Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 161925, 20 pages
doi:10.1155/2012/161925

Research Article

Positive Solutions for Nonlinear Singular Differential Systems Involving Parameter on the Half-Line

Lishan Liu,[1] Ying Wang,[1,2] Xinan Hao,[1] and Yonghong Wu[3]

[1] *School of Mathematical Sciences, Qufu Normal University, Qufu 273165, Shandong, China*
[2] *School of Science, Linyi University, Linyi 276005, Shandong, China*
[3] *Department of Mathematics and Statistics, Curtin University of Technology, Perth, WA 6845, Australia*

Correspondence should be addressed to Lishan Liu, lls@mail.qfnu.edu.cn

Received 16 May 2012; Accepted 14 June 2012

Academic Editor: Chaitan Gupta

By using the upper-lower solutions method and the fixed-point theorem on cone in a special space, we study the singular boundary value problem for systems of nonlinear second-order differential equations involving two parameters on the half-line. Some results for the existence, nonexistence and multiplicity of positive solutions for the problem are obtained.

1. Introduction

In this paper, we are concerned with the following boundary value problem for systems of nonlinear singular second-order ordinary differential equations on the half-line:

$$\begin{gathered}
-\left(p_1(t)u'(t)\right)' = \lambda\phi_1(t) f_1(t, u(t), v(t), a, b), \quad 0 < t < +\infty, \\
-\left(p_2(t)v'(t)\right)' = \lambda\phi_2(t) f_2(t, u(t), v(t), a, b), \quad 0 < t < +\infty, \\
\alpha_{11} u(0) - \beta_{11} \lim_{t\to 0^+} p_1(t)u'(t) = 0, \\
\alpha_{12} \lim_{t\to +\infty} u(t) + \beta_{12} \lim_{t\to +\infty} p_1(t)u'(t) = 0, \\
\alpha_{21} v(0) - \beta_{21} \lim_{t\to 0^+} p_2(t)v'(t) = 0, \\
\alpha_{22} \lim_{t\to +\infty} v(t) + \beta_{22} \lim_{t\to +\infty} p_2(t)v'(t) = 0,
\end{gathered} \tag{$P_{a,b}$}$$

where $\lambda > 0$ is a parameter, $a \geq 0$, $b \geq 0$ are constants; $f_1, f_2 : \mathbb{R}_+^5 \to \mathbb{R}_+$ are continuous, $\phi_1, \phi_2 : (0, +\infty) \to \mathbb{R}_+$ are continuous and may have singularity at $t = 0$; $p_i \in C(\mathbb{R}_+) \cap C^1(0, +\infty)$ with $p_i(t) > 0$ on $(0, +\infty)$ and $\int_0^{+\infty}(1/p_i(s))ds < +\infty$ $(i = 1, 2)$; $\alpha_{ij}, \beta_{ij} \geq 0$ $(i, j = 1, 2)$ with $\rho_i = \alpha_{i2}\beta_{i1} + \alpha_{i1}\beta_{i2} + \alpha_{i1}\alpha_{i2}\int_0^{+\infty}(1/p_i(s))ds > 0$ $(i = 1, 2)$, in which $\mathbb{R}_+ = [0, +\infty)$ is the set of nonnegative real numbers.

Boundary value problems (BVP for short) on infinite interval arise in many applications (see [1, 2] and the references therein). Over the last couple of decades, a great deal of results have been developed for differential, difference, and integral BVPs on the infinite interval, including those by Agarwal and O'Regan [1], O'Regan [2], and many others (see [3–17]). For the study of boundary value problems, Agarwal and O'Regan [1] adopted mainly the method of the nonlinear alternative theorem together with a wonderful diagonalization process and the fixed-point theorem in the Frechet space.

Boundary value problems on the half-line arise naturally in the study of radially symmetric solutions of nonlinear elliptic equations (see, [18–20]). Recently, by using the Krasnosel'skii fixed-point theorem, Lian and Ge [6] obtained the criteria for the existence of at least one positive solution, a unique positive solution, and multiple positive solutions of the following BVP

$$\begin{gathered} \left(p(t)x'(t)\right)' + \lambda\phi(t)f(t, x(t)) = 0, \quad 0 < t < +\infty, \\ \alpha_1 x(0) - \beta_1 \lim_{t \to 0^+} p(t)x'(t) = 0, \\ \alpha_2 \lim_{t \to +\infty} x(t) + \beta_2 \lim_{t \to +\infty} p(t)x'(t) = 0, \end{gathered} \tag{1.1}$$

where $\lambda > 0$ is a parameter, $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R} = (-\infty, +\infty)$ and $\phi(t) : (0, +\infty) \to (0, +\infty)$ are continuous. More recently, by employing the method of varying in translation together with the fixed-point theorem in cone, Zhang et al. [14] established the existence of positive solution for the following semipositone singular Sturm-Liouville boundary value problem on the half-line

$$\begin{gathered} \left(p(t)x'(t)\right)' + f(t, x) + q(t) = 0, \quad 0 < t < +\infty, \\ \alpha_1 x(0) - \beta_1 \lim_{t \to 0^+} p(t)x'(t) = 0, \\ \alpha_2 \lim_{t \to +\infty} x(t) + \beta_2 \lim_{t \to +\infty} p(t)x'(t) = 0, \end{gathered} \tag{1.2}$$

where $f : (0, +\infty) \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and allows the nonlinearity to have singularity at $t = 0$, $q : (0, +\infty) \to \mathbb{R}$ is a Lebesgue integrable function. As far as we know, there is very few work concerning the systems of BVPs on the half-line, although the study for the systems of BVPs $(P_{a,b})$ on the half-line is very important.

Using the fixed-point theorem of cone expansion and compression type, the upper-lower solutions method, and degree arguments, do Ó et al. [21] studied the existence, nonexistence, and multiplicity of positive solutions for the following class of systems of second-order ordinary differential equations on the finite interval $[0, 1]$:

$$\begin{gathered} -u''(t) = f(t, u, v, a, b), \quad 0 < t < 1, \\ -v''(t) = g(t, u, v, a, b), \quad 0 < t < 1, \\ u(0) = u(1) = 0, \\ v(0) = v(1) = 0, \end{gathered} \tag{1.3}$$

where $f, g : [0,1] \times \mathbb{R}_+^4 \to \mathbb{R}_+$ are continuous and nondecreasing with respect to the last four variables.

Motivated by the above works, in this paper, we extend the results of [6, 14, 21, 22] to $(P_{a,b})$ and also expand the domain from finite intervals to the infinite interval—the half line.

There are two aims in this paper. The first aim is to obtain the existence of positive solutions for the system $(P_{a,b})$. For this purpose, we solve the fixed point of an operator F instead of the positive solutions for the system $(P_{a,b})$. The main difficulty for this is to testify that the operator F is completely continuous, as the Ascoli-Arzela theorem cannot be used in infinite interval $\mathbb{R}_+$. Some modification of the compactness criterion in infinite interval $\mathbb{R}_+$ (Lemma 2.4) has thus been made to resolve this problem. The second aim is to show that there exists a continuous curve Γ which splits the positive quadrant of the (a,b)-plane into two disjoint sets Q_1 and Q_2 such that the system $(P_{a,b})$ has at least two positive solutions in Q_1, at least one positive solution on the boundary of Q_1, and no positive solutions in Q_2.

The rest of the paper is organized as follows. In Section 2, we present some necessary definitions and lemmas that will be used to prove our main results. In Section 3, first, we give Lemma 3.1, which is a result of completely continuous operator, then we discuss our main results.

2. Preliminaries and Lemmas

In this section, we present some notations and lemmas that will be used in the proof of our main results.

Throughout this paper, the space $X = E \times E$ will be the basic space to study $(P_{a,b})$, where the Banach space E is denoted by $E = \{u \in C(\mathbb{R}_+) : \lim_{t\to+\infty} u(t) \text{ exists}\}$ with the supremum norm $\|u\|_\infty = \sup_{t\in\mathbb{R}_+}|u(t)|$. Clearly, $(X, \|\cdot\|)$ is a Banach space with the norm $\|(u,v)\| = \|u\|_\infty + \|v\|_\infty$ for $(u,v) \in X$. For convenience, let

$$\begin{gathered}
\overline{a}_i = \beta_{i1} + \alpha_{i1}\int_0^{+\infty}\frac{1}{p_i(s)}ds, \quad \overline{b}_i = \beta_{i2} + \alpha_{i2}\int_0^{+\infty}\frac{1}{p_i(s)}ds, \quad i = 1,2, \\
\overline{a}_i(t) = \beta_{i1} + \alpha_{i1}\int_0^{t}\frac{1}{p_i(s)}ds, \quad \overline{b}_i(t) = \beta_{i2} + \alpha_{i2}\int_t^{+\infty}\frac{1}{p_i(s)}ds, \quad i = 1,2.
\end{gathered} \tag{2.1}$$

Then, it is obvious that $\alpha_{i2}\overline{a}_i(t) + \alpha_{i1}\overline{b}_i(t) = \rho_i$ $(i = 1,2)$ is a constant for any $t \in \mathbb{R}_+$ and $\overline{a}_i(t)$ is increasing on $t \in \mathbb{R}_+$, $\overline{b}_i(t)$ is decreasing on $t \in \mathbb{R}_+$ for $i = 1,2$.

