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Research Article

Iterative Algorithms with Perturbations for Solving the Systems of Generalized Equilibrium Problems and the Fixed Point Problems of Two Quasi-Nonexpansive Mappings

Rabian Wangkeeree and Uraiwan Boonkong

Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand

Correspondence should be addressed to Rabian Wangkeeree, rabianw@nu.ac.th

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We introduce new iterative algorithms with perturbations for finding a common element of the set of solutions of the system of generalized equilibrium problems and the set of common fixed points of two quasi-nonexpansive mappings in a Hilbert space. Under suitable conditions, strong convergence theorems are obtained. Furthermore, we also consider the iterative algorithms with perturbations for finding a common element of the solution set of the systems of generalized equilibrium problems and the common fixed point set of the super hybrid mappings in Hilbert spaces.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot,\cdot \rangle$ and norm $\|\cdot\|$ and C a nonempty closed convex subset of H and let T be a mapping of C into H. Then, $T : C \to H$ is said to be nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to H$ is said to be quasi-nonexpansive if $\|Tx - y\| \le \|x - y\|$ for all $x \in C$ and $y \in F(T) := \{x \in C : Tx = x\}$. It is well known that the set $F(T)$ of fixed points of a quasi-nonexpansive mapping T is closed and convex; see Itoh and Takahashi [1]. A mapping $T : C \to H$ is called nonspreading [2] if

$$2\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|Ty - x\|^2, \tag{1.1}$$

for all $x, y \in C$. We remark that nonlinear every nonspreading mappings are quasi-nonexpansive mappings if the set of fixed points is nonempty.

Recall that a mapping $\Psi : C \to H$ is said to be μ-inverse strongly monotone if there exists a positive real number μ such that

$$\langle \Psi x - \Psi y,\ x - y \rangle \geq \mu \|\Psi x - \Psi y\|^2, \quad \forall x, y \in C. \tag{1.2}$$

If Ψ is a μ-inverse strongly monotone mapping of C into H, then it is obvious that Ψ is $1/\mu$-Lipschitz continuous.

Let $G : C \times C \to \mathbb{R}$ be a bifuction and $\Psi : C \to H$ be μ-inverse strongly monotone mapping. The generalized equilibrium problem (for short, GEP) for F and Ψ is to find $z \in C$ such that

$$G(z, y) + \langle \Psi z, y - z \rangle \geq 0, \quad \forall y \in C. \tag{1.3}$$

The problem (1.3) was studied by Moudafi [3]. The set of solutions for problem (1.3) is denoted by $\mathrm{GEP}(F, \Psi)$, that is,

$$\mathrm{GEP}(F, \Psi) = \{z \in C : G(z, y) + \langle \Psi z, y - z \rangle \geq 0,\ \forall y \in C\}. \tag{1.4}$$

If $\Psi \equiv 0$ in (1.3), then GEP reduces to the classical equilibrium problem and $\mathrm{GEP}(G, 0)$ is denoted by $\mathrm{EP}(G)$, that is,

$$\mathrm{EP}(G) = \{z \in C : G(z, y) \geq 0,\ \forall y \in C\}. \tag{1.5}$$

If $G \equiv 0$ in (1.3), then GEP reduces to the classical variational inequality and $\mathrm{GEP}(0, \Psi)$ is denoted by $\mathrm{VI}(\Psi, C)$, that is,

$$\mathrm{VI}(\Psi, C) = \{z \in C : \langle \Psi z, y - z \rangle \geq 0,\ \forall y \in C\}. \tag{1.6}$$

The problem (1.3) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, Min–Max problems, the Nash equilibrium problems in noncooperative games, and others; see, for example, Blum and Oettli [4] and Moudafi [3].

In 2005, Combettes and Hirstoaga [5] introduced an iterative algorithm of finding the best approximation to the initial data and proved a strong convergence theorem. In 2007, by using the viscosity approximation method, S. Takahashi and W. Takahashi [6] introduced another iterative scheme for finding a common element of the set of solutions of the equilibrium problem and the set of fixed points of a nonexpansive mapping. Subsequently, algorithms constructed for solving the equilibrium problems and fixed point problems have further developed by some authors. In particular, Ceng and Yao [7] introduced an iterative scheme for finding a common element of the set of solutions of the mixed equilibrium problem and the set of common fixed points of finitely many nonexpansive mappings. Mainge and Moudafi [8] introduced an iterative algorithm for equilibrium problems and fixed point problems. Wangkeeree [9] introduced a new iterative scheme for finding the common element of the set of common fixed points of nonexpansive mappings, the set of solutions of an equilibrium problem, and the set of solutions of the variational inequality. Wangkeeree and Kamraksa [10] introduced an iterative algorithm for finding a common element of the set of solutions of a mixed equilibrium problem, the set of fixed points of an infinite family of nonexpansive mappings and the set of solutions of a general system of variational inequalities for a cocoercive mapping in a real Hilbert space. Their results extend and improve many results in the literature.

In 1967, Wittmann [11] (see also [12]) proved the strong convergence theorem of Halpern's type [13] $\{x_n\}$ defined by, for any $x_1 = x \in C$,

$$x_{n+1} = \alpha_n x + (1-\alpha_n)Tx_n, \quad \forall n \in \mathbb{N}, \tag{1.7}$$

where $\{\alpha_n\} \subset (0,1)$ satisfies $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, and $\sum_{n=1}^{\infty}|\alpha_n - \alpha_{n+1}| < \infty$. In [14], Kurokawa and Takahashi also studied the following Halpern's type for nonspreading mappings in a Hilbert space; see also Hojo and Takahashi [15]. Let T be a nonspreading mapping of C into itself. Let $u \in C$ and define two sequences $\{x_n\}$ and $\{z_n\}$ in C as follows: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n u + (1-\alpha_n)z_n, \quad \text{where } z_n = \frac{1}{n}\sum_{k=0}^{n-1}T^k x_n \tag{1.8}$$

for all $n = 1, 2, \ldots$, where $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$. If $F(T)$ is nonempty, they proved that $\{x_n\}$ and $\{z_n\}$ converge strongly to $P_{F(T)}u$, where $P_{F(T)}$ is the metric projection of H onto $F(T)$. Recently, Yao and Shahzad [16] gave the following iteration process for nonexpansive mappings with perturbation: $x_1 \in C$ and

$$x_{n+1} = (1-\beta_n)x_n + \beta_n P_C(\alpha_n u_n + (1-\alpha_n)Tx_n), \quad \forall n \in \mathbb{N}, \tag{1.9}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$, and the sequence $\{u_n\} \subseteq H$ is a small perturbation for the n-step iteration satisfying $\|u_n\| \to 0$ as $n \to \infty$. In fact, there are perturbations always occurring in the iterative processes because the manipulations are inaccurate.

On the other hand, very recently, Chuang et al. [17] considered the following iteration process for finding a common element of the set of solutions of the equilibrium problem and the set of common fixed points for a quasi-nonexpansive mapping $T : C \to H$ with perturbation

$$\begin{gathered} q_1 \in H, \\ x_n \in C, \quad \text{such that } G(x_n, y) + \frac{1}{r_n}\langle y - x_n,\ x_n - q_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ q_{n+1} = \alpha_n u_n + (1-\alpha_n)y_n, \quad \forall n \in \mathbb{N}, \end{gathered} \tag{1.10}$$

where C is a nonempty closed convex subset of H, $G : C\times C \to \mathbb{R}$ is a function, $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$, and $\{u_n\} \subset H$ is a convergent sequence and $\{r_n\} \subset [a,\infty)$ for some $a > 0$. They obtained a strong convergence theorem for such iterations.

