Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 154307, 18 pages
doi:10.1155/2012/154307

Research Article

Study on the Existence and Uniqueness of Solution of Generalized Capillarity Problem

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Received 6 September 2012; Accepted 22 November 2012

Academic Editor: Jiaxin Hu

By using the perturbation theories on sums of ranges of nonlinear accretive mappings of Calvert and Gupta (1978), the abstract result on the existence and uniqueness of the solution in $L^p(\Omega)$ of the generalized Capillarity equation with nonlinear Neumann boundary value conditions, where $2N/(N+1) < p < +\infty$ and $N(\geq 1)$ denotes the dimension of R^N, is studied. The equation discussed in this paper and the methods here are a continuation of and a complement to the previous corresponding results. To obtain the results, some new techniques are used in this paper.

1. Introduction and Preliminary

Since the p-Laplacian operator $-\Delta_p$ with $p \neq 2$ arises from a variety of physical phenomena, such as nonNewtonian fluids, reaction-diffusion problems, and petroleum extraction, it becomes a very popular topic in mathematical fields.

We began our study on this topic in 1995. We used a perturbation result of ranges for m-accretive mappings in Calvert and Gupta [1] to obtain a sufficient condition in Wei and He [2] so that the following zero boundary value problem,

$$\begin{aligned} -\Delta_p u + g(x, u(x)) &= f(x), \quad \text{a.e. on } \Omega, \\ -\frac{\partial u}{\partial n} &= 0, \quad \text{a.e. on } \Gamma, \end{aligned} \tag{1.1}$$

has solutions in $L^p(\Omega)$, where $2 \leq p < +\infty$. Later on, a series work of ours has been done from different angles on this kind of equations, cf. [3–7], and so forth.

Especially, in 2008, as a summary of the work done in [2–6], we use some new techniques to work for the following problem with so-called generalized p-Laplacian operator:

$$
\begin{gathered}
-\operatorname{div}\left[\left(C(x)+|\nabla u|^2\right)^{(p-2)/2}\nabla u\right]+\varepsilon|u|^{q-2}u+g(x,u(x))=f(x), \quad \text{a.e. in } \Omega,\\
-\left\langle \vartheta, \left(C(x)+|\nabla u|^2\right)^{(p-2)/2}\nabla u\right\rangle \in \beta_x(u(x)), \quad \text{a.e. on } \Gamma,
\end{gathered} \tag{1.2}
$$

where $0 \leq C(x) \in L^p(\Omega)$, ε is a nonnegative constant and ϑ denotes the exterior normal derivative of Γ. We showed in Wei and Agarwal [7] that (1.2) has solutions in $L^s(\Omega)$ under some conditions, where $2N/(N+1) < p \leq s < +\infty$, $1 \leq q < +\infty$ if $p \geq N$, and $1 \leq q \leq Np/(N-p)$ if $p < N$, for $N \geq 1$.

Capillarity equation is another important equation appeared in the capillarity phenomenon and we notice that in Chen and Luo [8], the authors studied the eigenvalue problem for the following generalized Capillarity equations:

$$
\begin{gathered}
-\operatorname{div}\left[\left(1+\frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u\right]=\lambda\left(|u|^{q-2}u+|u|^{r-2}u\right), \quad \text{in } \Omega,\\
u=0, \quad \text{a.e. on } \partial\Omega.
\end{gathered} \tag{1.3}
$$

Their work inspired us and one idea came to our mind. Can we borrow the main ideas dealing with the nonlinear elliptic boundary value problems with the generalized p-Laplacian operator to study the nonlinear generalized Capillarity equation with Neumann boundary conditions?

We will answer the question in this paper. By using the perturbation results of ranges for m-accretive mappings in Calvert and Gupta [1] again, we will study the following one:

$$
\begin{aligned}
&-\operatorname{div}\left[\left(1+\frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u\right]+\lambda\left(|u|^{q-2}u+|u|^{r-2}u\right)+g(x,u(x))\\
&\quad = f(x), \quad \text{a.e. in } \Omega,\\
&-\left\langle \vartheta, \left(1+\frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u\right\rangle \in \beta_x(u(x)), \quad \text{a.e. on } \Gamma.
\end{aligned} \tag{1.4}
$$

More details on (1.4) will be given in Section 2. Our methods and techniques are different from those in Chen and Luo [8].

Now, we list some basic knowledge we need in sequel.

Let X be a real Banach space with a strictly convex dual space X^*. We use "$\to$" and "$w-\lim$" to denote strong and weak convergence, respectively. For any subset G of X, we denote by $\operatorname{int} G$ its interior and $\overline{G}$ its closure, respectively. Let "$X \hookrightarrow\hookrightarrow Y$" denote that space X is embedded compactly in space Y and "$X \hookrightarrow Y$" denote that space X is embedded continuously in space Y. A mapping $T : D(T) = X \to X^*$ is said to be hemicontinuous on X if $w - \lim_{t\to 0} T(x + ty) = Tx$, for any $x, y \in X$. Let J denote the *duality mapping* from X into 2^{X^*} defined by

$$J(x) = \{f \in X^* : (x, f) = \|x\| \cdot \|f\|, \|f\| = \|x\|\}, \quad x \in X, \tag{1.5}$$

where $(\cdot,\cdot)$ denotes the generalized duality pairing between X and X^*. It is wellknown that J is a single-valued mapping since X^* is strictly convex.

Let $A : X \to 2^X$ be a given multivalued mapping. We say that A is boundedly-inversely compact if for any pair of bounded subsets G and G' of X, the subset $G \bigcap A^{-1}(G')$ is relatively compact in X. The mapping $A : X \to 2^X$ is said to be accretive if $(v_1 - v_2, J(u_1 - u_2)) \geq 0$, for any $u_i \in D(A)$ and $v_i \in Au_i$, $i = 1, 2$. The accretive mapping A is said to be m-accretive if $R(I + \mu A) = X$, for some $\mu > 0$.

Let $B : X \to 2^{X^*}$ be a given multi-valued mapping. The graph of B, $G(B)$, is defined by $G(B) = \{[u, w] \mid u \in D(B),\ w \in Bu\}$. Then $B : X \to 2^{X^*}$ is said to be monotone if $G(B)$ is a monotone subset of $X \times X^*$ in the sense that

$$(u_1 - u_2, w_1 - w_2) \geq 0, \tag{1.6}$$

for any $[u_i, w_i] \in G(B)$, $i = 1, 2$. The monotone operator B is said to be maximal monotone if $G(B)$ is maximal among all monotone subsets of $X \times X^*$ in the sense of inclusion. The mapping B is said to be strictly monotone if the equality in (1.6) implies that $u_1 = u_2$. The mapping B is said to be coercive if $\lim_{n\to+\infty}((x_n, x_n^*)/\|x_n\|) = \infty$ for all $[x_n, x_n^*] \in G(B)$ such that $\lim_{n\to+\infty}\|x_n\| = +\infty$.

