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Research Article

Univalence Conditions Related to a General Integral Operator

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We consider a general integral operator based on two types of analytic functions, namely, regular functions and, respectively, functions having a positive real part. Some univalence conditions for this integral operator are obtained.

1. Introduction

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{i=2}^{\infty} a_i z^i, \tag{1.1}$$

which are analytic in the open unit disk, $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. Also, let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of univalent functions, in the open unit disk. We denote by $\mathcal{P}$, the class of functions p which are analytic in $\mathcal{U}$, $\operatorname{Re} p(z) > 0$, for all $z \in \mathcal{U}$ and $p(0) = 1$, the so-called Caratheodory functions.

In [1], Pescar introduced and studied the following integral operator:

$$E_n(z) = \int_0^z \prod_{j=1}^{n} \left(\frac{f_j(u)}{u} \right)^{\alpha_j} \left(g_j(u) \right)^{\beta_j} du, \tag{1.2}$$

where α_j, β_j are complex numbers, $f_j \in \mathcal{A}$, $g_j \in \mathcal{P}$, $j = \overline{1, n}$.

In this paper, we generalize this integral operator, by considering the general integral operator defined as follows:

$$K_n(z) = \left(\delta \int_0^z u^{\delta-1} \prod_{j=1}^{n} \left(\frac{f_j(u)}{u} \right)^{\alpha_j} (g_j(u))^{\beta_j} du \right)^{1/\delta}, \tag{1.3}$$

where $\delta, \alpha_j, \beta_j$ are complex numbers, $\delta \neq 0$, $f_j \in \mathcal{A}$, $g_j \in \mathcal{P}$, $j = \overline{1,n}$.

Remarks. This integral operator extends many other integral operators from related works on this field as those given by Srivastava, Mocanu, Owa, Pescar, Orhan, Breaz, and others. Using this new operator and the related univalence conditions that are going to be proved here, one can study some other already known operators in a unified perspective. Thus, for different particular cases of the parameters δ, α_j, β_j, our integral operator K_n is shown to be an extension of the following integral operators.

(i) For $\delta = 1$, the integral operator K_n is the operator E_n from (1.2), introduced by Pescar in [1].

(ii) For $n = 1$, $\alpha_j = \delta$, $\beta_j = 0$, the integral operator K_n was studied in [2], by Miller and Mocanu.

(iii) For $\beta_j = 0$, $j = \overline{1,n}$, the integral operator K_n was studied in [3], by Pescar and Breaz.

(iv) For $\beta_j = 0$, $j = \overline{1,n}$, $\alpha_j = \alpha - 1$, and $\delta = n(\alpha - 1) + 1$, the integral operator K_n was studied in [4], by Breaz et al. and also in [5], by Srivastava et al.

(v) For $\delta = 1$, $\beta_j = 0$, $j = \overline{1,n}$, we get the integral operator defined by D. Breaz and N. Breaz, in the paper [6].

(vi) For $\delta = 1$, $\alpha_j = 0$, $\beta_j > 0$, and $g_j = f_j'$, the operator was studied by Breaz et al., in the paper [7].

More precisely, if we are interested to study various properties of two or more different operators reminded above (and other similar operators which are not mentioned here), we can do this in an integrated manner by simply allowing the parameters involved in the definition, to be more general and consequently by studying only one operator, having the form (1.3). In this paper, we will study some univalence criteria for this new operator.

The following known results will be used in order to prove our results.

Lemma 1.1 (see [8]). *Let γ be a complex number,* $\operatorname{Re}\gamma > 0$ *and $f \in \mathcal{A}$. If*

$$\frac{1 - |z|^{2\operatorname{Re}\gamma}}{\operatorname{Re}\gamma} \left| \frac{z f''(z)}{f'(z)} \right| \leq 1, \tag{1.4}$$

for all $z \in \mathcal{U}$, then for any complex number δ, $\operatorname{Re}\delta \geq \operatorname{Re}\gamma$,

$$F_\delta(z) = \left[\delta \int_0^z u^{\delta-1} f'(u) du \right]^{1/\delta} \in \mathcal{S}. \tag{1.5}$$

Lemma 1.2 (see [9]). *Let f be a regular function in the open disk $\mathcal{U}_R = \{z \in \mathbb{C} : |z| < R\}$ with $|f(z)| < M$, M fixed. If f has in $z = 0$ one zero with multiplicity $\geq m$, then*

$$\left|f(z)\right| \leq \frac{M}{R^m}|z|^m, \quad (z \in \mathcal{U}_R), \tag{1.6}$$

the equality for $(z \neq 0)$ can hold only if

$$f(z) = e^{i\theta}\frac{M}{R^m}z^m, \tag{1.7}$$

where θ is constant.