In this paper, inspired and motivated by Yao and Shahzad [16], S. Takahashi and W. Takahashi [18] and Chuang et al. [17], we introduce a new iterative algorithms with perturbations for finding a common element of the set of solutions of the system of generalized equilibrium problems and the set of common fixed points of two quasi-nonexpansive mappings in a Hilbert space. Under suitable conditions, strong convergence theorems are obtained. Furthermore, we also consider the iterative algorithms with perturbations for finding a common element of the solution set of the system of generalized equilibrium problems and the common fixed point set of the super hybrid mappings in a Hilbert space.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. We denote the strongly convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. In a Hilbert space, it is known that

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2, \tag{2.1}$$

for all $x, y \in H$ and $\lambda \in \mathbb{R}$; see [19]. Furthermore, we have that for any $x, y, u, v \in H$

$$2\langle x-y,\ u-v\rangle = \|x-v\|^2 + \|y-u\|^2 - \|x-u\|^2 - \|y-v\|^2. \tag{2.2}$$

Let C be a nonempty closed convex subset of H and $x \in H$. We know that there exists a unique nearest point $z \in C$ such that $\|x-z\| = \inf_{y\in C}\|x-y\|$. We denote such a correspondence by $z = P_C x$. The mapping P_C is called the metric projection of H onto C. It is known that P_C is nonexpansive and

$$\langle x - P_C x,\ P_C x - u\rangle \geq 0; \tag{2.3}$$

for all $x \in H$ and $u \in C$; see [19, 20] for more details.

Let C be a nonempty, closed and convex subset of H and let $G : C \times C \to \mathbb{R}$ be a bifunction. For solving the generalized equilibrium problem, let us assume that the bifunction $G : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $G(x,x) = 0$ for a ll $x \in C$;

(A2) G is monotone, that is, $G(x,y) + G(y,x) \leq 0$ for any $x, y \in C$;

(A3) for each $x, y, z \in C$

$$\lim_{t\downarrow 0} G(tz + (1-t)x, y) \leq G(x,y); \tag{2.4}$$

(A4) for each $x \in C$, $G(x,\cdot)$ is convex and lower semicontinuous.

We know the following lemma which appears implicitly in Blum and Oettli [4].

Lemma 2.1 (see [4]). *Let C be a nonempty closed convex subset of H and let G be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying* (A1)–(A4). *Let $r > 0$ and $x \in H$. Then, there exists a unique $z \in C$ such that*

$$G(z,y) + \frac{1}{r}\langle y - z,\ z - x\rangle \geq 0, \quad \forall y \in C. \tag{2.5}$$

The following lemma was also given in Combettes and Hirstoaga [5].

Lemma 2.2 (see [5]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $G : C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions* (A1)–(A4). *For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : G(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C \right\} \tag{2.6}$$

for all $x \in H$. Then the following hold:

(i) *T_r is single-valued;*

(ii) *T_r is firmly nonexpansive, that is, for any $x, y \in C$,*

$$\left\|T_r x - T_r y\right\|^2 \leq \langle T_r x - T_r y, \ x - y\rangle; \tag{2.7}$$

(iii) *(EP) is a closed convex subset of C;*

(iv) *$F(T_r) = EP(G)$.*

Remark 2.3. For any $x \in H$ and $r > 0$, by Lemma 2.2 (i), there exists $u \in H$ such that

$$G(u, y) + \frac{1}{r}\langle y - u, u - x\rangle \geq 0, \quad \forall y \in H. \tag{2.8}$$

Replacing x with $x - r\Psi x \in H$ in (2.8), we have

$$G(u, y) + \langle \Psi x, y - u\rangle + \frac{1}{r}\langle y - u, u - x\rangle \geq 0, \quad \forall y \in H, \tag{2.9}$$

where $\Psi : H \to H$ is an inverse strongly monotone mapping.

Lemma 2.4 (see [21]). *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Define the sequence $\{\tau(n)\}_{n \geq n_0}$ of integers as follows:*

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}, \tag{2.10}$$

where $n_0 \in \mathbb{N}$ such that $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then, the following hold:

(i) *$\tau(1) \leq \tau(2) \leq \cdots$ and $\tau(n) \to \infty$;*

(ii) *$\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ and $\Gamma_n \leq \Gamma_{\tau(n)+1}$, $\forall n \in \mathbb{N}$.*

Lemma 2.5 (see [22]). *Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ a sequence of real numbers in $[0, 1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{u_n\}$ a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} u_n < \infty$, $\{t_n\}$ a sequence of real numbers with $\limsup t_n \leq 0$. Suppose that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n t_n + u_n, \quad \forall n \in \mathbb{N}. \tag{2.11}$$

Then $\lim_{n \to \infty} a_n = 0$.

3. Main Results

Let C be a nonempty closed convex subset of a Hilbert space H. For each $i = 1, 2, \ldots, k$, let $G_i : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4) and Ψ_i a μ_i-inverse strongly monotone mapping. For each $j = 1, 2$, let $T_j : C \to H$ be two mappings. Let $\{x_n\}$ be a sequence generated in the following manner:

$$\begin{gathered} x_1 \in H, \\ G_1(u_{n,1}, y) + \langle \Psi_1 x_n,\ y - u_{n,1} \rangle + \frac{1}{r_n} \langle y - u_{n,1},\ u_{n,1} - x_n \rangle \geq 0, \quad \forall y \in C, \\ G_2(u_{n,2}, y) + \langle \Psi_2 x_n,\ y - u_{n,2} \rangle + \frac{1}{r_n} \langle y - u_{n,2},\ u_{n,2} - x_n \rangle \geq 0, \quad \forall y \in C, \\ \vdots \\ G_k(u_{n,k}, y) + \langle \Psi_k x_n,\ y - u_{n,k} \rangle + \frac{1}{r_n} \langle y - u_{n,k},\ u_{n,k} - x_n \rangle \geq 0, \quad \forall y \in C, \\ \omega_n = \frac{1}{k} \sum_{i=1}^{k} u_{n,i}, \\ y_n = \gamma_n \omega_n + (1 - \gamma_n) T_1 \omega_n, \\ z_n = \beta_n y_n + (1 - \beta_n) T_2 \omega_n, \\ x_{n+1} = \alpha_n u_n + (1 - \alpha_n) z_n, \quad \forall n \in \mathbb{N}, \end{gathered} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $(0, 1)$ and $\{u_n\} \subset H$ is a sequence and $\{r_n\} \subset [a, 2\mu_i)$ for some $a > 0$ and for all $i \in \{1, 2, \ldots, k\}$. Under certain appropriate assumptions imposed on the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, the strong convergence theorem of $\{x_n\}$ defined by (3.1) is studied in the following theorem.

Theorem 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H. For each $i = 1, 2, \ldots, k$, let $G_i : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)–(A4) *and Ψ_i a μ_i-inverse strongly monotone mapping. For each $j = 1, 2$, let $T_j : C \to H$ be two quasi-nonexpansive mappings such that $I - T_j$ are demiclosed at zero with $\Omega := F(T_1) \cap F(T_2) \cap (\cap_{i=1}^{k} \mathrm{GEP}(G_i, \Psi_i)) \neq \emptyset$. Let the sequences $\{x_n\}, \{y_n\}$, and $\{z_n\}$ be defined by* (3.1)*, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{u_n\}$ satisfy the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $\liminf_{n\to\infty} \beta_n (1 - \beta_n) > 0$;

(C3) $\liminf_{n\to\infty} \gamma_n (1 - \gamma_n) > 0$;

(C4) $\lim_{n\to\infty} u_n = u$ *for some* $u \in H$.

Then $\{x_n\}$ converges strongly to x^*, where $x^* = P_\Omega u$.