Definition 1.1. The *duality mapping* $J : X \to 2^{X^*}$ is said to be satisfying Condition (I) if there exists a function $\eta : X \to [0, +\infty)$ such that

$$\|Ju - Jv\| \leq \eta(u - v), \quad \text{for } \forall\ u, \quad v \in X. \tag{I}$$

Definition 1.2. Let $A : X \to 2^X$ be an accretive mapping and $J : X \to X^*$ be a duality mapping. We say that A satisfies Condition $(*)$ if, for any $f \in R(A)$ and $a \in D(A)$, there exists a constant $C(a, f)$ such that

$$(v - f, J(u - a)) \geq C(a, f), \quad \text{for any } u \in D(A),\ v \in Au. \tag{$*$}$$

Lemma 1.3 (Li and Guo [9]). *Let Ω be a bounded conical domain in R^N. Then we have the following results.*

(a) *If $mp > N$, then $W^{m,p}(\Omega) \hookrightarrow C_B(\Omega)$; if $mp < N$ and $q = Np/(N - mp)$, then $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$; if $mp = N$ and $p > 1$, then for $1 \leq q < +\infty$, $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$.*

(b) *If $mp > N$, then $W^{m,p}(\Omega) \hookrightarrow\hookrightarrow C_B(\Omega)$; if $0 < mp \leq N$ and $q_0 = Np/(N - mp)$, then $W^{m,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$, $1 \leq q < q_0$.*

Lemma 1.4 (Pascali and Sburlan [10]). *If $B : X \to 2^{X^*}$ is an everywhere defined, monotone and hemicontinuous operator, then B is maximal monotone. If $B : X \to 2^{X^*}$ is maximal monotone and coercive, then $R(B) = X^*$.*

Lemma 1.5 (Pascali and Sburlan [10]). *If $\Phi : X \to (-\infty, +\infty]$ is a proper convex and lower-semicontinuous function, then $\partial\Phi$ is maximal monotone from X to X^*.*

Lemma 1.6 (Pascali and Sburlan [10]). *If A and B are two maximal monotone operators in X such that $(\operatorname{int} D(A)) \bigcap D(B) \neq \emptyset$, then $A + B$ is maximal monotone.*

Proposition 1.7 (Calvert and Gupta [1]). *Let $X = L^p(\Omega)$ and Ω be a bounded domain in R^N. For $2 \leq p < +\infty$, the duality mapping $J_p : L^p(\Omega) \to L^{p'}(\Omega)$ defined by $J_p u = |u|^{p-1} \operatorname{sgn} u \|u\|_p^{2-p}$, for $u \in L^p(\Omega)$, satisfies Condition (I); for $2N/(N+1) < p \leq 2$ and $N \geq 1$, the duality mapping $J_p : L^p(\Omega) \to L^{p'}(\Omega)$ defined by $J_p u = |u|^{p-1} sgnu$, for $u \in L^p(\Omega)$, satisfies Condition (I), where $(1/p) + (1/p') = 1$.*

Lemma 1.8 (see Calvert and Gupta [1]). *Let Ω be a bounded domain in R^N and $g : \Omega \times R \to R$ be a function satisfying Caratheodory's conditions such that*

(i) *$g(x, \cdot)$ is monotonically increasing on R;*

(ii) *the mapping $u \in L^p(\Omega) \to g(x, u(x)) \in L^p(\Omega)$ is well defined, where $2N/(N+1) < p < +\infty$ and $N \geq 1$.*

Let $J_p : L^p(\Omega) \to L^{p'}(\Omega)$, $(1/p) + (1/p') = 1$ be the duality mapping defined by

$$J_p u = \begin{cases} |u|^{p-1} \operatorname{sgn} u, & \text{if } \dfrac{2N}{N+1} < p \leq 2, \\ |u|^{p-1} \operatorname{sgn} u \|u\|_p^{2-p}, & \text{if } 2 \leq p < +\infty, \end{cases} \tag{1.7}$$

for $u \in L^p(\Omega)$. Then the mapping $B : L^p(\Omega) \to L^p(\Omega)$ defined by $(Bu)(x) = g(x, u(x))$, for any $x \in \Omega$ satisfies Condition $(*)$.

Theorem 1.9 (Calvert and Gupta [1]). *Let X be a real Banach space with a strictly convex dual X^*. Let $J : X \to X^*$ be a duality mapping on X satisfying Condition (I). Let $A, B_1 : X \to 2^X$ be accretive mappings such that*

(i) *either both A, B_1 satisfy Condition $(*)$ or $D(A) \subset D(B_1)$ and B_1 satisfies Condition $(*)$,*

(ii) *$A + B_1$ is m-accretive and boundedly inversely compact.*

If $B_2 : X \to X$ be a bounded continuous mapping such that, for any $y \in X$, there is a constant $C(y)$ satisfying $(B_2(u + y), Ju) \geq -C(y)$ for any $u \in X$. Then.

(a) $\overline{[R(A) + R(B_1)]} \subset \overline{R(A + B_1 + B_2)}$.

(b) $\operatorname{int}[R(A) + R(B_1)] \subset \operatorname{int} R(A + B_1 + B_2)$.

2. The Main Results

2.1. Notations and Assumptions of (1.4)

Next in this paper, we assume $2N/(N+1) < p < +\infty$, $1 \le q, r < +\infty$ if $p \ge N$, and $1 \le q, r \le Np/(N-p)$ if $p < N$, where $N \ge 1$. We use $\|\cdot\|_p$, $\|\cdot\|_q$, $\|\cdot\|_r$, and $\|\cdot\|_{1,p,\Omega}$ to denote the norms in $L^p(\Omega)$, $L^q(\Omega)$, $L^r(\Omega)$ and $W^{1,p}(\Omega)$. Let $(1/p) + (1/p') = 1$, $(1/q) + (1/q') = 1$, and $(1/r) + (1/r') = 1$.

In (1.4), Ω is a bounded conical domain of a Euclidean space R^N with its boundary $\Gamma \in C^1$, (c.f. [4]). We suppose that the Green's Formula is available. Let $|\cdot|$ denote the Euclidean norm in R^N, $\langle \cdot, \cdot \rangle$ the Euclidean inner-product and ϑ the exterior normal derivative of Γ. λ is a nonnegative constant.

Let $\varphi : \Gamma \times R \to R$ be a given function such that, for each $x \in \Gamma$,

(i) $\varphi_x = \varphi(x, \cdot) : R \to R$ is a proper, convex, lower-semicontinuous function with $\varphi_x(0) = 0$.

(ii) $\beta_x = \partial\varphi_x$ (: subdifferential of φ_x) is maximal monotone mapping on R with $0 \in \beta_x(0)$ and for each $t \in R$, the function $x \in \Gamma \to (I + \mu\beta_x)^{-1}(t) \in R$ is measurable for $\mu > 0$.

Let $g : \Omega \times R \to R$ be a given function satisfying Caratheodory's conditions such that for $2N/(N+1) < p < +\infty$ and $N \ge 1$, the mapping $u \in L^p(\Omega) \to g(x, u(x)) \in L^p(\Omega)$ is defined. Further, suppose that there is a function $T(x) \in L^p(\Omega)$ such that $g(x,t)t \ge 0$, for $|t| \ge T(x)$, $x \in \Omega$.

2.2. Existence and Uniqueness of the Solution of (1.4)

Definition 2.1 (Calvert and Gupta [1]). Define $g_+(x) = \liminf_{t\to+\infty} g(x,t)$ and $g_-(x) = \limsup_{t\to-\infty} g(x,t)$.