Lemma 1.3 (see [10]). *For each $f \in \mathcal{S}$,*

$$\frac{1-r}{1+r} \leq \left|\frac{zf'(z)}{f(z)}\right| \leq \frac{1+r}{1-r}, \quad |z| = r < 1. \tag{1.8}$$

Lemma 1.4 (see [11]). *If $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ is regular in $|z| < 1$ and $\operatorname{Re}(g(z)) > 0$, then*

$$\left|\frac{zg'(z)}{g(z)}\right| \leq \frac{2|z|}{1-|z|^2}. \tag{1.9}$$

Also, for the statement of our main results, we need to define the following classes:

$$\begin{aligned} \mathcal{A}_M &= \left\{f \in \mathcal{A} : \left|\frac{zf'(z)}{f(z)} - 1\right| \leq M, M \geq 1\right\}, \\ \mathcal{P}_L &= \left\{f \in \mathcal{P} : \left|\frac{zf'(z)}{f(z)}\right| \leq L, L > 0\right\}. \end{aligned} \tag{1.10}$$

2. Main Results

Theorem 2.1. *Let δ, α_j, β_j be complex numbers, $\operatorname{Re}\delta \geq 1$, M_j, L_j positive real numbers, $M_j \geq 1$, $j = \overline{1,n}$, and the functions $f_j \in \mathcal{A}_{M_j}$, $g_j \in \mathcal{P}_{L_j}$, $j = \overline{1,n}$.*

If

$$\sum_{j=1}^{n}\left[|\alpha_j|M_j + |\beta_j|L_j\right] \leq \frac{3\sqrt{3}}{2}, \tag{2.1}$$

then the general integral operator K_n, defined by (1.3)*, is in the class $\mathcal{S}$.*

Proof. We consider the regular function

$$h_n(z) = \int_0^z \left(\frac{f_1(u)}{u}\right)^{\alpha_1} \cdots \left(\frac{f_n(u)}{u}\right)^{\alpha_n} \cdot (g_1(u))^{\beta_1} \cdots (g_n(u))^{\beta_n} du. \tag{2.2}$$

After some calculus, we have

$$\left(1-|z|^2\right)\left|\frac{zh_n''(z)}{h_n'(z)}\right| \le \left(1-|z|^2\right)\sum_{j=1}^{n}\left[|\alpha_j|\left|\frac{zf_j'(z)}{f_j(z)}-1\right| + |\beta_j|\left|\frac{zg_j'(z)}{g_j(z)}\right|\right], \tag{2.3}$$

for all $z \in \mathcal{U}$.

Since $f_j \in A_{M_j}$, by applying Lemma 1.2 for $m = 1$, we get

$$\left|\frac{zf_j'(z)}{f_j(z)}-1\right| \le M_j|z|, \quad \left(j=\overline{1,n}; z \in \mathcal{U}\right). \tag{2.4}$$

Also since $g_j \in P_{L_j}$, from Lemma 1.2, we have

$$\left|\frac{zg_j'(z)}{g_j(z)}\right| \le L_j|z|, \quad \left(j=\overline{1,n}; z \in \mathcal{U}\right). \tag{2.5}$$

If we put these last two inequalities in (2.3), together with the inequality from the hypothesis, we get

$$\left(1-|z|^2\right)\left|\frac{zh_n''(z)}{h_n'(z)}\right| \le \left(1-|z|^2\right)|z| \cdot \frac{3\sqrt{3}}{2}. \tag{2.6}$$

Now we take into account the fact that

$$\max_{|z|<1}\left[\left(1-|z|^2\right)|z|\right] = \frac{2}{3\sqrt{3}}, \tag{2.7}$$

thus obtaining

$$\left(1-|z|^2\right)\left|\frac{zh_n''(z)}{h_n'(z)}\right| \le 1, \quad (z \in \mathcal{U}). \tag{2.8}$$