Proof. We first have that for all $i = 1, 2, \dots, k$, $I - r_n\Psi_i$ is a nonexpansive mapping. Indeed, for all $x, y \in C$, we obtain

$$\begin{aligned}\left\|(I - r_n\Psi_i)x - (I - r_n\Psi_i)y\right\|^2 &= \left\|(x-y) - r_n(\Psi_i x - \Psi_i y)\right\|^2 \\ &= \left\|x-y\right\|^2 - 2r_n\langle \Psi_i x - \Psi_i y, x - y\rangle + r_n^2\left\|\Psi_i x - \Psi_i y\right\|^2 \\ &\le \left\|x-y\right\|^2 - r_n(2\mu_i - r_n)\left\|\Psi_i x - \Psi_i y\right\|^2 \\ &\le \left\|x-y\right\|^2.\end{aligned} \tag{3.2}$$

Thus $I - r_n\Psi_i$ is nonexpansive for each $i \in \{1, 2, \dots, k\}$. Now, let $w \in \Omega$ be arbitrary. By (C4), $\{u_n\}$ is a bounded sequence, there exists $M \le 0$ such that

$$\sup_{n\in\mathbb{N}}\|u_n - w\| \le M. \tag{3.3}$$

For each $i = 1, 2, \dots, k$ and $n \in \mathbb{N}$, we have from $u_{n,i} = T_{r_{n,i}}(x_n - r_n\Psi_i x_n)$ that

$$\begin{aligned}\|u_{n,i} - w\| &= \left\|T_{r_{n,i}}(x_n - r_n\Psi_i x_n) - T_{r_{n,i}}(w - r_n\Psi_i w)\right\| \\ &\le \|(x_n - r_n\Psi_i x_n) - (w - r_n\Psi_i w)\| \\ &\le \|x_n - w\|,\end{aligned} \tag{3.4}$$

which gives also that

$$\|\omega_n - w\| \le \frac{1}{k}\sum_{i=1}^{k}\|u_{n,i} - w\| \le \|x_n - w\| \quad \forall w \in \Omega. \tag{3.5}$$

Since T_1 is quasi-nonexpansive, we have

$$\begin{aligned}\left\|y_n - w\right\| &= \left\|\gamma_n\omega_n + (1-\gamma_n)T_1\omega_n - w\right\| \\ &= \left\|\gamma_n(\omega_n - w) + (1-\gamma_n)(T_1\omega_n - w)\right\| \\ &\le \gamma_n\|\omega_n - w\| + (1-\gamma_n)\|T_1\omega_n - w\| \\ &\le \|\omega_n - w\|.\end{aligned} \tag{3.6}$$

So, we have from (3.5) and (3.6) and the quasi-nonexpansiveness of T_2 that

$$\begin{aligned}\|x_{n+1} - w\| &= \|\alpha_n(u_n - w) + (1-\alpha_n)(z_n - w)\| \\ &\le \alpha_n\|u_n - w\| + (1-\alpha_n)\|z_n - w\| \\ &\le \alpha_n\|u_n - w\| + (1-\alpha_n)\left\{\beta_n\left\|y_n - w\right\| + (1-\beta_n)\|T_2\omega_n - w\|\right\} \\ &\le \alpha_n\|u_n - w\| + (1-\alpha_n)\left\{\beta_n\|\omega_n - w\| + (1-\beta_n)\|\omega_n - w\|\right\} \\ &\le \alpha_n\|u_n - w\| + (1-\alpha_n)\|\omega_n - w\| \\ &\le \alpha_n\|u_n - w\| + (1-\alpha_n)\|x_n - w\| \\ &\le \max\{M, \|x_n - w\|\}.\end{aligned} \tag{3.7}$$

By Induction, we have that

$$\|x_n - w\| \leq \max\{\|x_1 - w\|, M\}, \quad \forall n \in \mathbb{N}. \tag{3.8}$$

Thus we obtain that $\{\|x_n - w\|\}$ is bounded, so also $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{\omega_n\}$, $\{T_1\omega_n\}$, and $\{T_2\omega_n\}$ are bounded. Since Ω is closed and convex, we can take $x^* = P_\Omega u$. It follows that

$$\begin{aligned}
\left\|y_n - x^*\right\|^2 &= \left\|\gamma_n(\omega_n - x^*) + (1-\gamma_n)(T_1\omega_n - x^*)\right\|^2 \\
&= \gamma_n\|\omega_n - x^*\|^2 + (1-\gamma_n)\|T_1\omega_n - x^*\|^2 - \gamma_n(1-\gamma_n)\|\omega_n - T_1\omega_n\|^2 \\
&\leq \gamma_n\|\omega_n - x^*\|^2 + (1-\gamma_n)\|\omega_n - x^*\|^2 - \gamma_n(1-\gamma_n)\|\omega_n - T_1\omega_n\|^2 \\
&= \|\omega_n - x^*\|^2 - \gamma_n(1-\gamma_n)\|\omega_n - T_1\omega_n\|^2 \\
&\leq \|\omega_n - x^*\|^2.
\end{aligned} \tag{3.9}$$

From (3.9), we have

$$\begin{aligned}
\|z_n - x^*\|^2 &= \left\|\beta_n(y_n - x^*) + (1-\beta_n)(T_2\omega_n - x^*)\right\|^2 \\
&= \beta_n\left\|y_n - x^*\right\|^2 + (1-\beta_n)\|T_2\omega_n - x^*\|^2 - \beta_n(1-\beta_n)\left\|y_n - T_2\omega_n\right\|^2 \\
&\leq \beta_n\|\omega_n - x^*\|^2 + (1-\beta_n)\|\omega_n - x^*\|^2 - \beta_n(1-\beta_n)\left\|y_n - T_2\omega_n\right\|^2 \\
&= \|\omega_n - x^*\|^2 - \beta_n(1-\beta_n)\left\|y_n - T_2\omega_n\right\|^2 \\
&\leq \|\omega_n - x^*\|^2.
\end{aligned} \tag{3.10}$$

Hence we have from (3.5), (3.9), and (3.10) that

$$\begin{aligned}
\|\omega_{n+1} - x^*\|^2 &\leq \|x_{n+1} - x^*\|^2 \\
&= \|\alpha_n(u_n - x^*) + (1-\alpha_n)(z_n - x^*)\|^2 \\
&= \alpha_n\|u_n - x^*\|^2 + (1-\alpha_n)\|z_n - x^*\|^2 - \alpha_n(1-\alpha_n)\|u_n - z_n\|^2 \\
&\leq \alpha_n\|u_n - x^*\|^2 + (1-\alpha_n)\|z_n - x^*\|^2 \\
&= \alpha_n\|u_n - x^*\|^2 + (1-\alpha_n)\Big\{\beta_n\left\|y_n - x^*\right\|^2 + (1-\beta_n)\|T_2\omega_n - x^*\|^2 \\
&\qquad - \beta_n(1-\beta_n)\left\|y_n - T_2\omega_n\right\|^2\Big\} \\
&\leq \alpha_n\|u_n - x^*\|^2 + \beta_n\Big\{\gamma_n\|\omega_n - x^*\|^2 + (1-\gamma_n)\|T_1\omega_n - x^*\|^2 \\
&\qquad - \gamma_n(1-\gamma_n)\|\omega_n - T_1\omega_n\|^2\Big\} + (1-\beta_n)\|T_2\omega_n - x^*\|^2 \\
&\quad - \beta_n(1-\beta_n)\left\|y_n - T_2\omega_n\right\|^2
\end{aligned}$$

$$
\begin{aligned}
&\le \alpha_n\|u_n - x^*\|^2 + \beta_n\Big(\gamma_n\|\omega_n - x^*\|^2 + (1-\gamma_n)\|\omega_n - x^*\|^2 \\
&\qquad\qquad - \gamma_n(1-\gamma_n)\|\omega_n - T_1\omega_n\|^2\Big) + (1-\beta_n)\|\omega_n - x^*\|^2 \\
&\quad - \beta_n(1-\beta_n)\left\|y_n - T_2\omega_n\right\|^2 \\
&= \alpha_n\|u_n - x^*\|^2 + \|\omega_n - x^*\|^2 - \gamma_n(1-\gamma_n)\|\omega_n - T_1\omega_n\|^2 \\
&\quad - \beta_n(1-\beta_n)\left\|y_n - T_2\omega_n\right\|^2.
\end{aligned}
\tag{3.11}
$$

