot)) - g(\cdot, \alpha_x(\cdot)), f(\cdot, y(\cdot)) - g(\cdot, \alpha_y(\cdot)) \in \mathrm{PAP}_0(\mathbb{X},\rho). \end{gathered} \tag{3.40}$$

Observe

$$\begin{aligned} \mathcal{F}x(t) - \mathcal{F}y(t) &= \int_{-\infty}^{t} T(t-s)\big(g(s,\alpha_x(s)) - g(s,\alpha_y(s))\big) ds \\ &\quad + \int_{-\infty}^{t} T(t-s)\big[\big(f(s,x(s)) - g(s,\alpha_x(s))\big) - \big(f(s,y(s)) - g(s,\alpha_y(s))\big)\big] ds. \end{aligned} \tag{3.41}$$

We know from the arguments in the above paragraph that the two integrals on the right side are in, respectively, $\mathrm{AP}(\mathbb{X})$ and $\mathrm{PAP}_0(\mathbb{X},\rho)$. Therefore,

$$\begin{aligned} \|\mathcal{F}x - \mathcal{F}y\|_{\rho} &\leq \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} \big\|T(t-s)\big(g(s,\alpha_x(s)) - g(s,\alpha_y(s))\big)\big\| ds \\ &\quad + \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} \big\|T(t-s)\big[\big(f(s,x(s))\big) - \big(g(s,\alpha_x(s))\big) - \big(f(s,y(s)) - g(s,\alpha_y(s))\big)\big]\big\| ds \end{aligned}$$

$$\begin{aligned}
&\leq \left(2L_g\left\|\alpha_x-\alpha_y\right\|+L_f\|x-y\|\right)\sup_{t\in\mathbb{R}}\int_{-\infty}^{t}\|T(t-s)\|ds\\
&\leq \left(2L_g+L_f\right)\left(\left\|\alpha_x-\alpha_y\right\|+\left\|\beta_x-\beta_y\right\|\right)\sup_{t\in\mathbb{R}}\int_{-\infty}^{t}Ke^{-\omega(t-s)}ds\\
&=\frac{\left(2L_g+L_f\right)K}{\omega}\left(\left\|\alpha_x-\alpha_y\right\|+\left\|\beta_x-\beta_y\right\|\right).
\end{aligned} \tag{3.42}$$

Since the inequality is true for any decomposition of $x(t)$, $y(t)$, we can take the infimum and then obtain

$$\left\|\mathcal{F}x-\mathcal{F}y\right\|_{\rho}\leq\frac{\left(2L_g+L_f\right)K}{\omega}\|x-y\|_{\rho}. \tag{3.43}$$

Hence $\mathcal{F}$ is a contraction under the assumption that $(2L_g+L_f)K/\omega<1$.

By the contraction mapping theorem, the mapping $\mathcal{F}$ has a unique fixed point $x(t)\in\mathrm{PAP}(\mathbb{X},\rho)$, and this fixed point satisfies the integral equation

$$x(t)=\int_{-\infty}^{t}T(t-s)f(s,x(s))ds, \tag{3.44}$$

for all $t\in\mathbb{R}$. Fixing $a\in\mathbb{R}$, we have

$$x(a)=\int_{-\infty}^{a}T(a-s)f(s,x(s))ds. \tag{3.45}$$

Since

$$T(t-s)=T(t-a)(a-s),\quad t\geq a\geq s, \tag{3.46}$$

it follows that $x(t)$ satisfies (3.22). Hence $x(t)$ is a mild solution to (1.1).

On the other hand, let $y(t)$ be a pseudo almost-periodic mild solution to (1.1). Then $y(t)$ satisfies (3.22), with x replaced by y. Letting $a\to-\infty$ yields

$$y(t)=\int_{-\infty}^{t}T(t-s)f(s,y(s))ds,\quad t\in\mathbb{R}. \tag{3.47}$$

Since $y(t)$ is bounded on $\mathbb{R}$ and $\{T(t)\}_{t\geq 0}$ is exponentially stable, hence $x(t)\equiv y(t)$ on $\mathbb{R}$.

In conclusion, $x(t)$ is the unique mild solution to (1.1), which ends the proof. □

Example 3.6. Let $\mathbb{X} = L^2(0,1)$, and

$$D(A) := \left\{x \in C^1[0,1];\ x' \text{ is absolutely continuous on } [0,1],\ x'' \in \mathbb{X},\ x(0) = x(1) = 0\right\},$$
$$Ax(t) = x''(t), \quad t \in (0,1),\ x \in D(A). \tag{3.48}$$

Then, A generates a C_0-semigroup $T(t)$ on $\mathbb{X}$ with

$$\|T(t)\| \le e^{-\pi^2 t}, \quad t \ge 0. \tag{3.49}$$

Let

$$\begin{aligned} f(t,x) &= x\left(\cos t + \cos\sqrt{2}t\right) + \frac{1}{1+t^2}\cos x \\ &= g(t,x) + \phi(t,x), \quad t \in \mathbb{R},\ x \in \mathbb{X}, \\ \rho(t) &= \left(1+t^2\right). \end{aligned} \tag{3.50}$$

Then, $g \in \mathrm{AP}(\mathbb{R}\times\mathbb{X},\mathbb{X})$, $\phi \in \mathrm{PAP}_0(\mathbb{R}\times\mathbb{X},\mathbb{X},\rho)$, and f, g satisfy (H_2) with

$$L_f = 2, \quad L_g = 2, \quad K = 1, \quad \omega = \pi^2. \tag{3.51}$$

By Theorem 3.5, the corresponding equation (1.1) has a unique weighted pseudo almost-periodic mild solution under the weight $\rho(t) = (1+t^2)$.

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