Lemma 2.1 (see [6]). *Under the condition $\int_0^{+\infty}(1/p_i(s))ds < +\infty$ and $\rho_i > 0$ for $i = 1,2$, the linear boundary value problem*

$$\begin{gathered}
(p_i(t)x'(t))' + v(t) = 0, \quad 0 < t < +\infty, \\
\alpha_{i1}x(0) - \beta_{i1}\lim_{t\to0^+}p_i(t)x'(t) = 0, \\
\alpha_{i2}\lim_{t\to+\infty}x(t) + \beta_{i2}\lim_{t\to+\infty}p_i(t)x'(t) = 0, \quad i = 1,2
\end{gathered} \tag{2.2}$$

has a unique solution for any $v \in L^1(0,+\infty)$. Moreover, this unique solution can be expressed in the form

$$x_i(t) = \int_0^{+\infty} G_i(t,s)v(s)ds, \quad i = 1,2, \tag{2.3}$$

where the Green function $G_i(t,s)$ is defined by

$$G_i(t,s)=\frac{1}{\rho_i}\begin{cases}\overline{a}_i(t)\overline{b}_i(s), & 0\le t\le s<+\infty,\\ \overline{a}_i(s)\overline{b}_i(t), & 0\le s\le t<+\infty, \quad i=1,2.\end{cases} \tag{2.4}$$

Remark 2.2. From (2.4), we can get the properties of $G_i(t,s)$ as follows.

(1) $G_i(t,s)$ is continuous and nonnegative on $\mathbb{R}_+\times\mathbb{R}_+$.

(2) For each $s\in\mathbb{R}_+$, $G_i(t,s)$ is continuously differentiable on $\mathbb{R}_+$ except for $t=s$.

(3) $\partial G_i(t,s)/\partial t|_{t=s^+}-\partial G_i(t,s)/\partial t|_{t=s^-}=-1/p_i(s)$.

(4) For each $s\in\mathbb{R}_+$, $G_i(t,s)$ satisfies the corresponding homogeneous BVP (i.e., the BVP (2.2) with $v(t)\equiv 0$) on $\mathbb{R}_+$ except for $t=s$. In other words, $G_i(t,s)$ is the Green function of BVP (2.2) on the half-line.

(5) $G_i(t,s)\le G_i(s,s)\le(1/\rho_i)\overline{a}_i\overline{b}_i<+\infty,\ i=1,2$.

(6) $\overline{G}_i(s)=\lim_{t\to+\infty}G_i(t,s)=(1/\rho_i)\beta_{i2}\overline{a}_i(s)\le G_i(s,s)<+\infty$, $i=1,2$.

(7) For any $t\in[\delta,1/\delta]$ and $s\in\mathbb{R}_+, G_i(t,s)\ge\omega_i G_i(s,s)$, where

$$\delta\in(0,1),\quad \omega_i=\min\left\{\frac{\overline{b}_i(1/\delta)}{\overline{b}_i},\frac{\overline{a}_i(\delta)}{\overline{a}_i}\right\},\quad i=1,2,\quad 0<\omega=\min\{\omega_1,\omega_2\}<1. \tag{2.5}$$

For any $(x_1,x_2,\dots,x_m)\in\mathbb{R}^m$, define $|(x_1,x_2,\dots,x_m)|=|x_1|+|x_2|+\cdots|x_m|$. In what follows, we list some conditions for convenience.

(H_1) The function $f_i:\mathbb{R}_+^5\to\mathbb{R}_+$ is continuous and nondecreasing with respect to the last four variables. In other words,

$$f_i(t,x_1,y_1,a_1,b_1)\le f_i(t,x_2,y_2,a_2,b_2),\qquad i=1,2, \tag{2.6}$$

for any $t\in\mathbb{R}_+,(x_1,y_1,a_1,b_1)\le(x_2,y_2,a_2,b_2)$, where the order is understood to apply to every component. And there exists (a_0,b_0) such that $f_i(t,x,y,a,b)$ is bounded for any (a,b) satisfying $(0,0)\le(a,b)\le(a_0,b_0)$, $t\in\mathbb{R}_+$, x and y in any bounded set of $\mathbb{R}_+$.

(H_2) The function $\phi_i(t):(0,+\infty)\to\mathbb{R}_+$ is continuous and singular at $t=0$, $\phi_i(t)\not\equiv 0$ on $\mathbb{R}_+$ satisfying $0<\int_0^{+\infty}G_i(s,s)\phi_i(s)ds<+\infty,\ i=1,2$.

(H_3) For the above a_0 and b_0 in (H_1),

$$\begin{aligned} f_i^0&=\limsup_{x+y\to0^+}\ \sup_{t\in\mathbb{R}_+}\frac{f_i(t,x,y,a_0,b_0)}{x+y}<L,\\ f_{i\infty}&=\liminf_{x+y\to+\infty}\ \inf_{t\in[\delta,1/\delta]}\frac{f_i(t,x,y,0,0)}{x+y}>l,\quad i=1,2,\end{aligned} \tag{2.7}$$

where

$$L = \frac{1}{2}\left(\max\left\{\int_0^{+\infty} G_1(s,s)\phi_1(s)ds, \int_0^{+\infty} G_2(s,s)\phi_2(s)ds\right\}\right)^{-1},$$
$$l = \frac{1}{2}\left(\min\left\{\min_{t\in[\delta,1/\delta]}\int_\delta^{1/\delta} G_1(t,s)\phi_1(s)ds, \min_{t\in[\delta,1/\delta]}\int_\delta^{1/\delta} G_2(t,s)\phi_2(s)ds\right\}\right)^{-1}. \tag{2.8}$$

(H_4)

$$\lim_{|(a,b)|\to+\infty} f_i(t,x,y,a,b) = +\infty, \quad \text{uniformly for } t \in \left[\delta, \frac{1}{\delta}\right],\ x \geq 0,\ y \geq 0,\ i = 1,2. \tag{2.9}$$

From the above assumptions (H_1) and (H_2), it is not difficult to show that the pair $(u,v) \in X$ is a solution of the system $(P_{a,b})$ if and only if $(u,v) \in X$ is a solution of the following system of nonlinear integral equations:

$$u(t) = \lambda\int_0^{+\infty} G_1(t,s)\phi_1(s)f_1(s,u(s),v(s),a,b)ds,$$
$$v(t) = \lambda\int_0^{+\infty} G_2(t,s)\phi_2(s)f_2(s,u(s),v(s),a,b)ds, \quad t \in \mathbb{R}_+. \tag{$S_{a,b}$}$$

Define operators $A_i : X \to E$ and $F : X \to X$ as follows:

$$A_i(u,v)(t) = \lambda\int_0^{+\infty} G_i(t,s)\phi_i(s)f_i(s,u(s),v(s),a,b)ds, \quad i = 1,2, \tag{2.10}$$

$$F(u,v) = (A_1(u,v), A_2(u,v)). \tag{2.11}$$

Then, the solution of the system $(P_{a,b})$ is equivalent to the fixed point of the operator F. Define a cone K in the Banach space X as follows:

$$K = \left\{(u,v) \in X : u(t) \geq 0,\ v(t) \geq 0, t \in \mathbb{R}_+,\ u(t) \geq \omega\|u\|_\infty, v(t) \geq \omega\|v\|_\infty,\ t \in \left[\delta, \frac{1}{\delta}\right]\right\}, \tag{2.12}$$

which induces a partial order "$\leq$": $(u_1,v_1) \leq (u_2,v_2)$ if and only if $u_1(t) \leq u_2(t), v_1(t) \leq v_2(t)$ for any $t \in \mathbb{R}_+$.