We also have that

$$
\gamma_n(1-\gamma_n)\|\omega_n - T_1\omega_n\|^2 \le \alpha_n\|u_n - x^*\|^2 + \|\omega_n - x^*\|^2 - \|\omega_{n+1} - x^*\|^2, \tag{3.12}
$$

$$
\beta_n(1-\beta_n)\left\|y_n - T_2\omega_n\right\|^2 \le \alpha_n\|u_n - x^*\|^2 + \|\omega_n - x^*\|^2 - \|\omega_{n+1} - x^*\|^2. \tag{3.13}
$$

Furthemore, we have from $y_n = \gamma_n\omega_n + (1-\gamma_n)T_1\omega_n$ that

$$
\begin{aligned}
\|\omega_n - T_2\omega_n\| &\le \left\|\omega_n - y_n\right\| + \left\|y_n - T_2\omega_n\right\| \\
&= \left\|\omega_n - \gamma_n\omega_n - (1-\gamma_n)T_1\omega_n\right\| + \left\|y_n - T_2x_n\right\| \\
&= (1-\gamma_n)\|\omega_n - T_1\omega_n\| + \left\|y_n - T_2\omega_n\right\|.
\end{aligned}
\tag{3.14}
$$

On the other hand, since $x_{n+1} - x^* = \alpha_n(u_n - x^*) + (1-\alpha_n)(z_n - x^*)$, we have

$$
\begin{aligned}
\|\omega_{n+1} - x^*\|^2 &\le \|x_{n+1} - x^*\|^2 \\
&\le (1-\alpha_n)\|z_n - x^*\|^2 + 2\alpha_n\langle u_n - x^*, x_{n+1} - x^*\rangle \\
&\le (1-\alpha_n)\|\omega_n - x^*\|^2 + 2\alpha_n\langle u_n - x^*, x_{n+1} - x^*\rangle \\
&= (1-\alpha_n)\|\omega_n - x^*\|^2 + 2\alpha_n\langle u_n - u, x_{n+1} - x^*\rangle \\
&\quad + 2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle \\
&= (1-\alpha_n)\|\omega_n - x^*\|^2 + 2\alpha_n\langle u_n - u, x_{n+1} - x^*\rangle \\
&\quad + 2\alpha_n\langle u - x^*, x_{n+1} - \omega_n\rangle + 2\alpha_n\langle u - x^*, \omega_n - x^*\rangle.
\end{aligned}
\tag{3.15}
$$

We also have that

$$
\begin{aligned}
\|x_{n+1} - \omega_n\| &\le \left\|x_{n+1} - y_n\right\| + \left\|y_n - \omega_n\right\| \\
&= \left\|\alpha_n(u_n - y_n) + (1-\alpha_n)(z_n - y_n)\right\| + \left\|(1-\gamma_n)(\omega_n - T_1\omega_n)\right\| \\
&\le \alpha_n\left\|u_n - y_n\right\| + (1-\alpha_n)\left\|\beta_n y_n + (1-\beta_n)T_2\omega_n - y_n\right\| \\
&\quad + (1-\gamma_n)\|\omega_n - T_1\omega_n\| \\
&= \alpha_n\left\|u_n - y_n\right\| + (1-\alpha_n)(1-\beta_n)\left\|y_n - T_2\omega_n\right\| \\
&\quad + (1-\gamma_n)\|\omega_n - T_1\omega_n\|.
\end{aligned}
\tag{3.16}
$$

Moreover, for any $i \in \{1, 2, \ldots, k\}$, we have from $u_{n,i} = T_{r_{n,i}}(x_n - r_n \Psi_i x_n)$ that

$$\begin{aligned}
\|u_{n,i} - x^*\|^2 &\leq \|(x_n - x^*) - r_n(\Psi_i x_n - \Psi_i x^*)\|^2 \\
&= \|x_n - x^*\|^2 - 2r_n\langle x_n - x^*, \Psi_i x_n - \Psi_i x^*\rangle + r_n^2\|\Psi_i x_n - \Psi_i x^*\|^2 \\
&\leq \|x_n - x^*\|^2 - r_n(2\mu_i - r_n)\|\Psi_i x_n - \Psi_i x^*\|^2.
\end{aligned} \tag{3.17}$$

It follows that

$$\begin{aligned}
\|\omega_n - x^*\|^2 &= \left\| \sum_{i=1}^{k} \frac{1}{k}(u_{n,i} - x^*) \right\|^2 \\
&\leq \frac{1}{k}\sum_{i=1}^{k}\|u_{n,i} - x^*\|^2 \\
&\leq \|x_n - x^*\|^2 - \frac{1}{k}\sum_{i=1}^{k} r_n(2\mu_i - r_n)\|\Psi_i x_n - \Psi_i x^*\|^2.
\end{aligned} \tag{3.18}$$

This implies that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n(u_n - x^*) + (1 - \alpha_n)(z_n - x^*)\|^2 \\
&\leq \alpha_n\|u_n - x^*\|^2 + (1 - \alpha_n)\|\omega_n - x^*\|^2 \\
&\leq \alpha_n\|u_n - x^*\|^2 + (1 - \alpha_n)\|x_n - x^*\|^2 \\
&\quad - (1 - \alpha_n)\frac{1}{k}\sum_{i=1}^{k} r_n(2\mu_i - r_n)\|\Psi_i x_n - \Psi_i x^*\|^2,
\end{aligned} \tag{3.19}$$

and hence

$$(1 - \alpha_n)\frac{1}{k}\sum_{i=1}^{k} r_n(2\mu_i - r_n)\|\Psi_i x_n - \Psi_i x^*\|^2 \leq \alpha_n\|u_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2. \tag{3.20}$$

Furthermore, we have from Lemma 2.2 that for any $i \in 1, 2, \ldots, k$, we have

$$\begin{aligned}
\|u_{n,i} - x^*\|^2 &\leq \langle (x_n - r_n\Psi_i x_n) - (x^* - r_n\Psi_i x^*), u_{n,i} - x^*\rangle \\
&= \frac{1}{2}\Big\{ \|(x_n - r_n\Psi_i x_n) - (x^* - r_n\Psi_i x^*)\|^2 + \|u_{n,i} - x^*\|^2 \\
&\quad - \|(x_n - r_n\Psi_i x_n) - (x^* - r_n\Psi_i x^*) - (u_{n,i} - x^*)\|^2 \Big\} \\
&\leq \frac{1}{2}\Big\{ \|x_n - x^*\|^2 + \|u_{n,i} - x^*\|^2 - \|(x_n - u_{n,i}) - r_n(\Psi_i x_n - \Psi_i x^*)\|^2 \Big\} \\
&= \frac{1}{2}\Big\{ \|x_n - x^*\|^2 + \|u_{n,i} - x^*\|^2 - \|x_n - u_{n,i}\|^2 - r_n^2\|\Psi_i x_n - \Psi_i x^*\|^2 \\
&\quad + 2r_n\langle x_n - u_{n,i},\ \Psi_i x_n - \Psi_i x^*\rangle \Big\}.
\end{aligned} \tag{3.21}$$

This implies that

$$\|u_{n,i} - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_n - u_{n,i}\|^2 + 2r_n\|x_n - u_{n,i}\|\|\Psi_i x_n - \Psi_i x^*\|. \tag{3.22}$$