Further, define a function $g_1 : \Omega \times R \to R$ by

$$g_1(x,t) = \begin{cases} \left(\inf\limits_{s\ge t} g(x,s)\right) \wedge (t - T(x)), & \forall t \ge T(x), \\ 0, & \forall t \in [-T(x), T(x)], \\ \left(\sup\limits_{s\le t} g(x,s)\right) \vee (t + T(x)), & \forall t \le -T(x). \end{cases} \tag{2.1}$$

Then for all $x \in \Omega$, $g_1(x,t)$ is increasing in t and $\lim_{t\to\pm\infty} g_1(x,t) = g_\pm(x)$. Moreover, $g_1 : \Omega \times R \to R$ satisfies Caratheodory's conditions and the functions $g_\pm(x)$ are measurable on Ω. And, if $g_2(x,t) = g(x,t) - g_1(x,t)$ then $g_2(x,t)t \ge 0$, for $|t| \ge T(x)$, $x \in \Omega$.

Proposition 2.2 (see Calvert and Gupta [1]). *For $2N/(N+1) < p < +\infty$ and $N \ge 1$, define the mapping $B_1 : L^p(\Omega) \to L^p(\Omega)$ by $(B_1u)(x) = g_1(x, u(x))$, for all $u \in L^p(\Omega)$ and $x \in \Omega$, then B_1 is a bounded, continuous, and m-accretive mapping.*

Moreover, Lemma 1.8 implies that B_1 satisfies Condition $(*)$.

Define $B_2 : L^p(\Omega) \to L^p(\Omega)$ by $(B_2u)(x) = g_2(x, u(x))$, where $g_2(x,t) = g(x,t) - g_1(x,t)$, then B_2 satisfies the inequality:

$$(B_2(u+y), J_p u) \geq -C(y), \tag{2.2}$$

for any $u, y \in L^p(\Omega)$, where $C(y)$ is a constant depending on y and $J_p : L^p(\Omega) \to L^{p'}(\Omega)$ denotes the duality mapping, where $(1/p) + (1/p') = 1$.

Lemma 2.3 (Wei and Agarwal [7]). *The mapping $\Phi_p : W^{1,p}(\Omega) \to R$ defined by*

$$\Phi_p(u) = \int_\Gamma \varphi_x(u|_\Gamma(x))\, d\Gamma(x), \tag{2.3}$$

for any $u \in W^{1,p}(\Omega)$, is a proper, convex, and lower-semicontinuous mapping on $W^{1,p}(\Omega)$.

Moreover, Lemma 1.5 implies that $\partial\Phi_p$, the subdifferential of Φ_p, is maximal monotone.

Lemma 2.4 (Wei and He [2]). *Let X_0 denote the closed subspace of all constant functions in $W^{1,p}(\Omega)$. Let X be the quotient space $W^{1,p}(\Omega)/X_0$. For $u \in W^{1,p}(\Omega)$, define the mapping $P : W^{1,p}(\Omega) \to X_0$ by $Pu = (1/\operatorname{meas}(\Omega)) \int_\Omega u\,dx$. Then, there is a constant $C > 0$, such that for all $u \in W^{1,p}(\Omega)$,*

$$\|u - Pu\|_p \leq C \|\nabla u\|_{(L^p(\Omega))^N}. \tag{2.4}$$

Lemma 2.5. *Define the mapping $B_{p,q,r} : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ by*

$$\begin{aligned}(v, B_{p,q,r}u) = &\int_\Omega \left\langle \left(1 + \frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right) |\nabla u|^{p-2}\nabla u, \nabla v \right\rangle dx \\ &+ \lambda \int_\Omega |u(x)|^{q-2} u(x) v(x)\, dx + \lambda \int_\Omega |u(x)|^{r-2} u(x) v(x)\, dx,\end{aligned} \tag{2.5}$$

for any $u, v \in W^{1,p}(\Omega)$. Then $B_{p,q,r}$ is everywhere defined, strictly monotone, hemicontinuous, and coercive.

Proof. Step 1. $B_{p,q,r}$ is everywhere defined.

From Lemma 1.3, we know that $W^{1,p}(\Omega) \hookrightarrow C_B(\Omega)$, when $p > N$. And, $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, $W^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$, when $p \leq N$. Thus, for all $v \in W^{1,p}(\Omega)$, $\|v\|_q \leq k_1 \|v\|_{1,p,\Omega}$, $\|v\|_r \leq k_2 \|v\|_{1,p,\Omega}$, where k_1, k_2 are positive constants.

For $u, v \in W^{1,p}(\Omega)$, we have

$$\begin{aligned}\left|(v, B_{p,q,r}u)\right| &\leq 2\int_\Omega |\nabla u|^{p-1}|\nabla v|dx + \lambda\int_\Omega |u|^{q-1}|v|dx + \lambda\int_\Omega |u|^{r-1}|v|dx \\ &\leq 2\||\nabla u|\|_p^{p/p'}\||\nabla v|\|_p + \lambda\|v\|_q\|u\|_q^{q/q'} + \lambda\|v\|_r\|u\|_r^{r/r'} \\ &\leq 2\|u\|_{1,p,\Omega}^{p/p'}\|v\|_{1,p,\Omega} + k_1'\lambda\|v\|_{1,p,\Omega}\|u\|_{1,p,\Omega}^{q/q'} + k_2'\lambda\|v\|_{1,p,\Omega}\|u\|_{1,p,\Omega}^{r/r'},\end{aligned} \tag{2.6}$$

where k_1' and k_2' are positive constants. Thus $B_{p,q,r}$ is everywhere defined.

Step 2. $B_{p,q,r}$ is strictly monotone.

For $u, v \in W^{1,p}(\Omega)$, we have

$$\begin{aligned}&\left|(u-v, B_{p,q,r}u - B_{p,q,r}v)\right| \\ &= \int_\Omega \left\langle \left(1 + \frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u - \left(1 + \frac{|\nabla v|^p}{\sqrt{1+|\nabla v|^{2p}}}\right)|\nabla v|^{p-2}\nabla v, \nabla u - \nabla v\right\rangle dx \\ &\quad + \lambda\int_\Omega \left(|u|^{q-2}u - |v|^{q-2}v\right)(u-v)dx + \lambda\int_\Omega \left(|u|^{r-2}u - |v|^{r-2}v\right)(u-v)dx \\ &= \int_\Omega \left\{\left(1 + \frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^p - \left(1 + \frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u\nabla v \right. \\ &\qquad \left. - \left(1 + \frac{|\nabla v|^p}{\sqrt{1+|\nabla v|^{2p}}}\right)|\nabla v|^{p-2}\nabla u\nabla v + \left(1 + \frac{|\nabla v|^p}{\sqrt{1+|\nabla v|^{2p}}}\right)|\nabla v|^p\right\} dx \\ &\quad + \lambda\int_\Omega \left(|u|^{q-2}u - |v|^{q-2}v\right)(u-v)dx + \lambda\int_\Omega \left(|u|^{r-2}u - |v|^{r-2}v\right)(u-v)dx \\ &\geq \int_\Omega \left\{\left(1 + \frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{p-1} - \left(1 + \frac{|\nabla v|^p}{\sqrt{1+|\nabla v|^{2p}}}\right)|\nabla v|^{p-1}\right\}(|\nabla u| - |\nabla v|)dx \\ &\quad + \lambda\int_\Omega \left(|u|^{q-1} - |v|^{q-1}\right)(|u| - |v|)dx + \lambda\int_\Omega \left(|u|^{r-1} - |v|^{r-1}\right)(|u| - |v|)dx.\end{aligned} \tag{2.7}$$

If, we let $h(t) = (1 + (t/\sqrt{1+t^2}))t^{(p-1)/p}$, for $t \geq 0$. Then we know that

$$h'(t) = \frac{t^{(p-1)/p}}{(1+t^2)^{3/2}} + t^{-(1/p)}\left(1 + \frac{t}{\sqrt{1+t^2}}\right)\frac{p-1}{p} \geq 0, \tag{2.8}$$

since $t \geq 0$. And, $h'(t) = 0$ if and only if $t = 0$. Then $h(t)$ is strictly monotone. Thus we can easily know that $B_{p,q,r}$ is strictly monotone.