Consider the following system:

$$\begin{aligned}
&-\left(p_1(t)u'(t)\right)' = \lambda\phi_1(t)g_1(t,u(t),v(t)),\\
&-\left(p_2(t)v'(t)\right)' = \lambda\phi_2(t)g_2(t,u(t),v(t)), \quad 0<t<+\infty,\\
&\alpha_{11}u(0) - \beta_{11}\lim_{t\to 0^+} p_1(t)u'(t) = 0,\\
&\alpha_{12}\lim_{t\to +\infty} u(t) + \beta_{12}\lim_{t\to +\infty} p_1(t)u'(t) = 0,\\
&\alpha_{21}v(0) - \beta_{21}\lim_{t\to 0^+} p_2(t)v'(t) = 0,\\
&\alpha_{22}\lim_{t\to +\infty} v(t) + \beta_{22}\lim_{t\to +\infty} p_2(t)v'(t) = 0,
\end{aligned} \tag{S}$$

where $g_i : \mathbb{R}_+^3 \to \mathbb{R}_+$ $(i=1,2)$ are nonnegative continuous functions and are nondecreasing with respect to the last two variables.

Definition 2.3. The pair (u,v) is said to be a lower solution for the system (S) if the pair (u,v) satisfies the following inequality system:

$$\begin{aligned}
&-\left(p_1(t)u'(t)\right)' \le \lambda\phi_1(t)g_1(t,u(t),v(t)),\\
&-\left(p_2(t)v'(t)\right)' \le \lambda\phi_2(t)g_2(t,u(t),v(t)), \quad 0<t<+\infty,\\
&\alpha_{11}u(0) - \beta_{11}\lim_{t\to 0^+} p_1(t)u'(t) \le 0,\\
&\alpha_{12}\lim_{t\to +\infty} u(t) + \beta_{12}\lim_{t\to +\infty} p_1(t)u'(t) \le 0,\\
&\alpha_{21}v(0) - \beta_{21}\lim_{t\to 0^+} p_2(t)v'(t) \le 0,\\
&\alpha_{22}\lim_{t\to +\infty} v(t) + \beta_{22}\lim_{t\to +\infty} p_2(t)v'(t) \le 0.
\end{aligned} \tag{T}$$

Similarly, we define the upper solution for the system (S) by replacing the $\le$ in (T) by $\ge$.

Lemma 2.4 (see [1, 23]). *Let E be defined as above and $M \subset E$. Then, M is relatively compact in E if the following conditions hold.*

(1) *M is uniformly bounded in E.*

(2) *The functions in M are equicontinuous on any bounded interval of $\mathbb{R}_+$.*

(3) *The functions in M are equiconvergent at $+\infty$, that is, for any given $\varepsilon > 0$, there exists a $T = T(\varepsilon) > 0$ such that $|f(t) - f(+\infty)| < \varepsilon$, for any $t > T$, $f \in M$.*

Lemma 2.5 (see [24, 25]). *Let P be a positive cone in a real Banach space $(E, \|\cdot\|)$, $P_r = \{x \in P : \|x\| < r\}$, $\overline{P}_{r,R} = \{x \in P : r \le \|x\| \le R\}(0 < r < R < +\infty)$, and let $A : \overline{P}_{r,R} \to P$ be a completely continuous operator. If the following conditions are satisfied,*

(1) *$\|Ax\| \le \|x\|$, for all $x \in \partial P_R$,*

(2) *there exists a* $e \in \partial P_1$ *such that* $x \neq Ax + me$ *for any* $x \in \partial P_r$ *and* $m > 0$, *then* A *has fixed points in* $\overline{P}_{r,R}$.

Remark 2.6. If (1) and (2) are satisfied for $x \in \partial P_r$ and $x \in \partial P_R$, respectively, then Lemma 2.5 still holds.

Lemma 2.7 (see [25]). *Let* $(E, \|\cdot\|)$ *be a Banach space,* K *a cone in* E*, and let* Ω *be a bounded open set in* E *with* $\theta \in \Omega$*. Suppose that* $A : K \cap \overline{\Omega} \to K$ *is a completely continuous operator. If* $Ax \neq \mu x$ *for* $x \in K \cap \partial\Omega$ *and* $\mu \geq 1$*, then the fixed-point index*

$$i(A, K \cap \Omega, K) = 1. \tag{2.13}$$

Lemma 2.8 (see [25]). *Let* $(E, \|\cdot\|)$ *be a Banach space,* K *be a cone in* E*. For* $r > 0$*, define* $K_r = \{x \in K : \|x\| < r\}$*. Suppose that* $A : \overline{K}_r \to K$ *is a completely continuous operator such that* $Ax \neq x$ *for* $x \in \partial K_r$.

(i) *If* $\|Ax\| \geq \|x\|$ *for* $x \in \partial K_r$*, then* $i(A, K_r, K) = 0$.

(ii) *If* $\|Ax\| \leq \|x\|$ *for* $x \in \partial K_r$*, then* $i(A, K_r, K) = 1$.

3. Main Results

3.1. The Complete Continuity of the Operator F

Before presenting the main results, we give a lemma.

Lemma 3.1. *Assume that* (H_1) *and* (H_2) *hold. Then, for any* (a,b) *satisfying* $(0,0) \leq (a,b) \leq (a_0, b_0)$*,* $F : X \to X$ *is a completely continuous operator and* $F(K) \subseteq K$.

Proof. We divide the proof into four steps.

(i) Firstly, we show that $F : X \to X$ is well defined. For any fixed $(u,v) \in X$, there exists $r_1 > 0$ such that $|u(t)| \leq r_1$ and $|v(t)| \leq r_1$ for any $t \in \mathbb{R}_+$. It follows from (H_1) and the property (1) of the Green function $G_i(t,s)$ that $A_i(u,v) \geq 0$ and

$$B_{f_i}^{r_1} = \sup\{f_i(t,x,y,a,b) : t \in \mathbb{R}_+,\ x \in [0,r_1],\ y \in [0,r_1],\ a \in [0,a_0], b \in [0,b_0]\} < +\infty. \tag{3.1}$$

Thus, by (H_1) and (H_2), for any $t \in \mathbb{R}_+$, $a \in [0,a_0]$ and $b \in [0,b_0]$, we obtain

$$0 \leq \lambda \int_0^{+\infty} G_i(t,s)\phi_i(s) f_i(s,u(s),v(s),a,b)ds \leq \lambda B_{f_i}^{r_1} \int_0^{+\infty} G_i(s,s)\phi_i(s)ds < +\infty. \tag{3.2}$$

Hence, the operator $F(u,v) = (A_1(u,v), A_2(u,v))$ is well defined for any $(u,v) \in X$.

For any $t_1, t_2, s \in \mathbb{R}_+$, by the property (5) of the Green function $G_i(t,s)$, we have

$$|G_i(t_1,s) - G_i(t_2,s)|\phi_i(s) \leq 2G_i(s,s)\phi_i(s), \quad i = 1,2. \tag{3.3}$$

So, by (H_2), the Lebesgue dominated convergence theorem and the continuity of $G_i(t,s)$, for any $t_1, t_2 \in \mathbb{R}_+$, $a \in [0,a_0], b \in [0,b_0]$, we get

$$\begin{aligned}
&|A_i(u,v)(t_1) - A_i(u,v)(t_2)| \\
&\quad \le \lambda \int_0^{+\infty} |G_i(t_1,s) - G_i(t_2,s)| \phi_i(s) f_i(s,u(s),v(s),a,b)\,ds \\
&\quad \le \lambda B_{f_i}^{r_1} \int_0^{+\infty} |G_i(t_1,s) - G_i(t_2,s)| \phi_i(s)\,ds \\
&\quad \longrightarrow 0, \quad t_1 \longrightarrow t_2.
\end{aligned} \tag{3.4}$$

Therefore, $A_i(u,v) \in C(\mathbb{R}_+)$, and so $F(u,v) \in C(\mathbb{R}_+) \times C(\mathbb{R}_+)$. By the property (6) of $G_i(t,s)$ and the Lebesgue dominated convergence theorem, we obtain

$$\lim_{t\to+\infty} A_i(u,v)(t) = \lambda \int_0^{+\infty} \overline{G}_i(s)\phi_i(s) f_i(s,u(s),v(s),a,b)\,ds < +\infty. \tag{3.5}$$

Hence, $F : X \to X$ is well defined.

(ii) Next we show that $F : X \to X$ is continuous. Let $(u_n,v_n), (u,v) \in X, n \in \mathbb{N}$, and $\|(u_n,v_n) - (u,v)\| \to 0$ $(n \to +\infty)$, we will prove that $\|F(u_n,v_n) - F(u,v)\| \to 0$ $(n \to +\infty)$. By (2.10), (H_1), and (H_2), for any $t \in \mathbb{R}_+$, $a \in [0,a_0], b \in [0,b_0]$ and any natural number n, we have

$$\begin{aligned}
&|A_i(u_n,v_n)(t) - A_i(u,v)(t)| \\
&\quad \le \lambda \int_0^{+\infty} G_i(s,s)\phi_i(s)\Big(\big|f_i(s,u_n(s),v_n(s),a,b)\big| + \big|f_i(s,u(s),v(s),a,b)\big|\Big)ds \\
&\quad \le 2\lambda B_{f_i}^{r_2} \int_0^{+\infty} G_i(s,s)\phi_i(s)\,ds < +\infty, \quad i = 1,2,
\end{aligned} \tag{3.6}$$

where $B_{f_i}^{r_2} = \sup\{f_i(t,x,y,a,b) : t \in \mathbb{R}_+,\ x \in [0,r_2],\ y \in [0,r_2],\ a \in [0,a_0], b \in [0,b_0]\} < +\infty$ by (H_1), r_2 is a real number such that $r_2 \ge \max_{n\in\mathbb{N}}\{\|(u,v)\|, \|(u_n,v_n)\|\}$, in which $\mathbb{N}$ is the natural number set.