Then we have from (3.22) that

$$\begin{aligned}\|\omega_n - x^*\|^2 &\leq \frac{1}{k}\sum_{i=1}^{k}\|u_{n,i} - x^*\|^2 \\ &\leq \|x_n - x^*\|^2 - \frac{1}{k}\sum_{i=1}^{k}\|u_{n,i} - x_n\|^2 \\ &\quad + \frac{1}{k}\sum_{i=1}^{k}2r_n\|x_n - u_{n,i}\|\|\Psi_i x_n - \Psi_i x^*\|.\end{aligned} \tag{3.23}$$

Hence we have from (3.23) that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \alpha_n\|u_n - x^*\|^2 + (1-\alpha_n)\|\omega_n - x^*\|^2 \\ &\leq \alpha_n\|u_n - x^*\|^2 + (1-\alpha_n)\left(\|x_n - x^*\|^2 - \frac{1}{k}\sum_{i=1}^{k}\|u_{n,i} - x_n\|^2\right) \\ &\quad + (1-\alpha_n)\left(\frac{1}{k}\sum_{i=1}^{k}2r_n\|x_n - u_{n,i}\|\|\Psi_i x_n - \Psi_i x^*\|\right).\end{aligned} \tag{3.24}$$

It follows that

$$\begin{aligned}(1-\alpha_n)\frac{1}{k}\sum_{i=1}^{k}\|u_{n,i} - x_n\|^2 &\leq \alpha_n\|u_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\ &\quad + (1-\alpha_n)\left(\frac{1}{k}\sum_{i=1}^{k}2r_n\|x_n - u_{n,i}\|\|\Psi_i x_n - \Psi_i x^*\|\right).\end{aligned} \tag{3.25}$$

Next, we will consider the following two cases.

Case A. Put $\Gamma_n = \|\omega_n - x^*\|^2$ for all $n \in \mathbb{N}$. Suppose that $\Gamma_{n+1} \leq \Gamma_n$ for all $n \in \mathbb{N}$. In this case $\lim_{n\to\infty}\Gamma_n$ exists and then $\lim_{n\to\infty}(\Gamma_{n+1} - \Gamma_n) = 0$. By (C1), (C3), and (3.12), we have

$$\lim_{n\to\infty}\|\omega_n - T_1\omega_n\| = 0. \tag{3.26}$$

Similarly by (C1), (C2), and (3.13), we also have

$$\lim_{n\to\infty}\|y_n - T_2\omega_n\| = 0. \tag{3.27}$$

So, we have from (3.14), (3.26), and (3.27) that

$$\lim_{n\to\infty} \|\omega_n - T_2\omega_n\| = 0. \tag{3.28}$$

Since $\lim_{n\to\infty}\|\omega_n - x^*\|$ exists, we have from (3.11) and (3.26)

$$\lim_{n\to\infty} \|\omega_n - x^*\| = \lim_{n\to\infty} \|x_n - x^*\|. \tag{3.29}$$

We also have from (C1), (3.16), (3.26), and (3.27) that

$$\lim_{n\to\infty} \|x_{n+1} - \omega_n\| = 0. \tag{3.30}$$

Since $\lim_{n\to\infty}\|x_n - x^*\|$ exists we have from (C1) and (3.20) that

$$\lim_{n\to\infty} \|\Psi_i x_n - \Psi_i x^*\| = 0, \quad \forall i = 1, 2, \ldots, k. \tag{3.31}$$

This together with (3.25) and the existence of $\lim_{n\to\infty}\|x_n - x^*\|$ implies that

$$\lim_{n\to\infty} \|u_{n,i} - x_n\| = 0, \quad \forall i = 1, 2, \ldots, k, \tag{3.32}$$

which gives that

$$\|\omega_n - x_n\| \le \frac{1}{k}\sum_{i=1}^{k}\|u_{n,i} - x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.33}$$

So, from (3.30), $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Furthermore, we have from (3.33) that

$$\|\omega_{n+1} - \omega_n\| \le \|\omega_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - \omega_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty; \tag{3.34}$$

that is

$$\lim_{n\to\infty} \|\omega_{n+1} - \omega_n\| = 0. \tag{3.35}$$

Now, since $\{\omega_n\}$ is a bounded sequence, there exists a subsequence $\{\omega_{n_j}\}$ of $\{\omega_n\}$ such that

$$\limsup_{n\to\infty} \langle u - x^*, \omega_n - x^*\rangle = \lim_{j\to\infty} \left\langle u - x^*, \omega_{n_j} - x^*\right\rangle. \tag{3.36}$$

Without loss of generality, we may assume that $\omega_{n_j} \rightharpoonup v$. Since T_1 is demiclosed at zero and by (3.26), we conclude that $v \in F(T_1)$. Similarly, since T_2 is demiclosed at zero and by (3.28), we have $v \in F(T_2)$. Therefore, we get that

$$v \in F(T_1) \cap F(T_2). \tag{3.37}$$

Next, we show that $v \in \cap_{i=1}^{k} \mathrm{GEP}(G_i, \Psi_i)$. For each $i \in \{1, 2, \ldots, k\}$, since $u_{n,i} = T_{r_{n,i}}(x_n - r_n \Psi_i x_n)$, we have

$$G_i(u_{n,i}, y) + \langle \Psi_i x_n,\ y - u_{n,i} \rangle + \frac{1}{r_n}\langle y - u_{n,i},\ u_{n,i} - x_n \rangle \geq 0, \quad \forall y \in C. \tag{3.38}$$

From (A2), we also have

$$\langle \Psi_i x_n, y - u_{n,i} \rangle + \frac{1}{r_n}\langle y - u_{n,i}, u_{n,i} - x_n \rangle \geq G_i(y, u_{n,i}). \tag{3.39}$$

Replacing n by n_j, we have

$$\left\langle \Psi_i x_{n_j}, y - u_{n_j,i} \right\rangle + \left\langle y - u_{n_j,i}, \frac{u_{n_j,i} - x_{n_j}}{r_{n_j}} \right\rangle \geq G_i\left(y, u_{n_j,i}\right). \tag{3.40}$$

Put $y_t = ty + (1-t)v$ for all $t \in (0,1]$ and $y \in C$. Since $v \in C$, then $y_t \in C$ and

$$\begin{aligned}
\left\langle y_t - u_{n_j,i}, \Psi_i y_t \right\rangle &\geq \left\langle y_t - u_{n_j,i}, \Psi_i y_t \right\rangle - \left\langle y_t - u_{n_j,i}, \Psi_i x_{n_j} \right\rangle \\
&\quad - \left\langle y_t - u_{n_j,i}, \frac{u_{n_j,i} - x_{n_j}}{r_{n_j}} \right\rangle + G_i\left(y_t, u_{n_j,i}\right) \\
&= \left\langle y_t - u_{n_j,i}, \Psi_i y_t - \Psi_i u_{n_j,i} \right\rangle + \left\langle y_t - u_{n_j,i}, \Psi_i u_{n_j,i} - \Psi_i x_{n_j} \right\rangle \\
&\quad - \left\langle y_t - u_{n_j,i}, \frac{u_{n_j,i} - x_{n_j}}{r_{n_j}} \right\rangle + G_i\left(y_t, u_{n_j,i}\right).
\end{aligned} \tag{3.41}$$

Since $\|u_{n_j,i} - x_{n_j}\| \to 0$ as $j \to \infty$, we obtain that $\|\Psi_i u_{n_j,i} - \Psi_i x_{n_j}\| \to 0$ as $j \to \infty$. Furthermore, by the monotonicity of Ψ_i, we obtain that

$$\left\langle y_t - u_{n_j,i}, \Psi_i y_t - \Psi_i u_{n_j,i} \right\rangle \geq 0. \tag{3.42}$$