Step 3. $B_{p,q,r}$ is hemicontinuous.

In fact, it suffices to show that, for any $u, v, w \in W^{1,p}(\Omega)$ and $t \in [0,1]$, $(w, B_{p,q,r}(u + tv) - B_{p,q,r}u) \to 0$, as $t \to 0$.

By Lebesque's dominated convergence theorem, it follows that

$$\begin{aligned}
0 &\leq \lim_{t\to 0}\left|(w, B_{p,q,r}(u+tv) - B_{p,q,r}u)\right| \\
&\leq \int_\Omega \lim_{t\to 0}\Bigg|\left(1 + \frac{|\nabla u + t\nabla v|^p}{\sqrt{1+|\nabla u + t\nabla v|^{2p}}}\right)|\nabla u + t\nabla v|^{p-2}(\nabla u + t\nabla v) \\
&\qquad\qquad - \left(1 + \frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u\Bigg||\nabla w|\,dx \\
&\quad + \lambda\int_\Omega \lim_{t\to 0}\left||u+tv|^{q-2}(u+tv) - |u|^{q-2}u\right||w|\,dx \\
&\quad + \lambda\int_\Omega \lim_{t\to 0}\left||u+tv|^{r-2}(u+tv) - |u|^{r-2}u\right||w|\,dx = 0,
\end{aligned} \tag{2.9}$$

and hence $B_{p,q,r}$ is hemicontinuous.

Step 4. $B_{p,q,r}$ is coercive.

Now, for $u \in W^{1,p}(\Omega)$, Lemma 2.4 implies that $\|u\|_{1,p,\Omega} \to \infty$ is equivalent to $\|u - (1/\operatorname{meas}(\Omega))\int_\Omega u\,dx\|_{1,p,\Omega} \to \infty$ and hence we have the following result:

$$\begin{aligned}
\frac{(u, B_{p,q,r}u)}{\|u\|_{1,p,\Omega}} &= \frac{\int_\Omega\left(1 + \left(|\nabla u|^p/\sqrt{1+|\nabla u|^{2p}}\right)\right)|\nabla u|^p\,dx}{\|u\|_{1,p,\Omega}} + \lambda\frac{\int_\Omega |u|^q\,dx}{\|u\|_{1,p,\Omega}} + \lambda\frac{\int_\Omega |u|^r\,dx}{\|u\|_{1,p,\Omega}} \\
&= \frac{\int_\Omega\left(|\nabla u|^p + \sqrt{1+|\nabla u|^{2p}}\right)dx - \int_\Omega\left(1/\sqrt{1+|\nabla u|^{2p}}\right)dx}{\|u\|_{1,p,\Omega}} \\
&\quad + \lambda\frac{\int_\Omega |u|^q\,dx}{\|u\|_{1,p,\Omega}} + \lambda\frac{\int_\Omega |u|^r\,dx}{\|u\|_{1,p,\Omega}} \\
&\geq \frac{2\int_\Omega |\nabla u|^p\,dx - \int_\Omega\left(1/\sqrt{1+|\nabla u|^{2p}}\right)dx}{\|u\|_{1,p,\Omega}} + \lambda\frac{\int_\Omega |u|^q\,dx}{\|u\|_{1,p,\Omega}} + \lambda\frac{\int_\Omega |u|^r\,dx}{\|u\|_{1,p,\Omega}} \longrightarrow +\infty,
\end{aligned} \tag{2.10}$$

as $\|u\|_{1,p,\Omega} \to +\infty$, which implies that $B_{p,q,r}$ is coercive.

This completes the proof. □

Remark 2.6. Lemma 2.5 is a key result for later use.

Definition 2.7. Define a mapping $A_p : L^p(\Omega) \to 2^{L^p(\Omega)}$ as follows:

$$D(A_p) = \{u \in L^p(\Omega) \mid \text{there exists an } f \in L^p(\Omega), \text{ such that } f \in B_{p,q,r}u + \partial\Phi_p(u)\}. \tag{2.11}$$

For $u \in D(A_p)$, let $A_p u = \{f \in L^p(\Omega) \mid f \in B_{p,q,r}u + \partial\Phi_p(u)\}$.

Proposition 2.8. *The mapping $A_p : L^p(\Omega) \to 2^{L^p(\Omega)}$ is m-accretive.*

Proof. Step 1. A_p is accretive.
Case 1. If $p \geq 2$, the duality mapping $J_p : L^p(\Omega) \to L^{p'}(\Omega)$ is defined by $J_p u = |u|^{p-1} \operatorname{sgn} u \|u\|_p^{2-p}$ for $u \in L^p(\Omega)$. It then suffices to prove that for any $u_i \in D(A_p)$ and $v_i \in A_p u_i$, $i = 1, 2$,

$$(v_1 - v_2, J_p(u_1 - u_2)) \geq 0. \tag{2.12}$$

To this, we are left to prove that both

$$\begin{gathered} \left(|u_1 - u_2|^{p-1} \operatorname{sgn}(u_1 - u_2)\|u_1 - u_2\|_p^{2-p}, B_{p,q,r}u_1 - B_{p,q,r}u_2\right) \geq 0, \\ \left(|u_1 - u_2|^{p-1} \operatorname{sgn}(u_1 - u_2)\|u_1 - u_2\|_p^{2-p}, \partial\Phi_p(u_1) - \partial\Phi_p(u_2)\right) \geq 0 \end{gathered} \tag{2.13}$$

are available.

Now take for a constant $k > 0$, $\chi_k : R \to R$ is defined by $\chi_k(t) = |(t \wedge k) \vee (-k)|^{p-1} \operatorname{sgn} t$. Then χ_k is monotone, Lipschitz with $\chi_k(0) = 0$ and χ_k' is continuous except at finitely many points on R. This gives that

$$\begin{aligned} &\left(|u_1 - u_2|^{p-1} \operatorname{sgn}(u_1 - u_2)\|u_1 - u_2\|_p^{2-p},\ \partial\Phi_p(u_1) - \partial\Phi_p(u_2)\right) \\ &\quad = \lim_{k \to +\infty} \|u_1 - u_2\|_p^{2-p} (\chi_k(u_1 - u_2),\ \partial\Phi_p(u_1) - \partial\Phi_p(u_2)) \geq 0. \end{aligned} \tag{2.14}$$

Also,

$$\begin{aligned} &\left(|u_1 - u_2|^{p-1} \operatorname{sgn}(u_1 - u_2)\|u_1 - u_2\|_p^{2-p},\ B_{p,q,r}u_1 - B_{p,q,r}u_2\right) \\ &\quad = \|u_1 - u_2\|_p^{2-p} \times \lim_{k \to +\infty} \int_\Omega \Bigg\langle \left(1 + \frac{|\nabla u_1|^p}{\sqrt{1 + |\nabla u_1|^{2p}}}\right) |\nabla u_1|^{p-2} \nabla u_1 \\ &\qquad\qquad - \left(1 + \frac{|\nabla u_2|^p}{\sqrt{1 + |\nabla u_2|^{2p}}}\right) |\nabla u_2|^{p-2} \nabla u_2, \nabla u_1 - \nabla u_2 \Bigg\rangle \end{aligned}$$

$$\times \chi_k'(u_1-u_2)dx + \lambda\|u_1-u_2\|_p^{2-p}\int_\Omega \left(|u_1|^{q-2}u_1 - |u_2|^{q-2}u_2\right)|u_1-u_2|^{p-1}\operatorname{sgn}(u_1-u_2)dx$$
$$+\lambda\|u_1-u_2\|_p^{2-p}\int_\Omega \left(|u_1|^{r-2}u_1 - |u_2|^{r-2}u_2\right)|u_1-u_2|^{p-1}\operatorname{sgn}(u_1-u_2)dx \ge 0, \tag{2.15}$$

the last inequality is available since χ_k is monotone and $\chi_k(0) = 0$.