For any $\varepsilon > 0$, by (H_2), there exists a sufficiently large $T_0 > 0$ such that

$$2\lambda B_{f_i}^{r_2} \int_{T_0}^{+\infty} G_i(s,s)\phi_i(s)\,ds < \frac{\varepsilon}{2}, \quad i = 1,2. \tag{3.7}$$

On the other hand, by the continuity of $f_i(t,x,y,a,b)$ on $[0,T_0]\times[0,r_2]\times[0,r_2]\times[0,a_0]\times[0,b_0]$, for the above $\varepsilon > 0$, there exists a $\delta > 0$ such that for any $s \in [0,T_0]$, $a \in [0,a_0], b \in [0,b_0]$ and $x,x',y,y' \in [0,r_2]$, when $|x - x'| < \delta$, $|y - y'| < \delta$, we have

$$\big|f_i(s,x,y,a,b) - f_i(s,x',y',a,b)\big| < \frac{\varepsilon}{2}\left(\lambda \int_0^{+\infty} G_i(s,s)\phi_i(s)\,ds\right)^{-1}, \quad i = 1,2. \tag{3.8}$$

From $\|(u_n, v_n) - (u, v)\| \to 0$ $(n \to +\infty)$ and the definition of the norm $\|\cdot\|$ in the space X, we can easily conclude that $\|u_n - u\|_\infty \to 0$, $\|v_n - v\|_\infty \to 0$ $(n \to +\infty)$. So, for the above $\delta > 0$, there exists a sufficiently large natural number N_0 such that, when $n > N_0$, for any $s \in [0, T_0]$, we have

$$|u_n(s) - u(s)| \le \|u_n - u\|_\infty < \delta, \qquad |v_n(s) - v(s)| \le \|v_n - v\|_\infty < \delta. \tag{3.9}$$

Hence, by (3.8), when $n > N_0$, for any $s \in [0, T_0]$, $a \in [0, a_0]$, $b \in [0, b_0]$, we get

$$\left| f_i(s, u_n(s), v_n(s), a, b) - f_i(s, u(s), v(s), a, b) \right| < \frac{\varepsilon}{2}\left(\lambda \int_0^{+\infty} G_i(s,s)\phi_i(s)ds \right)^{-1}. \tag{3.10}$$

Therefore, by (2.10), (3.7), and (3.10), when $n > N_0$, for any $t \in \mathbb{R}_+$, $a \in [0, a_0]$ and $b \in [0, b_0]$, we obtain

$$\begin{aligned}
&|A_i(u_n, v_n)(t) - A_i(u, v)(t)| \\
&\le \lambda \int_0^{+\infty} G_i(s,s)\phi_i(s)\left| f_i(s, u_n(s), v_n(s), a, b) - f_i(s, u(s), v(s), a, b) \right| ds \\
&\le \lambda \int_0^{T_0} G_i(s,s)\phi_i(s)\left| f_i(s, u_n(s), v_n(s), a, b) - f_i(s, u(s), v(s), a, b) \right| ds \\
&\qquad + 2\lambda B_{f_i}^{r_2} \int_{T_0}^{+\infty} G_i(s,s)\phi_i(s)ds < \varepsilon, \quad i = 1, 2.
\end{aligned} \tag{3.11}$$

This implies that the operator $A_i : X \to E$ is continuous. Therefore, the operator $F : X \to X$ is continuous.

(iii) We need to prove that the operator $F : X \to X$ is compact. Let D be any bounded subset of X. Then, there exists a constant $r_3 > 0$ such that $\|(u, v)\| \le r_3$ for any $(u, v) \in D$. So $\|u\|_\infty \le r_3$, $\|v\|_\infty \le r_3$ for any $(u, v) \in D$. By (2.10), (H_1), and (H_2), for any $(u, v) \in D$ and $t \in \mathbb{R}_+$, we have

$$\begin{aligned}
|F(u,v)(t)| &= |A_1(u,v)(t)| + |A_2(u,v)(t)| \\
&= \left| \lambda \int_0^{+\infty} G_1(t,s)\phi_1(s) f_1(s, u(s), v(s), a, b) ds \right| \\
&\quad + \left| \lambda \int_0^{+\infty} G_2(t,s)\phi_2(s) f_2(s, u(s), v(s), a, b) ds \right| \\
&\le \lambda B_{f_1}^{r_3} \int_0^{+\infty} G_1(s,s)\phi_1(s)ds + \lambda B_{f_2}^{r_3} \int_0^{+\infty} G_2(s,s)\phi_2(s)ds \\
&< +\infty,
\end{aligned} \tag{3.12}$$

where $B_{f_i}^{r_3} = \sup\{ f_i(t, x, y, a, b) : t \in \mathbb{R}_+,\ x \in [0, r_3],\ y \in [0, r_3],\ a \in [0, a_0],\ b \in [0, b_0] \} < +\infty$ by (H_1). Hence, $F(D)$ is uniformly bounded. By the similar proof as for (3.4), we can conclude that $A_i(D)$ is equicontinuous, and so $F(D)$ is also equicontinuous.

From (H_2) and the property (6) of the Green function $G_i(t,s)$, for any $t \in \mathbb{R}_+$, we have

$$\int_0^{+\infty} \left|G_i(t,s) - \overline{G}_i(s)\right| \phi_i(s)ds \leq 2\int_0^{+\infty} G_i(s,s)\phi_i(s)ds < +\infty, \quad i = 1,2. \tag{3.13}$$

By (2.10), (3.5), and the Lebesgue dominated convergence theorem, for any $(u,v) \in D$, $t \in \mathbb{R}_+$, $a \in [0,a_0]$, and $b \in [0,b_0]$, we obtain

$$\begin{aligned}
|A_i(u,v)(t) - A_i(u,v)(+\infty)| &= \left|\lambda\int_0^{+\infty} G_i(t,s)\phi_i(s)f_i(s,u(s),v(s),a,b)ds\right. \\
&\qquad \left. -\lambda\int_0^{+\infty} \overline{G}_i(s)\phi_i(s)f_i(s,u(s),v(s),a,b)ds\right| \\
&\leq \lambda\int_0^{+\infty} \left|G_i(t,s) - \overline{G}_i(s)\right|\phi_i(s)f_i(s,u(s),v(s),a,b)ds \\
&\leq \lambda B_{f_i}^{r_3}\int_0^{+\infty} \left|G_i(t,s) - \overline{G}_i(s)\right|\phi_i(s)ds \\
&\longrightarrow 0, \quad t \longrightarrow +\infty, \ i = 1,2.
\end{aligned} \tag{3.14}$$

This implies that $A_i(D)$ is equiconvergent at $+\infty$. Hence, $F(D)$ is equiconvergent at $+\infty$. Therefore, the above discussion and Lemma 2.4 imply that $F : X \to X$ is completely continuous.