Taking $j \to \infty$ in (3.41), we have from (A4) that

$$\langle y_t - v, \Psi_i y_t \rangle \geq G_i(y_t, v). \tag{3.43}$$

Now, from (A1), (A4), and (3.43), we also have

$$\begin{aligned}
0 = G_i(y_t, y_t) &\leq t G_i(y_t, y) + (1-t) G_i(y_t, v) \\
&\leq t G_i(y_t, y) + (1-t)\langle y_t - v, \Psi_i y_t \rangle \\
&= t G_i(y_t, y) + (1-t) t \langle y - v, \Psi_i y_t \rangle,
\end{aligned} \tag{3.44}$$

which yields that

$$G_i(y_t, y) + (1-t)\langle y - v, \Psi_i y_t\rangle \geq 0. \tag{3.45}$$

Taking $t \to 0$, we have, for each $y \in C$

$$G_i(v, y) + \langle y - v,\ \Psi_i v\rangle \geq 0, \quad \forall i \in \{1, 2, \dots, k\}. \tag{3.46}$$

This shows $v \in \mathrm{GEP}(G_i, \Psi_i)$, for all $i = 1, 2, \dots, k$. Then, $v \in \cap_{i=1}^{k} \mathrm{GEP}(G_i, \Psi_i)$. Hence we have $v \in F(T_1) \cap F(T_2) \cap (\cap_{i=1}^{k} \mathrm{GEP}(G_i, \Psi_i)) := \Omega$. So, we have from (3.36) that

$$\limsup_{n\to\infty}\langle u - x^*,\ \omega_n - x^*\rangle = \langle u - x^*,\ v - x^*\rangle \leq 0. \tag{3.47}$$

By (C1), (C4), (3.15), (3.30), (3.47), and Lemma 2.5, we obtain that $\lim_{n\to\infty}\|\omega_n - x^*\| = 0$. Hence we have from (3.29) that $\{x_n\}$ converges to x^*, where $x^* = P_\Omega u$.

Case B. Assume that there exists a subsequence $\{\Gamma_{n_i}\}_{i\geq 0}$ of $\{\Gamma_n\}_{n\geq 0}$ such that $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. In this case, it follows from Lemma 2.4 that there exists a subsequence $\{\Gamma_{\tau(n)}\}$ of $\{\Gamma_n\}$ such that $\Gamma_{\tau(n)+1} > \Gamma_{\tau(n)}$, where $\tau : \mathbb{N} \to \mathbb{N}$ is defined by

$$\tau(n) = \max\{k \leq n :\ \Gamma_k < \Gamma_{k+1}\}, \quad \forall n \in \mathbb{N}. \tag{3.48}$$

So, from (3.12), that

$$\left\|\omega_{\tau(n)+1} - x^*\right\|^2 - \left\|\omega_{\tau(n)} - x^*\right\|^2 + \gamma_{\tau(n)}\left(1 - \gamma_{\tau(n)}\right)\left\|\omega_{\tau(n)} - T_1\omega_{\tau(n)}\right\|^2 \leq \alpha_{\tau(n)}\left\|u_{\tau(n)} - x^*\right\|^2. \tag{3.49}$$

Since $\|\omega_{\tau(n)} - x^*\|^2 := \Gamma_{\tau(n)} < \Gamma_{\tau(n)+1} := \|\omega_{\tau(n)+1} - x^*\|^2$, we have

$$\gamma_{\tau(n)}\left(1 - \gamma_{\tau(n)}\right)\left\|\omega_{\tau(n)} - T_1\omega_{\tau(n)}\right\|^2 \leq \alpha_{\tau(n)}\left\|u_{\tau(n)} - x^*\right\|^2. \tag{3.50}$$

By (C1) and (C3), we have

$$\lim_{n\to\infty}\left\|\omega_{\tau(n)} - T_1\omega_{\tau(n)}\right\| = 0. \tag{3.51}$$

By (3.15), we have

$$\left\|\omega_{\tau(n)+1} - x^*\right\|^2 \leq \left(1 - \alpha_{\tau(n)}\right)\left\|\omega_{\tau(n)} - x^*\right\|^2 + 2\alpha_{\tau(n)}\left\langle u_{\tau(n)} - x^*,\ x_{\tau(n)+1} - x^*\right\rangle. \tag{3.52}$$

Now, in view of $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$, we see that

$$\begin{aligned}\left\|\omega_{\tau(n)} - x^*\right\|^2 &\leq 2\left\langle u_{\tau(n)} - x^*,\ x_{\tau(n)+1} - x^*\right\rangle \\ &= 2\left\langle u_{\tau(n)} - u,\ x_{\tau(n)+1} - x^*\right\rangle + 2\left\langle u - x^*,\ x_{\tau(n)+1} - \omega_{\tau(n)}\right\rangle \\ &\quad + 2\left\langle u - x^*,\ \omega_{\tau(n)} - x^*\right\rangle.\end{aligned} \tag{3.53}$$

Furthermore, we also have from (3.13) that

$$\begin{aligned}\beta_{\tau(n)}\left(1 - \beta_{\tau(n)}\right)\left\|y_{\tau(n)} - T_2\omega_{\tau(n)}\right\|^2 &\leq \alpha_{\tau(n)}\left\|u_{\tau(n)} - x^*\right\|^2 + \left\|\omega_{\tau(n)} - x^*\right\|^2 \\ &\quad - \left\|\omega_{\tau(n)+1} - x^*\right\|^2 \\ &\leq \alpha_{\tau(n)}\left\|u_{\tau(n)} - x^*\right\|^2.\end{aligned} \tag{3.54}$$

Applying (C1) and (C2) to the last inequality, we get that

$$\lim_{n\to\infty}\left\|y_{\tau(n)} - T_2\omega_{\tau(n)}\right\| = 0. \tag{3.55}$$

By (C1), (3.16), (3.51), and (3.55), we have

$$\lim_{n\to\infty}\left\|x_{\tau(n)+1} - \omega_{\tau(n)}\right\| = 0. \tag{3.56}$$

By (3.33), we have

$$\lim_{n\to\infty}\left\|\omega_{\tau(n)+1} - x_{\tau(n)+1}\right\| = 0. \tag{3.57}$$

It follows from (3.56) and (3.57) that

$$\lim_{n\to\infty}\left\|\omega_{\tau(n)+1} - \omega_{\tau(n)}\right\| = 0. \tag{3.58}$$

Since $\{\omega_{\tau(n)}\}$ is a bounded sequence, there exists a subsequence $\{\omega_{\tau(n_j)}\}$ such that

$$\limsup_{n\to\infty}\left\langle u - x^*, \omega_{\tau(n)} - x^*\right\rangle = \lim_{j\to\infty}\left\langle u - x^*, \omega_{\tau(n_j)} - x^*\right\rangle. \tag{3.59}$$

Following the same argument as the proof of Case A for $\{\omega_{\tau(n_j)}\}$, we have that

$$\limsup_{n\to\infty}\left\langle u - x^*,\ \omega_{\tau(n)} - x^*\right\rangle \leq 0. \tag{3.60}$$

Using (C4), (3.53), (3.56), and (3.60), we have that

$$\lim_{n\to\infty}\left\|\omega_{\tau(n)} - x^*\right\| = 0. \tag{3.61}$$

By (3.58) and (3.61), we have that

$$\lim_{n\to\infty}\left\|\omega_{\tau(n)+1}-x^*\right\|=0. \tag{3.62}$$

By Lemma 2.4 (ii), we get $\lim_{n\to\infty}\Gamma_n=0$; that is $\lim_{n\to\infty}\|\omega_n-x^*\|=0$. We observe that

$$\|x_{n+1}-x^*\|^2\leq\alpha_n\|u_n-x^*\|^2+(1-\alpha_n)\|\omega_n-x^*\|^2. \tag{3.63}$$

Applying (C1), (C4), and $\lim_{n\to\infty}\|\omega_n-x^*\|^2=0$, we have immediately

$$\lim_{n\to\infty}\|x_n-x^*\|=0; \tag{3.64}$$

that is, $\{x_n\}$ converges strongly to x^*, where $x^*=P_\Omega u$. This completes the proof. □

Setting $\Psi_i\equiv 0$ for all $i=1,2,\ldots,k$ in Theorem 3.1, we obtain the following result.