Further from Lemma 1.1, for $\operatorname{Re}\delta \ge \operatorname{Re}\gamma = 1$, it is obvious that $K_n \in \mathcal{S}$. □

Corollary 2.2. *Let δ, α_j be complex numbers, $\operatorname{Re}\delta \ge 1$, M_j positive real numbers, $M_j \ge 1$, $f_j \in \mathcal{A}_{M_j}$, $j = \overline{1,n}$. If*

$$\sum_{j=1}^{n}\left[|\alpha_j| M_j\right] \le \frac{3\sqrt{3}}{2} \tag{2.9}$$

then the function

$$K_n(z) = \left[\delta \int_0^z u^{\delta-1} \prod_{j=1}^{n} \left(\frac{f_j(u)}{u}\right)^{\alpha_j} du\right]^{1/\delta}, \tag{2.10}$$

is in the class $\mathcal{S}$.

Proof. In Theorem 2.1, we take $\beta_j = 0,\ j = \overline{1,n}$. □

Corollary 2.3. *Let δ, β_j be complex numbers, $\operatorname{Re}\delta \geq 1$, L_j positive real numbers and $g_j \in \mathcal{P}_{L_j}$, $j = \overline{1,n}$. If*

$$\sum_{j=1}^{n} \left[|\beta_j| L_j\right] \leq \frac{3\sqrt{3}}{2}, \tag{2.11}$$

then the function

$$K_n(z) = \left[\delta \int_0^z u^{\delta-1} \prod_{j=1}^{n} (g_j(u))^{\beta_j} du\right]^{1/\delta} \tag{2.12}$$

belongs to the class $\mathcal{S}$.

Proof. In Theorem 2.1, we take $\alpha_j = 0,\ j = \overline{1,n}$. □

Theorem 2.4. *Let γ, δ, α_j, β_j be complex numbers, $j = \overline{1,n}$, $\operatorname{Re}\delta \geq \operatorname{Re}\gamma > 0$, and the functions $f_j \in \mathcal{A}_\mu$, $g_j \in \mathcal{P}_\mu$, $\mu \geq 1$, with*

$$\mu = \frac{(2\operatorname{Re}\gamma + 1)^{1+(1/2\operatorname{Re}\gamma)}}{2}. \tag{2.13}$$

If

$$\sum_{j=1}^{n} \left[|\alpha_j| + |\beta_j|\right] \leq 1, \tag{2.14}$$

then the general integral operator K_n belongs to the class $\mathcal{S}$.

Proof. Let consider again the regular function:

$$h_n(z) = \int_0^z \left(\frac{f_1(u)}{u}\right)^{\alpha_1} \cdots \left(\frac{f_n(u)}{u}\right)^{\alpha_n} \cdot (g_1(u))^{\beta_1} \cdots (g_n(u))^{\beta_n} du. \tag{2.15}$$

After some calculus, we have

$$\frac{1-|z|^{2\operatorname{Re}\gamma}}{\operatorname{Re}\gamma}\left|\frac{zh_n''(z)}{h_n'(z)}\right| \le \frac{1-|z|^{2\operatorname{Re}\gamma}}{\operatorname{Re}\gamma}\sum_{j=1}^{n}\left[\left|\alpha_j\right|\left|\frac{zf_j'(z)}{f_j(z)}-1\right|+\left|\beta_j\right|\left|\frac{zg_j'(z)}{g_j(z)}\right|\right], \quad (z\in\mathcal{U}). \tag{2.16}$$

Since $f_j \in A_\mu$ and $g_j \in P_\mu$, for all $j=\overline{1,n}$, from Lemma 1.2, we obtain

$$\begin{aligned}\left|\frac{zf_j'(z)}{f_j(z)}-1\right| &\le \mu|z|, \quad \left(j=\overline{1,n}; z\in\mathcal{U}\right),\\ \left|\frac{zg_j'(z)}{g_j(z)}\right| &\le \mu|z|, \quad \left(j=\overline{1,n}; z\in\mathcal{U}\right).\end{aligned} \tag{2.17}$$

Using these last two inequalities, the inequality from the hypothesis, and the fact that

$$\max_{|z|<1}\left[\frac{\left(1-|z|^{2\operatorname{Re}\gamma}\right)|z|}{\operatorname{Re}\gamma}\right]=\frac{1}{\mu}, \tag{2.18}$$

from (2.16) we get

$$\frac{1-|z|^{2\operatorname{Re}\gamma}}{\operatorname{Re}\gamma}\left|\frac{zh_n''(z)}{h_n'(z)}\right| \le 1, \tag{2.19}$$

for all $z\in\mathcal{U}$.