(iv) Finally, we prove $F(K) \subseteq K$. By the property (1) of $G_i(t,s)$, (H_1), and (H_2), it is easy to see that, for any $(u,v) \in K$ and $t \in \mathbb{R}_+$, $A_i(u,v)(t) \geq 0$ and

$$\begin{aligned}
A_i(u,v)(t) &= \lambda\int_0^{+\infty} G_i(t,s)\phi_i(s)f_i(s,u(s),v(s),a,b)ds \\
&\leq \lambda\int_0^{+\infty} G_i(s,s)\phi_i(s)f_i(s,u(s),v(s),a,b)ds.
\end{aligned} \tag{3.15}$$

So

$$\|A_i(u,v)\|_\infty \leq \lambda\int_0^{+\infty} G_i(s,s)\phi_i(s)f_i(s,u(s),v(s),a,b)ds, \quad i = 1,2. \tag{3.16}$$

On the other hand, by the property (7) of $G_i(t,s)$, we have

$$A_i(u,v)(t) \geq \omega\lambda\int_0^{+\infty} G_i(s,s)\phi_i(s)f_i(s,u(s),v(s),a,b)ds, \quad t \in \left[\delta, \frac{1}{\delta}\right], \ i = 1,2. \tag{3.17}$$

It follows from (3.16) and (3.17) that

$$A_i(u,v)(t) \geq \omega\|A_i(u,v)\|_\infty, \quad t \in \left[\delta, \frac{1}{\delta}\right], \ i = 1,2. \tag{3.18}$$

Therefore $F(K) \subseteq K$. The proof of Lemma 3.1 is completed. □

3.2. The Positive Solution for System (P_{a_0,b_0})

Theorem 3.2. *Assume that* (H_1)–(H_3) *hold, then the system* (P_{a_0,b_0}) *has at least one positive solution for any*

$$\lambda \in \left(\frac{l}{\min(f_{1\infty}, f_{2\infty})}, \frac{L}{\max(f_1^0, f_2^0)} \right). \tag{3.19}$$

Proof. From (3.19), there exists $\varepsilon_0 > 0$ such that

$$\frac{l}{\min(f_{1\infty}, f_{2\infty}) - \varepsilon_0} \le \lambda \le \frac{L}{\max(f_1^0, f_2^0) + \varepsilon_0}. \tag{3.20}$$

By the first inequality of (H_3), there exists $r > 0$ such that

$$f_i(t, x, y, a_0, b_0) \le \left(f_i^0 + \varepsilon_0\right) r, \quad 0 \le x + y \le r,\ t \in \mathbb{R}_+,\ i = 1, 2. \tag{3.21}$$

Setting $K_{r_1} = \{(u, v) \in K : \|(u, v)\| < r_1\}$ $(r_1 \le r)$, by the definition of $\|\cdot\|$, we know that

$$u(t) + v(t) \le \|u\|_\infty + \|v\|_\infty = \|(u, v)\| = r_1 \le r, \quad \forall (u, v) \in \partial K_{r_1}, t \in \mathbb{R}_+. \tag{3.22}$$

Then, for any $(u, v) \in \partial K_{r_1}$,

$$\begin{aligned}
\|A_i(u, v)\|_\infty &= \lambda \sup_{t\in\mathbb{R}_+} \left| \int_0^{+\infty} G_i(t, s)\phi_i(s) f_i(s, u(s), v(s), a_0, b_0) ds \right| \\
&\le \lambda \sup_{t\in\mathbb{R}_+} \int_0^{+\infty} G_i(t, s)\phi_i(s) \left(f_i^0 + \varepsilon_0\right) r_1 ds \\
&\le \lambda \left(f_i^0 + \varepsilon_0\right) r_1 \int_0^{+\infty} G_i(s, s)\phi_i(s) ds \\
&\le r_1 L \int_0^{+\infty} G_i(s, s)\phi_i(s) ds \\
&\le \frac{1}{2} r_1 = \frac{1}{2}\|(u, v)\|, \quad i = 1, 2.
\end{aligned} \tag{3.23}$$

Thus,

$$\|F(u, v)\| = \|A_1(u, v)\|_\infty + \|A_2(u, v)\|_\infty \le \|(u, v)\|, \quad (u, v) \in \partial K_{r_1}. \tag{3.24}$$

On the other hand, by the second inequality of (H_3), there exists $r_0 > \omega r_1 > 0$ such that

$$f_i(t, x, y, a_0, b_0) \ge f_i(t, x, y, 0, 0) \ge (f_{i\infty} - \varepsilon_0)(x + y), \quad x + y \ge r_0,\ t \in \left[\delta, \frac{1}{\delta}\right],\ i = 1, 2. \tag{3.25}$$

Take $r_2 = r_0/\omega > r_1$, and let $K_{r_2} = \{(u,v) \in K : \|(u,v)\| < r_2\}$, $(u_0, v_0) = (1/2, 1/2) \in \partial K_1$. Then,

$$(u,v) \neq F(u,v) + \mu(u_0, v_0), \quad \forall (u,v) \in \partial K_{r_2},\ \forall \mu > 0. \tag{3.26}$$

Suppose that (3.26) is false, then there exist $(u_2, v_2) \in \partial K_{r_2}$ and $\mu_2 > 0$ such that $(u_2, v_2) = F(u_2, v_2) + \mu_2(u_0, v_0)$. From (3.25) and the fact that

$$u_2(t) + v_2(t) \geq \omega\|u_2\|_\infty + \omega\|v_2\|_\infty = \omega\|(u_2, v_2)\| = \omega r_2 = r_0, \quad t \in \left[\delta, \frac{1}{\delta}\right], \tag{3.27}$$

we have

$$f_i(t, u_2(t), v_2(t), a_0, b_0) \geq \left(f_{i\infty} - \varepsilon_0\right)(u_2(t) + v_2(t)), \quad t \in \left[\delta, \frac{1}{\delta}\right],\ i = 1, 2. \tag{3.28}$$

Set

$$\xi = \min\left\{u_2(t) + v_2(t) : t \in \left[\delta, \frac{1}{\delta}\right]\right\}. \tag{3.29}$$

Then, $u_2(t) + v_2(t) \geq \xi > 0$ for any $t \in [\delta, 1/\delta]$. Hence, for any $t \in [\delta, 1/\delta]$, by (3.28), we have

$$\begin{aligned}
u_2(t) + v_2(t) &= A_1(u_2(t), v_2(t)) + A_2(u_2(t), v_2(t)) + \mu_2(u_0 + v_0) \\
&= \lambda \int_0^{+\infty} G_1(t,s)\phi_1(s) f_1(s, u_2(s), v_2(s), a_0, b_0)\,ds \\
&\quad + \lambda \int_0^{+\infty} G_2(t,s)\phi_2(s) f_2(s, u_2(s), v_2(s), a_0, b_0)\,ds + \mu_2(u_0 + v_0) \\
&\geq \lambda \int_\delta^{1/\delta} G_1(t,s)\phi_1(s)\left(f_{1\infty} - \varepsilon_0\right)(u_2(s) + v_2(s))\,ds \\
&\quad + \lambda \int_\delta^{1/\delta} G_2(t,s)\phi_2(s)\left(f_{2\infty} - \varepsilon_0\right)(u_2(s) + v_2(s))\,ds + \mu_2 \\
&\geq \min_{s\in[\delta,1/\delta]}(u_2(s) + v_2(s))\lambda\left(f_{1\infty} - \varepsilon_0\right) \min_{t\in[\delta,1/\delta]} \int_\delta^{1/\delta} G_1(t,s)\phi_1(s)\,ds \\
&\quad + \min_{s\in[\delta,1/\delta]}(u_2(s) + v_2(s))\lambda\left(f_{2\infty} - \varepsilon_0\right) \min_{t\in[\delta,1/\delta]} \int_\delta^{1/\delta} G_2(t,s)\phi_2(s)\,ds + \mu_2 \\
&\geq \xi + \mu_2.
\end{aligned} \tag{3.30}$$

Then, we can obtain that

$$u_2(t) + v_2(t) > \xi + \mu_2, \quad t \in \left[\delta, \frac{1}{\delta}\right]. \tag{3.31}$$

It is clearly that (3.31) contradicts (3.29), which implies that (3.26) holds.

It follows from (3.24), (3.26), Lemmas 2.5 and 3.1 that the operator F has fixed-point $(u,v) \in K_{r_2} \setminus \overline{K}_{r_1}$ such that $0 < r_1 < \|(u,v)\| < r_2$. It is easy to see that (u,v) is a positive solution of the system (P_{a_0,b_0}). The proof of Theorem 3.2 is completed. □

Remark 3.3. Noticing that $l/f_{1\infty} < 1$, $l/f_{2\infty} < 1$ and $L/f_1^0 > 1$, $L/f_2^0 > 1$, we conclude that Theorem 3.2 also holds for $\lambda = 1$.

Remark 3.4. From Theorem 3.2, we can see that $f_i(t,x,y,a_0,b_0)$ $(i = 1,2)$ do not need to be superlinear or sublinear. In fact, Theorem 3.2 still holds, if (H_1)-(H_2) are satisfied and one of the following conditions is satisfied:

(1) $f_{1\infty} = f_{2\infty} = +\infty$, $L > f_1^0 > 0$, $L > f_2^0 > 0$, for each $\lambda \in (0, L/\max\{f_1^0, f_2^0\})$,

(2) $f_{1\infty} = f_{2\infty} = +\infty$, $f_1^0 = f_2^0 = 0$, for each $\lambda \in (0,+\infty)$,

(3) $f_{1\infty} > l$, $f_{2\infty} > l$, $f_1^0 = f_2^0 = 0$, for each $\lambda \in (l/\min\{f_{1\infty}, f_{2\infty}\}, +\infty)$.