Corollary 3.2. *Let C be a nonempty closed convex subset of a Hilbert space H. For each $i=1,2,\ldots,k$, let $G_i : C\times C\to\mathbb{R}$ be a bifunction satisfying (A1)–(A4). For each $j=1,2$, let $T_j : C\to H$ be two quasi-nonexpansive mappings such that $I-T_j$ are demiclosed at zero with $\Omega:=F(T_1)\cap F(T_2)\cap(\cap_{i=1}^k EP(G_i))\neq\emptyset$. Let the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be defined by*

$$\begin{gathered}
x_1\in H,\\
G_1(u_{n,1},y)+\frac{1}{r_n}\langle y-u_{n,1},u_{n,1}-x_n\rangle\geq 0,\quad\forall y\in C,\\
G_2(u_{n,2},y)+\frac{1}{r_n}\langle y-u_{n,2},u_{n,2}-x_n\rangle\geq 0,\quad\forall y\in C,\\
\vdots\\
G_k(u_{n,k},y)+\frac{1}{r_n}\langle y-u_{n,k},u_{n,k}-x_n\rangle\geq 0,\quad\forall y\in C,\\
\omega_n=\frac{1}{k}\sum_{i=1}^{k}u_{n,i},\\
y_n=\gamma_n\omega_n+(1-\gamma_n)T_1\omega_n,\\
z_n=\beta_n y_n+(1-\beta_n)T_2\omega_n,\\
x_{n+1}=\alpha_n u_n+(1-\alpha_n)z_n,\quad\forall n\in\mathbb{N},
\end{gathered} \tag{3.65}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ satisfy the following conditions.

(C1) $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=1}^{\infty}\alpha_n=\infty$;

(C2) $\liminf_{n\to\infty}\beta_n(1-\beta_n)>0$;

(C3) $\liminf_{n\to\infty}\gamma_n(1-\gamma_n)>0$;

(C4) $\lim_{n\to\infty}u_n=u$ *for some* $u\in H$.

Then $\{x_n\}$ *converges strongly to* x^*, *where* $x^* = P_\Omega u$.

In the next results, using Theorem 3.1, we have new strong convergence theorems for two nonexpansive mappings in a Hilbert space.

Corollary 3.3. *Let* C *be a nonempty closed convex subset of a Hilbert space* H. *For each* $i = 1, 2, \ldots, k$, *let* $G_i : C \times C \to \mathbb{R}$ *be a bifunction satisfying* (A1)–(A4) *and* Ψ_i *a* μ_i*-inverse strongly monotone mapping. For each* $j = 1, 2$, *let* $T_j : C \to H$ *be two nonexpansive mappings such that* $\Omega := F(T_1) \cap F(T_2) \cap (\cap_{i=1}^{k} GEP(G_i, \Psi_i)) \neq \emptyset$. *Let the sequences* $\{x_n\}$, $\{y_n\}$, *and* $\{z_n\}$ *be defined by* (3.1), *where* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *satisfy the following conditions.*

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$;

(C3) $\liminf_{n\to\infty}\gamma_n(1-\gamma_n) > 0$;

(C4) $\lim_{n\to\infty}u_n = u$ *for some* $u \in H$.

Then $\{x_n\}$ *converges strongly to* x^*, *where* $x^* = P_\Omega u$.

4. Applications

In this section, we present some convergence theorems deduced from the results in the previous section. Recall that a mapping $T : C \to H$ is said to be nonspreading [2] if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2 \tag{4.1}$$

for all $x, y \in C$. Further, a mapping $T : C \to H$ is said to be hybrid [23] if

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2 \tag{4.2}$$

for all $x, y \in C$. These mappings are deduced from a firmly nonexpansive mapping in a Hilbert space.

A mapping $F : C \to H$ is said to be firmly nonexpansive if

$$\|Fx - Fy\|^2 \leq \langle x - y,\ Fx - Fy\rangle \tag{4.3}$$

for all $x, y \in C$; see, for instance, Browder [24] and Goebel and Kirk [25]. We also know that a firmly nonexpansive mapping F can be deduced from an equilibrium problem in a Hilbert space.

Recently, Kocourek et al. [26] introduced a more broad class of nonlinear mappings call generalized hybrid if there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \tag{4.4}$$

for all $x, y \in C$. Very recently, they defined a more broad class of mappings than the class of generalized hybrid mappings in a Hilbert space. A mapping $S : C \to H$ is called *super hybrid* if there are $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\begin{aligned}\alpha\|Sx-Sy\|^2+(1-\alpha+\gamma)\|x-Sy\|^2 \le (\beta+(\beta-\alpha)\gamma)\|Sx-y\|^2+(1-\beta-(\beta-\alpha-1)\gamma)\|x-y\|^2 \\ + (\alpha-\beta)\gamma\|x-Sx\|^2+\gamma\|y-Sy\|^2,\end{aligned} \tag{4.5}$$

for all $x, y \in C$. We call such a mapping an (α, β, γ)-super hybrid mapping. We notice that an $(\alpha, \beta, 0)$-super hybrid mapping is (α, β)-generalized hybrid. So, the class of super hybrid mappings contains the class of generalized hybrid mappings. A super hybrid mapping is not quasi-nonexpansive generally. For more details, see [27]. Before proving, we need the following lemmas.

Lemma 4.1 (see [27]). *Let C be a nonempty subset of a Hilbert space H and let α, β and γ be real numbers with $\gamma \neq -1$. Let S and T be mappings of C into H such that $S = (1/(1+\gamma))T+(\gamma/(1+\gamma))I$. Then, T is (α, β, γ)-super hybrid if and only if S is (α, β)-generalized hybrid. In this case, $F(S) = F(T)$.*

Lemma 4.2 (see [27]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $S : C \to H$ be a generalized hybrid mapping. Then S is demiclosed on C.*

Setting $S_j := (1/(1+\gamma_j))T_j+(\gamma_j/(1+\gamma_j))I$ in Theorem 3.1, where T_j is a super hybrid mapping and γ_j is a real number, we obtain the following result.