Hence, by Lemma 1.1, we have that $K_n \in \mathcal{S}$. □

Corollary 2.5. *Let γ, δ, α_j be complex numbers, $\operatorname{Re}\delta \ge \operatorname{Re}\gamma > 0$, and $f_j \in \mathcal{A}_\mu$, $j=\overline{1,n}$, $\mu \ge 1$, with*

$$\mu=\frac{\left(2\operatorname{Re}\gamma+1\right)^{1+(1/2\operatorname{Re}\gamma)}}{2}. \tag{2.20}$$

If

$$\sum_{j=1}^{n}\left|\alpha_j\right| \le 1, \tag{2.21}$$

then the integral operator defined by

$$K_n(z)=\left[\delta\int_0^z u^{\delta-1}\prod_{j=1}^{n}\left(\frac{f_j(u)}{u}\right)^{\alpha_j}du\right]^{1/\delta}, \tag{2.22}$$

is in the class $\mathcal{S}$.

Proof. In Theorem 2.4, we take $\beta_1 = \cdots = \beta_n = 0$. □

Corollary 2.6. *Let γ, δ, β_j be complex numbers, $j = \overline{1,n}$, $\operatorname{Re}\delta \geq \operatorname{Re}\gamma > 0$, and $g_j \in \mathcal{P}_\mu$, with*

$$\mu = \frac{(2\operatorname{Re}\gamma + 1)^{1+(1/2\operatorname{Re}\gamma))}}{2}. \tag{2.23}$$

If

$$\sum_{j=1}^{n} |\beta_j| \leq 1, \tag{2.24}$$

then the integral operator defined by

$$K_n(z) = \left[\delta \int_0^z u^{\delta-1} \prod_{j=1}^{n} (g_j(u))^{\beta_j} du\right]^{1/\delta}, \tag{2.25}$$

is in the class $\mathcal{S}$.

Proof. We put in Theorem 2.4, $\alpha_1 = \cdots = \alpha_n = 0$. □

Theorem 2.7. *Let $\gamma, \delta, \alpha_j, \beta_j$ be complex numbers, $j = \overline{1,n}$, $\operatorname{Re}\delta \geq \operatorname{Re}\gamma > 0$ and $f_j \in \mathcal{S}$, $g_j \in \mathcal{P}$. If*

$$2\sum_{j=1}^{n} |\alpha_j| + \sum_{j=1}^{n} |\beta_j| \leq \min\left\{\frac{\operatorname{Re}\gamma}{2}, \frac{1}{2}\right\}, \tag{2.26}$$

then the general integral operator $K_n \in \mathcal{S}$.

Proof. We consider the same regular function h_n as in the proof of the previous theorems, and after some calculus we get

$$\frac{1-|z|^{2\operatorname{Re}\gamma}}{\operatorname{Re}\gamma}\left|\frac{zh_n''(z)}{h_n'(z)}\right| \leq \frac{1-|z|^{2\operatorname{Re}\gamma}}{\operatorname{Re}\gamma}\sum_{j=1}^{n}\left[|\alpha_j|\left(\left|\frac{zf_j'(z)}{f_j(z)}\right|+1\right)+|\beta_j|\left|\frac{zg_j'(z)}{g_j(z)}\right|\right], \tag{2.27}$$

for all $z \in \mathcal{U}$.