3.3. Lower and Upper Solutions

Theorem 3.5. *Assume that* (H_1)-(H_2) *hold. Let* $(\underline{u},\underline{v})$ *and* $(\overline{u},\overline{v})$ *be a lower solution and an upper solution, respectively, of the system* (S) *such that* $(0,0) \le (\underline{u},\underline{v}) \le (\overline{u},\overline{v})$. *Then, the system* (S) *has a nonnegative solution* (u,v) *satisfying* $(\underline{u},\underline{v}) \le (u,v) \le (\overline{u},\overline{v})$.

Proof. Let

$$\begin{gathered} M_i(u,v)(t) = \lambda \int_0^{+\infty} G_i(t,s)\phi_i(s) g_i(s,u(s),v(s))\,ds, \quad i = 1,2, \\ H(u,v) = (M_1(u,v), M_2(u,v)). \end{gathered} \tag{3.32}$$

Then, the solutions of the system (S) are equivalent to the fixed points of the operator H in K.

Now, we introduce the following auxiliary operator $\mathscr{H}$ defined by

$$\mathscr{H}(u,v) = (\mathscr{M}_1(u,v), \mathscr{M}_2(u,v)), \tag{3.33}$$

where

$$\mathscr{M}_i(u,v)(t) = \lambda \int_0^{+\infty} G_i(t,s)\phi_i(s) g_i(s,\zeta_1(s,u),\zeta_2(s,v))\,ds, \quad i = 1,2, \tag{3.34}$$

in which

$$\zeta_1(t,u) = \max\{\underline{u}(t), \min\{u(t),\overline{u}(t)\}\}, \qquad \zeta_2(t,v) = \max\{\underline{v}(t), \min\{v(t),\overline{v}(t)\}\}. \tag{3.35}$$

It is easy to prove that the operator $\mathscr{H}$ has the following properties.

(1) $\mathscr{H}$ is a completely continuous operator.

(2) If the pair $(u,v) \in K$ is a fixed point of $\mathscr{H}$, then (u,v) is a fixed point of H with $(\underline{u},\underline{v}) \leq (u,v) \leq (\overline{u},\overline{v})$.

(3) If $(u,v) = \mu\mathscr{H}(u,v)$ with $0 \leq \mu \leq 1$, then $\|(u,v)\| \leq C_{**}$, where C_{**} does not depend on μ and $(u,v) \in K$.

Therefore, by using the topological degree of Leray-Schauder (see [26, Corollary 8.1, page. 61]), we obtain a fixed point of the operator H. The proof of Theorem 3.5 is completed. □

Theorem 3.6. *Assume that* (H_1)–(H_3) *and* (3.19) *hold. Then for* $(0,0) \leq (a,b) \leq (a_0,b_0)$, *the system* $(P_{a,b})$ *has at least one positive solution.*

Proof. From Theorem 3.2, we can see that the system (P_{a_0,b_0}) has at least one positive solution (u_0,v_0). Since the functions f_i $(i=1,2)$ are increasing functions with respect to the last four variables, we conclude that (u_0,v_0) is an upper solution and $(0,0)$ is a lower solution for the system $(P_{a,b})$. Hence, by Theorem 3.5, we have that the system $(P_{a,b})$ has at least one positive solution. This completes the proof of Theorem 3.6. □

3.4. A Priori Estimate

Theorem 3.7. *Assume that* (H_1)–(H_3) *and* (3.19) *hold. Then for any* (a,b) *satisfying* $(0,0) \leq (a,b) \leq (a_0,b_0)$, *there exists a constant* $C_0 > 0$ *independent of* (a,b), *such that* $\|(u,v)\| \leq C_0$ *for all positive solutions* (u,v) *of the system* $(P_{a,b})$.

Proof. Assume by contradiction that there exists a sequence of positive solution $(u_n,v_n) \in X$ of system $(P_{a,b})$ such that $\|(u_n,v_n)\| \to \infty$. From (3.19), there exists $\varepsilon_0 > 0$ such that $\lambda > l/(\min(f_{1\infty},f_{2\infty}) - \varepsilon_0)$. From assumption (H_3), there exists $H > 0$ such that, for $(0,0) \leq (a,b) \leq (a_0,b_0)$,

$$f_i(t,x,y,a,b) \geq f_i(t,x,y,0,0) \geq (f_{i\infty} - \varepsilon_0)(x+y), \quad x+y \geq H,\ t \in \left[\delta, \frac{1}{\delta}\right],\ i=1,2. \tag{3.36}$$

Since $u_n(t)+v_n(t) \geq \omega\|u_n\|_\infty + \omega\|v_n\|_\infty = \omega\|(u_n,v_n)\|, t \in [\delta,1/\delta]$, there exists natural number N_0 such that, for $n > N_0$, we have $u_n(t)+v_n(t) \geq H$ for $t \in [\delta,1/\delta]$. It follows from (3.36) that when $n > N_0$, $f_i(t,u_n(t),v_n(t),a,b) \geq (f_{i\infty}-\varepsilon_0)(u_n(t)+v_n(t)), t \in [\delta,1/\delta], i=1,2$. Thus,

$$\begin{aligned}
u_n(t)+v_n(t) &= \lambda\int_0^{+\infty} G_1(t,s)\phi_1(s)f_1(s,u_n(s),v_n(s),a,b)ds \\
&\quad + \lambda\int_0^{+\infty} G_2(t,s)\phi_2(s)f_2(s,u_n(s),v_n(s),a,b)ds \\
&\geq \lambda\int_\delta^{1/\delta} G_1(t,s)\phi_1(s)(f_{1\infty}-\varepsilon_0)(u_n(s)+v_n(s))ds \\
&\quad + \lambda\int_\delta^{1/\delta} G_2(t,s)\phi_2(s)(f_{2\infty}-\varepsilon_0)(u_n(s)+v_n(s))ds
\end{aligned}$$

$$\begin{aligned}
&\geq \min_{s\in[\delta,1/\delta]}(u_n(s)+v_n(s))\lambda\left(f_{1\infty}-\varepsilon_0\right)\min_{t\in[\delta,1/\delta]}\int_{\delta}^{1/\delta}G_1(t,s)\phi_1(s)ds\\
&\quad+\min_{s\in[\delta,1/\delta]}(u_n(s)+v_n(s))\lambda\left(f_{2\infty}-\varepsilon_0\right)\min_{t\in[\delta,1/\delta]}\int_{\delta}^{1/\delta}G_2(t,s)\phi_2(s)ds\\
&> \min_{s\in[\delta,1/\delta]}(u_n(s)+v_n(s)),\quad t\in\left[\delta,\frac{1}{\delta}\right]
\end{aligned}\tag{3.37}$$

which yields

$$\min_{t\in[\delta,1/\delta]}u_n(t)+v_n(t)>\min_{t\in[\delta,1/\delta]}(u_n(t)+v_n(t)),\tag{3.38}$$

which is a contradiction. This completes the proof of Theorem 3.7. □

Remark 3.8. By the discussions in Sections 3.1, 3.2, 3.3 and 3.4, we can conclude that, for any (a,b) satisfying $|(0,0)|\leq|(a,b)|\leq|(a_0,b_0)|$, the system $(P_{a,b})$ has at least one positive solution (u,v) with $\|(u,v)\|\leq C_0$. In the following section, we will establish the nonexistence result for the system $(P_{a,b})$.

3.5. Non-Existence

Theorem 3.9. *Assume that* (H_1)–(H_4) *and* (3.19) *hold. Then, there exist* $\underline{a}>0$, $\underline{b}>0$ *such that, for all* (a,b) *with* $|(a,b)|>|(\underline{a},\underline{b})|$, *the system* $(P_{a,b})$ *has no solution.*

Proof. Suppose by contradiction that there exists a sequence (a_n,b_n) with $|(a_n,b_n)|\to+\infty$ $(n\to+\infty)$ such that, for each nature number n, the system (P_{a_n,b_n}) has a positive solution (u_n,v_n) in K. From assumption (H_4), for any $M>0$, there exists a constant $C>0$ such that, for any $|(a,b)|>C$,

$$f_i(t,x,y,a,b)\geq M,\qquad x,y\geq 0,\ t\in\left[\delta,\frac{1}{\delta}\right],\ i=1,2.\tag{3.39}$$

By $|(a_n,b_n)|\to+\infty$ $(n\to+\infty)$, for the above $C>0$, there exists a natural number n_0 such that, for $n>n_0$, $|(a_n,b_n)|>C$, then, for $n>n_0$ and $t\in[\delta,1/\delta]$,

$$\begin{aligned}
u_n(t)&=\lambda\int_0^{+\infty}G_1(t,s)\phi_1(s)f_1(s,u_n(s),v_n(s),a_n,b_n)ds\\
&>\lambda\int_{\delta}^{1/\delta}G_1(t,s)\phi_1(s)f_1(s,u_n(s),v_n(s),a_n,b_n)ds\\
&\geq M\lambda\omega\int_{\delta}^{1/\delta}G_1(s,s)\phi_1(s)ds.
\end{aligned}\tag{3.40}$$