Case 2. If $2N/(N+1) < p < 2$, the duality mapping $J_p : L^p(\Omega) \to L^{p'}(\Omega)$ is defined by $J_p u = |u|^{p-1}\operatorname{sgn} u$, for $u \in L^p(\Omega)$. It then suffices to prove that for any $u_i \in D(A_p)$ and $v_i \in A_p u_i$, $i = 1, 2$,

$$\left(v_1 - v_2,\ J_p(u_1-u_2)\right) \ge 0. \tag{2.16}$$

To this, we define the function $\chi_n : R \to R$ by

$$\chi_n(t) = \begin{cases} |t|^{p-1}\operatorname{sgn} t, & \text{if } |t| \ge \dfrac{1}{n}, \\ \left(\dfrac{1}{n}\right)^{p-2} t, & \text{if } |t| \le \dfrac{1}{n}. \end{cases} \tag{2.17}$$

Then χ_n is monotone, Lipschitz with $\chi_n(0) = 0$ and χ_n' is continuous except at finitely many points on R. So $(\chi_n(u_1-u_2), \partial\Phi_p(u_1) - \partial\Phi_p(u_2)) \ge 0$.

Then, for $u_i \in D(A_p)$, $v_i \in A_p u_i$, $i = 1, 2$, we have

$$\begin{aligned} &\left(v_1 - v_2, J_p(u_1-u_2)\right) \\ &\quad = \left(|u_1-u_2|^{p-1}\operatorname{sgn}(u_1-u_2), B_{p,q,r}u_1 - B_{p,q,r}u_2\right) \\ &\qquad + \left(|u_1-u_2|^{p-1}\operatorname{sgn}(u_1-u_2), \partial\Phi_p(u_1) - \partial\Phi_p(u_2)\right) \\ &\quad = \left(|u_1-u_2|^{p-1}\operatorname{sgn}(u_1-u_2), B_{p,q,r}u_1 - B_{p,q,r}u_2\right) \\ &\qquad + \lim_{n\to\infty}\left(\chi_n(u_1-u_2), \partial\Phi_p(u_1) - \partial\Phi_p(u_2)\right) \ge 0. \end{aligned} \tag{2.18}$$

Step 2. $R(I + \mu A_p) = L^p(\Omega)$, for every $\mu > 0$.

First, define the mapping $I_p : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ by $I_p u = u$ and $(v, I_p u)_{(W^{1,p}(\Omega))^* \times W^{1,p}(\Omega)} = (v, u)_{L^2(\Omega)}$ for $u, v \in W^{1,p}(\Omega)$, where $(\cdot,\cdot)_{L^2(\Omega)}$ denotes the inner product of $L^2(\Omega)$. Then I_p is maximal monotone [7].

Secondly, for any $\mu > 0$, define the mapping $T_\mu : W^{1,p}(\Omega) \to 2^{(W^{1,p}(\Omega))^*}$ by $T_\mu u = I_p u + \mu B_{p,q,r} u + \mu\partial\Phi_p(u)$, for $u \in W^{1,p}(\Omega)$. Then similar to that in [7], by using Lemmas 1.4, 1.6, 2.3, and 2.5, we know that T_μ is maximal monotone and coercive, so that $R(T_\mu) = (W^{1,p}(\Omega))^*$, for any $\mu > 0$.

Therefore, for any $f \in L^p(\Omega)$, there exists $u \in W^{1,p}(\Omega)$, such that

$$f = T_\mu u = u + \mu B_{p,q,r} u + \mu \partial \Phi_p(u). \tag{2.19}$$

From the definition of A_p, it follows that $R(I + \mu A_p) = L^p(\Omega)$, for all $\mu > 0$.

This completes the proof. □

Lemma 2.9. *The mapping $A_p : L^p(\Omega) \to 2^{L^p(\Omega)}$ has a compact resolvent for $2N/(N+1) < p < 2$ and $N \geq 1$.*

Proof. Since A_p is m-accretive by Proposition 2.8, it suffices to prove that if $u + \mu A_p u = f (\mu > 0)$ and $\{f\}$ is bounded in $L^p(\Omega)$, then $\{u\}$ is relatively compact in $L^p(\Omega)$. Now define functions $\chi_n, \xi_n : R \to R$ by

$$\chi_n(t) = \begin{cases} |t|^{p-1} \operatorname{sgn} t, & \text{if } |t| \geq \dfrac{1}{n} \\ \left(\dfrac{1}{n}\right)^{p-2} t, & \text{if } |t| \leq \dfrac{1}{n}, \end{cases}$$
$$\xi_n(t) = \begin{cases} |t|^{2-(2/p)} \operatorname{sgn} t, & \text{if } |t| \geq \dfrac{1}{n}, \\ \left(\dfrac{1}{n}\right)^{1-(2/p)} t, & \text{if } |t| \leq \dfrac{1}{n}. \end{cases} \tag{2.20}$$

Noticing that $\chi_n'(t) = (p-1) \times (p'/2)^p \times (\xi_n'(t))^p$, for $|t| \geq 1/n$, where $(1/p) + (1/p') = 1$ and $\chi_n'(t) = (\xi_n'(t))^p$, for $|t| \leq 1/n$. We know that $(\chi_n(u), \partial \Phi_p(u)) \geq 0$ for $u \in W^{1,p}(\Omega)$ since χ_n is monotone, Lipschitz with $\chi_n(0) = 0$, and χ_n' is continuous except at finitely many points on R. Then

$$\begin{aligned} &\left(|u|^{p-1} \operatorname{sgn} u, A_p u\right) \\ &\quad = \lim_{n\to\infty} (\chi_n(u), A_p u) \geq \lim_{n\to\infty} (\chi_n(u), B_{p,q,r} u) \\ &\quad = \lim_{n\to\infty} \int_\Omega \left(1 + \frac{|\nabla u|^p}{\sqrt{1 + |\nabla u|^{2p}}} \right) |\nabla u|^p \chi_n'(u) dx \\ &\qquad + \lambda \lim_{n\to\infty} \int_\Omega |u|^{q-2} u \chi_n(u) dx + \lambda \lim_{n\to\infty} \int_\Omega |u|^{r-2} u \chi_n(u) dx \\ &\quad \geq \lim_{n\to\infty} \int_\Omega |\nabla u|^p \chi_n'(u) dx \\ &\quad \geq \text{const} \cdot \lim_{n\to\infty} \int_\Omega \left| \operatorname{grad}(\xi_n(u)) \right|^p dx \\ &\quad \geq \text{const} \int_\Omega \left| \operatorname{grad}\left(|u|^{2-(2/p)} \operatorname{sgn} u \right) \right|^p dx. \end{aligned} \tag{2.21}$$