Theorem 4.3. *Let C be a nonempty closed convex subset of a Hilbert space H. For each $i = 1, 2, \ldots, k$, let $G_i : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)–(A4) *and Ψ_i a μ_i-inverse strongly monotone mapping. For each $j = 1, 2$, let $T_j : C \to H$ be $(\alpha_j, \beta_j, \gamma_j)$-super hybrid mappings such that $\Omega := F(T_1) \cap F(T_2) \cap (\cap_{i=1}^{k} GEP(G_i, \Psi_i)) \neq \emptyset$. Let the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be defined by*

$$\begin{gathered} x_1 \in H, \\ G_1(u_{n,1}, y) + \langle \Psi_1 x_n, y - u_{n,1}\rangle + \frac{1}{r_n}\langle y - u_{n,1}, u_{n,1} - x_n\rangle \ge 0, \quad \forall y \in C, \\ G_2(u_{n,2}, y) + \langle \Psi_2 x_n, y - u_{n,2}\rangle + \frac{1}{r_n}\langle y - u_{n,2}, u_{n,2} - x_n\rangle \ge 0, \quad \forall y \in C, \\ \vdots \\ G_k(u_{n,k}, y) + \langle \Psi_k x_n, y - u_{n,k}\rangle + \frac{1}{r_n}\langle y - u_{n,k}, u_{n,k} - x_n\rangle \ge 0, \quad \forall y \in C, \\ \omega_n = \frac{1}{k}\sum_{i=1}^{k} u_{n,i}, \\ y_n = \gamma_n \omega_n + (1-\gamma_n)\left(\frac{1}{1+\gamma_1}T_1\omega_n + \frac{\gamma_1}{1+\gamma_1}\omega_n\right), \\ z_n = \beta_n y_n + (1-\beta_n)\left(\frac{1}{1+\gamma_2}T_2\omega_n + \frac{\gamma_2}{1+\gamma_2}\omega_n\right), \\ x_{n+1} = \alpha_n u_n + (1-\alpha_n) z_n, \quad \forall n \in \mathbb{N}, \end{gathered} \tag{4.6}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *are sequences in* $(0,1)$ *and* $\{u_n\} \subset H$ *is a sequence and* $\{r_n\} \subset [a, 2\mu_i)$ *for some* $a > 0$ *and for all* $i \in \{1, 2, \ldots, k\}$. *Suppose the following conditions are satisfied.*

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$;

(C3) $\liminf_{n\to\infty}\gamma_n(1-\gamma_n) > 0$;

(C4) $\lim_{n\to\infty}u_n = u$ *for some* $u \in H$.

Then $\{x_n\}$ *converges strongly to* x^*, *where* $x^* = P_\Omega u$.

Proof. For each $j = 1, 2$, setting

$$S_j = \frac{1}{1+\gamma_j}T_j + \frac{\gamma_j}{1+\gamma_j}I, \tag{4.7}$$

we have from Lemma 4.1 that each S_j is a generalized hybrid mapping and $F(S_j) = F(T_j)$. Since $F(S_j) \neq \emptyset$, we have that each S_j is quasi-nonexpansive. Following the proof of Theorem 3.1 and applying Lemma 4.2, we have the desired result. This completes the proof. □

Setting $\Psi \equiv 0$ in Theorem 4.3, we obtains the following result.

Corollary 4.4. *Let* C *be a nonempty closed convex subset of a Hilbert space* H. *For each* $i = 1, 2, \ldots, k$, *let* $G_i : C \times C \to \mathbb{R}$ *be a bifunction satisfying* (A1)–(A4). *For each* $j = 1, 2$, *let* $T_j : C \to H$ *be* $(\alpha_j, \beta_j, \gamma_j)$*-super hybrid mappings such that* $\Omega := F(T_1) \cap F(T_2) \cap (\cap_{i=1}^{k} EP(G_i)) \neq \emptyset$. *Let the sequences* $\{x_n\}, \{y_n\}$, *and* $\{z_n\}$ *be defined by*

$$\begin{gathered}
x_1 \in H, \\
G_1(u_{n,1}, y) + \frac{1}{r_n}\langle y - u_{n,1}, u_{n,1} - x_n \rangle \geq 0, \quad \forall y \in C, \\
G_2(u_{n,2}, y) + \frac{1}{r_n}\langle y - u_{n,2}, u_{n,2} - x_n \rangle \geq 0, \quad \forall y \in C, \\
\vdots \\
G_k(u_{n,k}, y) + \frac{1}{r_n}\langle y - u_{n,k}, u_{n,k} - x_n \rangle \geq 0, \quad \forall y \in C, \\
\omega_n = \frac{1}{k}\sum_{i=1}^{k} u_{n,i}, \\
y_n = \gamma_n \omega_n + (1-\gamma_n)\left(\frac{1}{1+\gamma_1}T_1\omega_n + \frac{\gamma_1}{1+\gamma_1}\omega_n\right), \\
z_n = \beta_n y_n + (1-\beta_n)\left(\frac{1}{1+\gamma_2}T_2\omega_n + \frac{\gamma_2}{1+\gamma_2}\omega_n\right), \\
x_{n+1} = \alpha_n u_n + (1-\alpha_n) z_n, \quad \forall n \in \mathbb{N},
\end{gathered} \tag{4.8}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *are sequences in (0, 1) and* $\{u_n\} \subset H$ *is a sequence and* $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$*. Suppose the following conditions are satisfied.*

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$;

(C3) $\liminf_{n\to\infty}\gamma_n(1-\gamma_n) > 0$;

(C4) $\lim_{n\to\infty}u_n = u$ *for some* $u \in H$.

Then $\{x_n\}$ *converges strongly to* x^**, where* $x^* = P_\Omega u$.

In Corollary 4.4, put $G_i(x,y) = 0$ for all $x, y \in C$ and $r_n = 1$ for all $n \in \mathbb{N}$. Then we have that $u_{n,i} = x_n$ for all $i = 1, 2, \ldots, k$, which gives that $\omega_n = (1/k)\sum_{i=1}^{k} u_{n,i} = x_n$. Thus we obtain the following results from Corollary 4.4.

Corollary 4.5. *Let C be a nonempty closed convex subset of a Hilbert space H. For each* $j = 1, 2$*, let* $T_j : C \to H$ *be* $(\alpha_j, \beta_j, \gamma_j)$*-super hybrid mappings such that* $F(T_1) \cap F(T_2) \neq \emptyset$*. Let the sequences* $\{x_n\}, \{y_n\}$*, and* $\{z_n\}$ *be defined by*

$$\begin{gathered} x_1 \in H, \\ y_n = \gamma_n x_n + (1-\gamma_n)\left(\frac{1}{1+\gamma_1}T_1x_n + \frac{\gamma_1}{1+\gamma_1}x_n\right), \\ z_n = \beta_n y_n + (1-\beta_n)\left(\frac{1}{1+\gamma_2}T_2x_n + \frac{\gamma_2}{1+\gamma_2}x_n\right), \\ x_{n+1} = \alpha_n u_n + (1-\alpha_n)z_n, \quad \forall n \in \mathbb{N}, \end{gathered} \tag{4.9}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *are sequences in* $(0,1)$ *and* $\{u_n\} \subset H$ *is a sequence. Suppose the following conditions are satisfied.*

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$;

(C3) $\liminf_{n\to\infty}\gamma_n(1-\gamma_n) > 0$;

(C4) $\lim_{n\to\infty}u_n = u$ *for some* $u \in H$.

Then $\{x_n\}$ *converges strongly to* x^**, where* $x^* = P_{F(T_1)\cap F(T_2)}u$.

In Corollary 4.5, put $T_1 = I$, the identity mapping, and $T_2 := T$, an (α, β, γ)-super hybrid mapping. Thus we obtain the following results.

Corollary 4.6. *Let C be a nonempty closed convex subset of a Hilbert space H. Let T be an* (α, β, γ)*-super hybrid mapping such that* $F(T) \neq \emptyset$*. Let the sequences* $\{x_n\}$*,* $\{y_n\}$*, and* $\{z_n\}$ *be defined by*

$$x_1 \in H, \tag{4.10}$$

$$z_n = \beta_n x_n + (1-\beta_n)\left(\frac{1}{1+\gamma}Tx_n + \frac{\gamma}{1+\gamma}x_n\right), \tag{4.11}$$

$$x_{n+1} = \alpha_n u_n + (1-\alpha_n)z_n, \quad \forall n \in \mathbb{N}, \tag{4.12}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $(0,1)$ *and* $\{u_n\} \subset H$ *is a sequence. Suppose the following conditions are satisfied.*

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(C2) $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$;

(C3) $\lim_{n\to\infty}u_n = u$ *for some* $u \in H$.

Then $\{x_n\}$ *converges strongly to* x^*, *where* $x^* = P_{F(T)}u$.

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