By the same way, we can obtain

$$v_n(t)\geq M\lambda\omega\int_{\delta}^{1/\delta}G_2(s,s)\phi_2(s)ds.\tag{3.41}$$

Since we can choose M arbitrarily large, we conclude that u_n and v_n are unbounded sequences in K, then $\lim_{t\to+\infty} u_n(t)$ and $\lim_{t\to+\infty} v_n(t)$ do not exist, which contradicts the fact that $u_n, v_n \in K$. So Theorem 3.9 holds. $\square$

We next define the set

$$A = \{a > 0 : \text{ the system } (P_{a,b}) \text{ has a positive solution for some } b > 0\}. \tag{3.42}$$

From Theorems 3.2 and 3.9, we conclude that A is nonempty and bounded. Thus, $0 < \tilde{a} = \sup A < +\infty$. Using the upper-lower solutions method, we see that, for all $a \in (0, \tilde{a})$, there exists $b > 0$ such that the system $(P_{a,b})$ has a positive solution. We now define the function $\Gamma : [0, \tilde{a}] \to \mathbb{R}_+$ by

$$\Gamma(a) = \sup\{b > 0 : \text{ system } (P_{a,b}) \text{ has a positive solution }\}. \tag{3.43}$$

By Theorem 3.6, the function Γ is continuous and nonincreasing. We thus claim that $\Gamma(a)$ is attained. In fact, it suffices to use Theorem 3.7 and the compactness of the operator F. Finally, it follows from the definition of the function Γ that the system $(P_{a,b})$ has at least one positive solution for $0 \le b \le \Gamma(a)$ and has no positive solutions for $b > \Gamma(a)$.

3.6. Existence of Two Positive Solutions

In this section,we will assume that the nonlinearities f_1 and f_2 are strict-increasing with respect to the fifth variable. Fix $a \in [0, \tilde{a}]$, and let $(\overline{\phi}, \overline{\varphi})$ be the solution of the problem $(P_{a,\Gamma(a)})$ which is obtained using Theorem 3.6. Our next result allows us to establish another solution of the system $(P_{a,b})$ for $0 < b < \Gamma(a)$.

Lemma 3.10. *For each $0 < b < \Gamma(a)$, there exists $\epsilon_0 > 0$ so that, for all $0 < \epsilon \le \epsilon_0$ and all $t \in \mathbb{R}_+$, one has*

$$\begin{aligned} \overline{\phi}_\epsilon(t) &> \lambda \int_0^{+\infty} G_1(t,s)\phi_1(s) f_1\left(s, \overline{\phi}_\epsilon(s), \overline{\varphi}_\epsilon(s), a, b\right) ds, \\ \overline{\varphi}_\epsilon(t) &> \lambda \int_0^{+\infty} G_2(t,s)\phi_2(s) f_2\left(s, \overline{\phi}_\epsilon(s), \overline{\varphi}_\epsilon(s), a, b\right) ds, \end{aligned} \tag{3.44}$$

where $\overline{\phi}_\epsilon(t) = \overline{\phi}(t) + \epsilon$, $\overline{\varphi}_\epsilon(t) = \overline{\varphi}(t) + \epsilon$.

Proof. Fix $\delta \in (0,1)$. Since f_1 is strict-increasing with respect to the fifth variable, we have that for each $0 < b < \Gamma(a)$ we may find a positive constant $I = I(b)$ so that, for all $s \in [\delta, 1/\delta]$, we have

$$f_1\left(s, \overline{\phi}(s), \overline{\varphi}(s), a, \Gamma(a)\right) - f_1\left(s, \overline{\phi}(s), \overline{\varphi}(s), a, b\right) \ge I > 0. \tag{3.45}$$

By the uniform continuity of f_1, there exists $\epsilon_0 > 0$ so that, for all $s \in [\delta, 1/\delta]$ and all $0 < \epsilon \le \epsilon_0$, we have

$$\left| f_1\left(s, \overline{\phi}(s) + \epsilon, \overline{\varphi}(s) + \epsilon, a, b\right) - f_1\left(s, \overline{\phi}(s), \overline{\varphi}(s), a, b\right) \right| \le \frac{I}{2}. \tag{3.46}$$

Next, we define

$$\begin{aligned} \Phi_\epsilon(t,s) &= G_1(t,s)\phi_1(s)\left[f_1\left(s, \overline{\phi}_\epsilon(s), \overline{\varphi}_\epsilon(s), a, b\right) - f_1\left(s, \overline{\phi}(s), \overline{\varphi}(s), a, b\right)\right], \\ \Psi(t,s) &= G_1(t,s)\phi_1(s)\left[f_1\left(s, \overline{\phi}(s), \overline{\varphi}(s), a, \Gamma(a)\right) - f_1\left(s, \overline{\phi}(s), \overline{\varphi}(s), a, b\right)\right]. \end{aligned} \tag{3.47}$$

Assume $0 < \epsilon \le \epsilon_0$. Then,

$$\begin{aligned} \overline{\phi}_\epsilon(t) > \overline{\phi}(t) &= \lambda \int_0^{+\infty} G_1(t,s)\phi_1(s) f_1\left(s, \overline{\phi}(s), \overline{\varphi}(s), a, \Gamma(a)\right) ds \\ &= \lambda \int_0^{+\infty} G_1(t,s)\phi_1(s) f_1\left(s, \overline{\phi}_\epsilon(s), \overline{\varphi}_\epsilon(s), a, b\right) ds \\ &\quad - \lambda \int_0^{+\infty} \Phi_\epsilon(t,s) ds + \lambda \int_0^{+\infty} \Psi(t,s) ds. \end{aligned} \tag{3.48}$$

Since $\Psi(t,s)$ is positive and $\Psi(t,s) - \Phi_\epsilon(t,s) > (I/2) G_1(t,s)\phi_1(s)$ for $t \in [\delta, 1/\delta]$, we have

$$\begin{aligned} \overline{\phi}_\epsilon(t) > {} & \lambda \int_0^{+\infty} G_1(t,s)\phi_1(s) f_1\left(s, \overline{\phi}_\epsilon(s), \overline{\varphi}_\epsilon(s), a, b\right) ds \\ & - \lambda \int_0^{\delta} \Phi_\epsilon(t,s) ds - \lambda \int_{1/\delta}^{+\infty} \Phi_\epsilon(t,s) ds + \frac{I}{2}\lambda \int_\delta^{1/\delta} G_1(t,s)\phi_1(s) ds. \end{aligned} \tag{3.49}$$

It is not difficult to show that the Lebesgue dominated convergence theorem implies that $\lambda \int_0^{\delta} \Phi_\epsilon(t,s) ds + \lambda \int_{1/\delta}^{+\infty} \Phi_\epsilon(t,s) ds$ converges to zero, uniformly in t, as ϵ tends to zero. Thus, for ϵ sufficiently small, we have

$$\overline{\phi}_\epsilon(t) > \lambda \int_0^{+\infty} G_1(t,s)\phi_1(s) f_1\left(s, \overline{\phi}_\epsilon(s), \overline{\varphi}_\epsilon(s), a, b\right) ds \tag{3.50}$$

uniformly in $t \in \mathbb{R}_+$.

A similar computation holds for $\overline{\varphi}_\epsilon$. □

We are now in a position to show existence of two positive solutions of the system $(P_{a,b})$ for $0 < b < \Gamma(a)$, where $a \in [0, \tilde{a}]$ is fixed.

Theorem 3.11. *Assume that* (H_1)–(H_4) *and* (3.19) *hold. Then, for all* $a \in [0, \tilde{a}]$, *the system* $(P_{a,b})$ *has at least two positive solutions for* $0 < b < \Gamma(a)$.

Proof. Consider the set

$$\Omega = \left\{ (\phi, \psi) \in X : -\epsilon < \phi(t) < \overline{\phi}_\epsilon(t),\ -\epsilon < \psi(t) < \overline{\psi}_\epsilon(t),\ \text{for } t \in \mathbb{R}_+ \right\}, \tag{3.51}$$

where $\overline{\phi}_\epsilon$ and $\overline{\psi}_\epsilon$ are the functions of Lemma 3.10. It is not hard to see that Ω is bounded and open in X and that $(0,0) \in \Omega$. Note that one of the solutions of the system $(P_{a,b})$ belongs to $\overline{K \cap \Omega}$, we also know that $F : K \cap \overline{\Omega} \to K$ is a completely continuous operator.