We now have from $f = u + \mu A_p u$ that

$$\begin{aligned} &\|f\|_p \left\| |u|^{2-(2/p)} \operatorname{sgn} u \right\|_{p^2/2(p-1)}^{p^2/2(p-1)p'} \\ &\quad \geq \left(|u|^{p-1} \operatorname{sgn} u, f \right) = \left(|u|^{p-1} \operatorname{sgn} u, u \right) + \mu \left(|u|^{p-1} \operatorname{sgn} u, A_p u \right) \\ &\quad \geq \left\| |u|^{2-(2/p)} \operatorname{sgn} u \right\|_{p^2/2(p-1)}^{p^2/2(p-1)} + \mu \cdot \text{const} \cdot \left\| \operatorname{grad} |u|^{2-(2/p)} \operatorname{sgn} u \right\|_p^p, \end{aligned} \tag{2.22}$$

which gives that

$$\left\| |u|^{2-(2/p)} \operatorname{sgn} u \right\|_p^{p/2(p-1)} \leq \left\| |u|^{2-(2/p)} \operatorname{sgn} u \right\|_{p^2/2(p-1)}^{p/2(p-1)} \|f\|_p \leq \text{const}, \tag{2.23}$$

in view of the fact that $p < p^2/2(p-1)$ when $2N/(N+1) < p < 2$ for $N \geq 1$. Again from (2.22), we have $\|\operatorname{grad}(|u|^{2-(2/p)} \operatorname{sgn} u)\|_p \leq$ const. Hence $\{f\}$ bounded in $L^p(\Omega)$ implies that $\{|u|^{2-(2/p)} \operatorname{sgn} u\}$ is bounded in $W^{1,p}(\Omega)$.

We notice that $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^{p^2/2(p-1)}(\Omega)$ when $N \geq 2$ and $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow C_B(\Omega)$ when $N = 1$ by Lemma 1.3, hence$\{|u|^{2-(2/p)} \operatorname{sgn} u\}$ is relatively compact in $L^{p^2/2(p-1)}(\Omega)$. This gives that $\{u\}$ is relatively compact in $L^p(\Omega)$ since the Nemytskii mapping $u \in L^{p^2/2(p-1)}(\Omega) \to |u|^{p/2(p-1)} \operatorname{sgn} u \in L^p(\Omega)$ is continuous.

This completes the proof. □

Remark 2.10. Since $\Phi_p(u + \alpha) = \Phi_p(u)$, for any $u \in W^{1,p}(\Omega)$ and $\alpha \in C_0^\infty(\Omega)$, we have $f \in A_p u$ implies that $f = B_{p,q,r} u$ in the sense of distributions.

Proposition 2.11. *For $f \in L^p(\Omega)$, if there exists $u \in L^p(\Omega)$ such that $f \in A_p u$, then u is the unique solution of* (1.4).

Proof. First, we show that

$$-\operatorname{div}\left[\left(1 + \frac{|\nabla u|^p}{\sqrt{1 + |\nabla u|^{2p}}}\right) |\nabla u|^{p-2} \nabla u\right] + \lambda |u|^{q-2} u + \lambda |u|^{r-2} u = f(x), \quad \text{a.e. } x \in \Omega \tag{2.24}$$

is available.

Now $f \in A_p u$ implies that $f = B_{p,q,r}u + \partial\Phi_p(u)$. For all $\varphi \in C_0^\infty(\Omega)$, by Remark 2.10, we have

$$\begin{aligned}
(\varphi, f) &= (\varphi, B_{p,q,r}u + \partial\Phi_p(u)) \\
&= (\varphi, B_{p,q,r}u) = \int_\Omega \left\langle \left(1 + \frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u, \nabla\varphi \right\rangle dx \\
&\quad + \lambda \int_\Omega |u|^{q-2}u\varphi\, dx + \lambda \int_\Omega |u|^{r-2}u\varphi\, dx \\
&= \int_\Omega -\operatorname{div}\left[\left(1 + \frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u\right]\varphi\, dx \\
&\quad + \lambda \int_\Omega |u|^{q-2}u\varphi\, dx + \lambda \int_\Omega |u|^{r-2}u\varphi\, dx,
\end{aligned} \tag{2.25}$$

which implies that (2.24) is true.

Secondly, we show that

$$-\left\langle \vartheta, \left(1 + \frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u \right\rangle \in \beta_x(u(x)), \quad \text{a.e. } x \in \Gamma. \tag{2.26}$$

We will prove (2.26) under the additional condition $|\beta_x(u)| \leq a|u|^{p/p'} + b(x)$, where $b(x) \in L^{p'}(\Gamma)$ and $a \in R$. Refer to the result of Brezis [11] for the general case.

Now, from (2.24), $f \in A_p u$ implies that $f(x) = -\operatorname{div}[(1+|\nabla u|^p/\sqrt{1+|\nabla u|^{2p}})|\nabla u|^{p-2}\nabla u] + \lambda|u(x)|^{q-2}u(x) + \lambda|u|^{r-2}u \in L^p(\Omega)$. By using Green's Formula, we have that for any $v \in W^{1,p}(\Omega)$,

$$\begin{aligned}
&\int_\Gamma \left\langle \vartheta, \left(1 + \frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u \right\rangle v|_\Gamma\, d\Gamma(x) \\
&\quad = \int_\Omega \operatorname{div}\left[\left(1 + \frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u\right] v\, dx \\
&\quad\quad + \int_\Omega \left\langle \left(1 + \frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u, \nabla v \right\rangle dx.
\end{aligned} \tag{2.27}$$

Then $-\langle \vartheta, (1 + |\nabla u|^p/\sqrt{1+|\nabla u|^{2p}})|\nabla u|^{p-2}\nabla u \rangle \in W^{-(1/p),p'}(\Gamma) = (W^{1/p,p}(\Gamma))^*$, where $W^{1/p,p}(\Gamma)$ is the space of traces of $W^{1,p}(\Omega)$.

Now let the mapping $B : L^p(\Gamma) \to L^{p'}(\Gamma)$ be defined by $Bu = g(x)$, for any $u \in L^p(\Gamma)$, where $g(x) = \beta_x(u(x))$ a.e. on Γ. Clearly, $B = \partial\Psi$ where $\Psi(u) = \int_\Gamma \varphi_x(u(x))d\Gamma(x)$ is a proper, convex, and lower-semicontinuous function on $L^p(\Gamma)$. Now define the mapping $K : W^{1,p}(\Omega) \to L^p(\Gamma)$ by $K(v) = v|_\Gamma$ for any $v \in W^{1,p}(\Omega)$. Then $K^*BK : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ is maximal monotone since both K, B are continuous. Finally, for any $u, v \in W^{1,p}(\Omega)$, we have

$$\begin{aligned}\Psi(Kv) - \Psi(Ku) &= \int_\Gamma \left[\varphi_x(v|_\Gamma(x)) - \varphi_x(u|_\Gamma(x))\right] d\Gamma(x) \\ &\geq \int_\Gamma \beta_x(u|_\Gamma(x))(v|_\Gamma(x) - u|_\Gamma(x))d\Gamma(x) \\ &= (BKu, Kv - Ku) = (K^*BKu, v - u).\end{aligned} \tag{2.28}$$

Hence we get $K^*BK \subset \partial\Phi_p$ and so $K^*BK = \partial\Phi_p$. Therefore, we have

$$-\left\langle \vartheta, \left(1 + \frac{|\nabla u|^p}{\sqrt{1 + |\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u \right\rangle \in \beta_x(u(x)), \quad \text{a.e. on } \Gamma. \tag{2.29}$$

Finally, we will show that u is unique.