Let $(\phi, \psi) \in K \cap \partial\Omega$. It follows that there exists $t_0 \in (0, +\infty)$ so that one of the following two cases holds: $\phi(t_0) = \overline{\phi}_\epsilon(t_0)$ or $\psi(t_0) = \overline{\psi}_\epsilon(t_0)$. In the case $\phi(t_0) = \overline{\phi}_\epsilon(t_0)$, it follows from Lemma 3.10 that, for all $\mu \geq 1$, we have

$$\begin{aligned} A_1(\phi, \psi)(t_0) &= \lambda \int_0^{+\infty} G_1(t_0, s)\phi_1(s) f_1(s, \phi(s), \psi(s), a, b)\, ds \\ &\leq \lambda \int_0^{+\infty} G_1(t_0, s)\phi_1(s) f_1\left(s, \overline{\phi}_\epsilon(s), \overline{\psi}_\epsilon(s), a, b\right) ds \\ &< \overline{\phi}_\epsilon(t_0) = \phi(t_0) \leq \mu\phi(t_0). \end{aligned} \tag{3.52}$$

Similarly, $A_2(\phi, \psi)(t_0) < \mu\psi(t_0)$ in the case $\psi(t_0) = \overline{\psi}_\epsilon(t_0)$. Hence, $F(\phi, \psi) \neq \mu(\phi, \psi)$, for all $(\phi, \psi) \in K \cap \partial\Omega$ and all $\mu \geq 1$. Now, according to Lemma 2.7, we have

$$i(F, K \cap \Omega, K) = 1. \tag{3.53}$$

On the other hand, a slight change in the proof of Theorem 3.7 shows the existence of an $r_3 > 0$ sufficiently large, say $r_3 > r_2$, where r_2 is as in Theorem 3.2, so that

$$\|F(\phi, \psi)\| > \|(\phi, \psi)\| \tag{3.54}$$

for every $\|(\phi, \psi)\| = r_3$ and every $(\phi, \psi) \in K$.

Let $r_4 = \max\{C_0 + 1, r_3, \|(\overline{\phi}_\epsilon, \overline{\psi}_\epsilon)\|\}$, where C_0 is as that in Theorem 3.7. Set

$$K_{r_4} = \{(\phi, \psi) \in K : \ \|(\phi, \psi)\| < r_4\}. \tag{3.55}$$

Then Theorem 3.7 implies that $F(\phi, \psi) \neq (\phi, \psi)$, for $(\phi, \psi) \in \partial K_{r_4}$. Consequently, Lemma 2.8 implies $i(F, K_{r_4}, \mathrm{K}) = 0$.

Now, by the additivity property of the fixed-point index, we obtain

$$i\left(F, K_{r_4} \setminus \overline{K \cap \Omega}, K\right) = -1. \tag{3.56}$$

Therefore, F has another fixed point in $K_{r_4} \setminus \overline{K \cap \Omega}$. □

Then, from the above discussion, it is easy to obtain the following conclusion.

4. Conclusion

Assume that (H_1)–(H_4) and (3.19) hold. Then, there exist a constant $\widetilde{a} > 0$ and a non-increasing continuous function $\Gamma : [0, \widetilde{a}] \to \mathbb{R}_+$ so that, for all $a \in [0, \widetilde{a}]$, the system $(P_{a,b})$ has at least one positive solution for $0 \le b \le \Gamma(a)$, has no positive solutions for $b > \Gamma(a)$, and has at least two positive solutions for $0 < b < \Gamma(a)$ when f_1 and f_2 are strict-increasing with respect to the fifth variable.

Acknowledgments

L. Liu, Y. Wang, and X. Hao authors were supported financially by the National Natural Science Foundation of China (11071141 and 11126231), the Natural Science Foundation of Shandong Province of China (ZR2011AQ008) and a Project of Shandong Province Higher Educational Science and Technology Program (J11LA06). The Y. Wu author was supported financially by the Australia Research Council through an ARC Discovery Project Grant.

References

[1] R. P. Agarwal and D. O'Regan, *Infinite Interval Problems for Differential, Difference and Integral Equations*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2001.
[2] D. O'Regan, *Theory of Singular Boundary Value Problems*, World Science, Singapore, 1994.
[3] R. Enguiça, A. Gavioli, and L. Sanchez, "Solutions of second-order and fourth-order ODEs on the half-line," *Nonlinear Analysis A*, vol. 73, no. 9, pp. 2968–2979, 2010.
[4] S. Djebali and K. Mebarki, "Multiple positive solutions for singular BVPs on the positive half-line," *Computers & Mathematics with Applications*, vol. 55, no. 12, pp. 2940–2952, 2008.
[5] L. Liu, X. Hao, and Y. Wu, "Unbounded solutions of second-order multipoint boundary value problem on the half-line," *Boundary Value Problems*, vol. 2010, Article ID 236560, 15 pages, 2010.
[6] H. Lian and W. Ge, "Existence of positive solutions for Sturm-Liouville boundary value problems on the half-line," *Journal of Mathematical Analysis and Applications*, vol. 321, no. 2, pp. 781–792, 2006.
[7] B. Liu, L. Liu, and Y. Wu, "Multiple solutions of singular three-point boundary value problems on $[0, \infty)$," *Nonlinear Analysis A*, vol. 70, no. 9, pp. 3348–3357, 2009.
[8] L. Liu, Z. Wang, and Y. Wu, "Multiple positive solutions of the singular boundary value problems for second-order differential equations on the half-line," *Nonlinear Analysis A*, vol. 71, no. 7-8, pp. 2564–2575, 2009.
[9] R. Ma and B. Zhu, "Existence of positive solutions for a semipositone boundary value problem on the half-line," *Computers & Mathematics with Applications*, vol. 58, no. 8, pp. 1672–1686, 2009.
[10] M. Meehan and D. O'Regan, "Existence theory for equations on the half-line," *Nonlinear Analysis A*, vol. 35, pp. 355–387, 1999.
[11] X. Zhang, "Successive iteration and positive solutions for a second-order multi-point boundary value problem on a half-line," *Computers & Mathematics with Applications*, vol. 58, no. 3, pp. 528–535, 2009.
[12] Y. Tian and W. Ge, "Positive solutions for multi-point boundary value problem on the half-line," *Journal of Mathematical Analysis and Applications*, vol. 325, no. 2, pp. 1339–1349, 2007.
[13] C.-G. Kim, "Existence and iteration of positive solutions for multi-point boundary value problems on a half-line," *Computers & Mathematics with Applications*, vol. 61, no. 7, pp. 1898–1905, 2011.
[14] X. Zhang, L. Liu, and Y. Wu, "Existence of positive solutions for second-order semipositone differential equations on the half-line," *Applied Mathematics and Computation*, vol. 185, no. 1, pp. 628–635, 2007.
[15] J. Zhao, Z. Liu, and L. Liu, "The existence of solutions of infinite boundary value problems for first-order impulsive differential systems in Banach spaces," *Journal of Computational and Applied Mathematics*, vol. 222, no. 2, pp. 524–530, 2008.
[16] M. Zima, "On positive solutions of boundary value problems on the half-line," *Journal of Mathematical Analysis and Applications*, vol. 259, no. 1, pp. 127–136, 2001.
[17] B. Liu, J. Li, and L. Liu, "Existence and uniqueness for an m-point boundary value problem at resonance on infinite intervals," *Computers & Mathematics with Applications*. In press.

[18] F. V. Atkinson and L. A. Peletier, "Ground states of $-\Delta u = f(u)$ and the Emden-Fowler equation," *Archive for Rational Mechanics and Analysis*, vol. 93, no. 2, pp. 103–127, 1986.

[19] L. Erbe and K. Schmitt, "On radial solutions of some semilinear elliptic equations," *Differential and Integral Equations*, vol. 1, no. 1, pp. 71–78, 1988.

[20] N. Kawano, E. Yanagida, and S. Yotsutani, "Structure theorems for positive radial solutions to $\Delta u + K(|x|)u^p = 0$ in R^n," *Funkcialaj Ekvacioj*, vol. 36, no. 3, pp. 557–579, 1993.

[21] J. M. do Ó, S. Lorca, and P. Ubilla, "Local superlinearity for elliptic systems involving parameters," *Journal of Differential Equations*, vol. 211, no. 1, pp. 1–19, 2005.

[22] J. M. do Ó, S. Lorca, J. Sánchez, and P. Ubilla, "Positive solutions for a class of multiparameter ordinary elliptic systems," *Journal of Mathematical Analysis and Applications*, vol. 332, no. 2, pp. 1249–1266, 2007.

[23] C. Corduneanu, *Integral Equations and Stability of Feedback Systems*, Academic Press, New York, NY, USA, 1973.

[24] H. Amann, "Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces," *SIAM Review*, vol. 18, no. 4, pp. 620–709, 1976.

[25] D. J. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, vol. 5, Academic Press, New York, NY, USA, 1988.

[26] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, Germany, 1985.