If $f \in A_p u$ and $f \in A_p v$, where $u, v \in D(A_p)$. Then

$$0 \leq \left(u - v, B_{p,q,r}u - B_{p,q,r}v\right) = \left(u - v, \partial\Phi_p(v) - \partial\Phi_p(u)\right) \leq 0, \tag{2.30}$$

since $B_{p,q,r}$ is strictly monotone and $\partial\Phi_p$ is maximal monotone, which implies that $u(x) = v(x)$.

This completes the proof. □

Remark 2.12. If $\beta_x \equiv 0$ for any $x \in \Gamma$, then $\partial\Phi_p(u) \equiv 0$, for all $u \in W^{1,p}(\Omega)$.

Proposition 2.13. *If $\beta_x \equiv 0$ for any $x \in \Gamma$, then $\{f \in L^p(\Omega) \mid \int_\Omega f\,dx = 0\} \subset R(A_p)$.*

Proof. We can easily know that $R(B_{p,q,r}) = (W^{1,p}(\Omega))^*$ in view of Lemmas 1.4 and 2.5. Note that for any $f \in L^p(\Omega)$ with $\int_\Omega f\,dx = 0$, the linear function $u \in W^{1,p}(\Omega) \to \int_\Omega fu\,dx$ is an element of $(W^{1,p}(\Omega))^*$. So there exists a $u \in W^{1,p}(\Omega)$ such that

$$\begin{aligned}\int_\Omega fv\,dx = &\int_\Omega \left\langle \left(1 + \frac{|\nabla u|^p}{\sqrt{1 + |\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u, \nabla v \right\rangle dx \\ &+ \lambda \int_\Omega |u|^{q-2}uv\,dx + \lambda \int_\Omega |u|^{r-2}uv\,dx,\end{aligned} \tag{2.31}$$

for any $v \in W^{1,p}(\Omega)$. So $f = A_p u$ in view of Remark 2.12.

This completes the proof. □

Definition 2.14 (see [1, 7]). For $t \in R$, $x \in \Gamma$, let $\beta_x^0(t) \in \beta_x(t)$ be the element with least absolute value if $\beta_x(t) \neq \emptyset$ and $\beta_x^0(t) = \pm\infty$, where $t > 0$ or < 0, respectively, in case $\beta_x(t) = \emptyset$. Finally, let $\beta_\pm(x) = \lim_{t \to \pm\infty} \beta_x^0(t)$ (in the extended sense) for $x \in \Gamma$. $\beta_\pm(x)$ define measurable functions on Γ, in view of our assumptions on β_x.

Proposition 2.15. *Let $f \in L^p(\Omega)$ such that*

$$\int_\Gamma \beta_-(x) d\Gamma(x) < \int_\Omega f\, dx < \int_\Gamma \beta_+(x) d\Gamma(x). \tag{2.32}$$

Then $f \in \operatorname{Int} R(A_p)$.

Proof. Let $f \in L^p(\Omega)$ and satisfy (2.32), by Proposition 2.8, there exists $u_n \in L^p(\Omega)$ such that, for each $n \geq 1$, $f = (1/n)u_n + A_p u_n$. In the same reason as that in [1], we only need to prove that $\|u_n\|_p \leq$ const, for all $n \geq 1$.

Indeed, suppose to the contrary that $1 \leq \|u_n\|_p \to \infty$, as $n \to \infty$. Let $v_n = u_n / \|u_n\|_p$. Let $\varphi : R \to R$ be defined by $\varphi(t) = |t|^p$, $\partial\varphi : R \to R$ be its subdifferential and for $\mu > 0$, $\partial\varphi_\mu : R \to R$ denote the Yosida-approximation of $\partial\varphi$. Let $\theta_\mu : R \to R$ denote the indefinite integral of $[(\partial\varphi_\mu)']^{1/p}$ with $\theta_\mu(0) = 0$ so that $(\theta'_\mu)^p = (\partial\varphi_\mu)'$. In view of Calvert and Gupta [1], we have

$$(\partial\varphi_\mu(v_n), \partial\Phi_p(u_n)) \geq \int_\Gamma \beta_x\left((1 + \mu\partial\varphi)^{-1}(u_n|_\Gamma(x))\right) \times \partial\varphi_\mu(v_n|_\Gamma(x)) d\Gamma(x) \geq 0. \tag{2.33}$$

Now multiplying the equation $f = (1/n)u_n + A_p u_n$ by $\partial\varphi_\mu(v_n)$, we get that

$$(\partial\varphi_\mu(v_n), f) = \left(\partial\varphi_\mu(v_n), \frac{1}{n}u_n\right) + (\partial\varphi_\mu(v_n), B_{p,q,r}u_n) + (\partial\varphi_\mu(v_n), \partial\Phi_p(u_n)). \tag{2.34}$$

Since $\partial\varphi_\mu(0) = 0$, it follows that $(\partial\varphi_\mu(v_n), u_n) \geq 0$. Also, we can know that

$$\begin{aligned}
(\partial\varphi_\mu(v_n), B_{p,q,r}u_n) = {} & \int_\Omega \left\langle \left(1 + \frac{|\nabla u_n|^p}{\sqrt{1 + |\nabla u_n|^{2p}}}\right) |\nabla u_n|^{p-2} \nabla u_n, \nabla v_n \right\rangle (\partial\varphi_\mu)'(v_n) dx \\
& + \lambda \int_\Omega |u_n|^{q-2} u_n \partial\varphi_\mu(v_n) dx + \lambda \int_\Omega |u_n|^{r-2} u_n \partial\varphi_\mu(v_n) dx \\
\geq {} & \int_\Omega \frac{|\nabla u|^p}{\|u_n\|_p} (\partial\varphi_\mu)'(v_n) dx = \|u_n\|_p^{p-1} \int_\Omega \left|\operatorname{grad}(\theta_\mu(v_n))\right|^p dx.
\end{aligned} \tag{2.35}$$

Then we get from (2.33) that

$$\begin{aligned}
& \|u_n\|_p^{p-1} \int_\Omega \left|\operatorname{grad}(\theta_\mu(v_n))\right|^p dx + \int_\Gamma \beta_x\left((1 + \mu\partial\varphi)^{-1}(u_n|_\Gamma(x))\right) \times \partial\varphi_\mu(v_n|_\Gamma(x)) d\Gamma(x) \\
& \leq (\partial\varphi_\mu(v_n), f).
\end{aligned} \tag{2.36}$$

Since $|\partial\varphi_\mu(t)| \le |\partial\varphi(t)|$ for any $t \in R$ and $\mu > 0$, we see from $\|v_n\|_p = 1$ for $n \ge 1$, that $\|\partial\varphi_\mu(v_n)\|_{p'} \le C$, for $\mu > 0$, where C is a constant which does not depend on n or μ and $(1/p) + (1/p') = 1$.

From (2.36), we have

$$\int_\Omega \left|\operatorname{grad}(\theta_\mu(v_n))\right|^p dx \le \frac{C}{\|u_n\|_p^{p-1}}, \quad \text{for } \mu > 0,\ n \ge 1. \tag{2.37}$$

Now we easily know that $(\theta'_\mu)^p = (\partial\varphi_\mu)' \to (\partial\varphi)'$ as $\mu \to 0$ a.e. on R.

Letting $\mu \to 0$, we see from Fatou's lemma and (2.37) that

$$\int_\Omega \left|\operatorname{grad}\left(|v_n|^{2-(2/p)} \operatorname{sgn} v_n\right)\right|^p dx \le \frac{C}{\|u_n\|_p^{p-1}}. \tag{2.38}$$

From (2.38), we know that $|v_n|^{2-(2/p)} \operatorname{sgn} v_n \to k$ (a constant) in $L^p(\Omega)$, as $n \to +\infty$. Next we will show that $k \ne 0$ is in $L^p(\Omega)$ from two aspects.

(i) If $p \ge 2$, since $\||v_n|^{2-(2/p)} \operatorname{sgn} v_n\|_p = \|v_n\|_{2p-2}^{2-(2/p)} \ge \|v_n\|_p^{2-(2/p)} = 1$, it follows that $k \ne 0$ in $L^p(\Omega)$,

(ii) if $2N/(N+1) < p < 2$, $\||v_n|^{2-(2/p)} \operatorname{sgn} v_n\|_p = \|v_n\|_{2p-2}^{2-(2/p)} \ge \|v_n\|_p^{2-(2/p)} = 1$, then $\{|v_n|^{2-(2/p)} \operatorname{sgn} v_n\}$ is bounded in $W^{1,p}(\Omega)$. By Lemma 1.3, $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow C_B(\Omega)$ when $N = 1$ and $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^{p^2/2(p-1)}(\Omega)$, when $N \ge 2$. So $\{|v_n|^{2-(2/p)} \operatorname{sgn} v_n\}$ is relatively compact in $L^{p^2/2(p-1)}(\Omega)$. Then there exists a subsequence of $\{|v_n|^{2-(2/p)} \operatorname{sgn} v_n\}$, for simplicity, we denote it by $\{|v_n|^{2-(2/p)} \operatorname{sgn} v_n\}$, satisfying $|v_n|^{2-(2/p)} \operatorname{sgn} v_n \to g$ in $L^{p^2/2(p-1)}(\Omega)$. Noticing that $p \le p^2/2(p-1)$ when $2N/(N+1) < p < 2$, it follows that $k = g$ a.e. on Ω. Now,

$$\begin{aligned} 1 = \|v_n\|_p^p &= \int_\Omega \left||v_n|^{2-(2/p)} \operatorname{sgn} v_n\right|^{p^2/2(p-1)} dx \\ &\le \text{const} \int_\Omega \left||v_n|^{2-(2/p)} \operatorname{sgn} v_n - g\right|^{p^2/2(p-1)} dx + \text{const}\|g\|_{p^2/2(p-1)}^{p^2/2(p-1)}, \end{aligned} \tag{2.39}$$

it follows that $g \ne 0$ in $L^p(\Omega)$ and then $k \ne 0$ in $L^p(\Omega)$. Assume, now, $k > 0$, we see from (2.36) that

$$\int_\Gamma \beta_x\left((1+\mu\partial\varphi)^{-1}(u_n|_\Gamma(x))\right) \times \partial\varphi_\mu(v_n|_\Gamma(x))d\Gamma(x) \le \left(\partial\varphi_\mu(v_n), f\right). \tag{2.40}$$

Choosing a subsequence so that $u_n|_\Gamma(x) \to +\infty$ a.e. on Γ, we see letting $n \to +\infty$ that $\int_\Gamma \beta_+(x)d\Gamma(x) \le \int_\Omega f(x)dx$, which is a contradiction with (2.32). Similarly, if $k < 0$, it also leads to a contradiction. Thus $f \in \operatorname{int} R(A_p)$.

This completes the proof. □

Proposition 2.16. $A_p + B_1 : L^p(\Omega) \to L^p(\Omega)$ *is m-accretive and has a compact resolvent.*

Proof. Using a theorem in Corduneanu [12], we know that $A_p + B_1 : L^p(\Omega) \to L^p(\Omega)$ is m-accretive.

To show that $A_p + B_1 : L^p(\Omega) \to L^p(\Omega)$ has a compact resolvent, we only need to prove that if $w \in A_p u + B_1 u$ with $\{w\}$ and $\{u\}$ being bounded in $L^p(\Omega)$, then $\{u\}$ is relatively compact in $L^p(\Omega)$. Now we discuss it from two aspects.

(i) If $p \geq 2$, since

$$\begin{aligned}\int_\Omega |\nabla u|^p dx &\leq (u, B_{p,q,r}u) = (u, A_p u) - (u, \partial\Phi_p(u)) \\ &\leq (u, A_p u) + (u, B_1 u) = (u, w) \leq \|u\|_p \|u\|_{p'} \leq \text{const},\end{aligned} \tag{2.41}$$

it follows that $\{u\}$ is bounded in $W^{1,p}(\Omega)$, where $(1/p) + (1/p') = 1$. Then $\{u\}$ is relatively compact in $L^p(\Omega)$ since $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$;

(ii) if $2N/(N+1) < p < 2$, since $w \in A_p u + B_1 u$ with $\{w\}$ and $\{u\}$ being bounded in $L^p(\Omega)$, we have $w - B_1 u \in A_p u$ with $\{w - B_1 u\}$ and $\{u\}$ being bounded in $L^p(\Omega)$ which gives that $\{u\}$ is relatively compact in $L^p(\Omega)$ since A_p is m-accretive by Proposition 2.8 and has a compact resolvent by Lemma 2.9.

This completes the proof. □

Theorem 2.17. *Let $f \in L^p(\Omega)$ be such that*

$$\int_\Gamma \beta_-(x) d\Gamma(x) + \int_\Omega g_-(x) dx < \int_\Omega f(x) dx < \int_\Gamma \beta_+(x) d\Gamma(x) + \int_\Omega g_+(x) dx, \tag{2.42}$$

then (1.4) *has a unique solution in $L^p(\Omega)$, where $2N/(N+1) < p < +\infty$ and $N \geq 1$.*

Proof . We want to use Theorem 1.9 to finish our proof. From Propositions 1.7, 2.2, 2.8, and 2.16, we can see that all of the conditions in Theorem 1.9 are satisfied. It then suffices to show that $f \in \text{int}[R(A_p) + R(B_1)]$ which ensures that $f \in R(A_p + B_1 + B_2)$. Thus Proposition 2.11 tells us (1.4) has a unique solution in $L^p(\Omega)$.

Using the similar methods as those in [2, 4, 7], by dividing it into two cases and using Propositions 2.13 and 2.15, respectively, we know that $f \in \text{int}[R(A_p) + R(B_1)]$.

This completes the proof. □

Remark 2.18. Compared to the work done in [1–7], not only the existence of the solution of (1.4) is obtained but also the uniqueness of the solution is obtained.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (Grant no. 11071053), the Natural Science Foundation of Hebei Province (Grant no. A2010001482), the Key Project of Science and Research of Hebei Education Department (Grant no. ZH2012080), and the Youth Project of Science and Research of Hebei Education Department (Grant no. Q2012054).

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