Since $f_j \in \mathcal{S}$, from Lemma 1.3, we get

$$\left|\frac{zf_j'(z)}{f_j(z)}\right| \leq \frac{1+|z|}{1-|z|}, \tag{2.28}$$

$j = \overline{1,n}$, $z \in \mathcal{U}$, and since $g_j \in P$, from Lemma 1.4, we get

$$\left|\frac{zg_j'(z)}{g_j(z)}\right| \leq \frac{2|z|}{1-|z|^2} \tag{2.29}$$

$j = \overline{1,n}$, $z \in \mathcal{U}$.

Now if we use (2.28) and (2.29) in (2.27), we obtain

$$\frac{1-|z|^{2\operatorname{Re}\gamma}}{\operatorname{Re}\gamma}\left|\frac{zh_n''(z)}{h_n'(z)}\right| \leq \frac{1-|z|^{2\operatorname{Re}\gamma}}{\operatorname{Re}\gamma}\left[\frac{2}{1-|z|}\sum_{j=1}^{n}|\alpha_j| + \frac{2|z|}{1-|z|^2}\sum_{j=1}^{n}|\beta_j|\right], \tag{2.30}$$

for all $z \in \mathcal{U}$.

We consider two cases.

(1) If $\min\{\operatorname{Re}\gamma/2, 1/2\} = \operatorname{Re}\gamma/2$, we have

$$1-|z|^{2\operatorname{Re}\gamma} \leq 1-|z|^2 \tag{2.31}$$

and further if we use this inequality together with the inequality from the hypothesis, from (2.30), we get

$$\frac{1-|z|^{2\operatorname{Re}\gamma}}{\operatorname{Re}\gamma}\left|\frac{zh_n''(z)}{h_n'(z)}\right| \leq 1, \tag{2.32}$$

for all $z \in \mathcal{U}$, and with Lemma 1.1 the proof is complete.

(2) If $\min\{\operatorname{Re}\gamma/2, 1/2\} = 1/2$, we obtain

$$\frac{1-|z|^{2\operatorname{Re}\gamma}}{\operatorname{Re}\gamma} \leq 1-|z|^2, \tag{2.33}$$

$z \in \mathcal{U}$ and further if we put this last inequality in (2.30) we get

$$\frac{1-|z|^{2\operatorname{Re}\gamma}}{\operatorname{Re}\gamma}\left|\frac{zh_n''(z)}{h_n'(z)}\right| \leq 4\sum_{j=1}^{n}|\alpha_j| + 2\sum_{j=1}^{n}|\beta_j|, \tag{2.34}$$

for all $z \in \mathcal{U}$.

Now by applying first the inequality condition from the hypothesis and then Lemma 1.1 for h_n, the proof is complete. □

Corollary 2.8. *Let γ, δ, α_j be complex numbers, $j = \overline{1,n}$, $\operatorname{Re}\delta \geq \operatorname{Re}\gamma > 0$ and $f_j \in \mathcal{S}$. If*

$$\sum_{j=1}^{n}|\alpha_j| \leq \min\left\{\frac{\operatorname{Re}\gamma}{4}, \frac{1}{4}\right\}, \tag{2.35}$$

then the integral operator

$$K_n(z) = \left[\delta \int_0^z u^{\delta-1} \prod_{j=1}^{n} \left(\frac{f_j(u)}{u}\right)^{\alpha_j} du\right]^{1/\delta} \tag{2.36}$$

is in the class $\mathcal{S}$.

Proof. In Theorem 2.7, we take $\beta_1 = \cdots = \beta_n = 0$. □

Corollary 2.9. *Let* γ, δ, β_j *be complex numbers,* $j = \overline{1,n}$, $\operatorname{Re}\delta \geq \operatorname{Re}\gamma > 0$ *and* $g_j \in \mathcal{P}$. *If*

$$\sum_{j=1}^{n} |\beta_j| \leq \min\left\{\frac{\operatorname{Re}\gamma}{2}, \frac{1}{2}\right\} \tag{2.37}$$

then the integral operator

$$K_n(z) = \left[\delta \int_0^z u^{\delta-1} \prod_{j=1}^{n} (g_j(u))^{\beta_j} du\right]^{1/\delta} \tag{2.38}$$

is in the class $\mathcal{S}$.

Proof. In Theorem 2.7 we take $\alpha_1 = \cdots = \alpha_n = 0$. □

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References

[1] V. Pescar, "On an integral operator," *Bulletin of the Transilvania University of Braşov, Series 3*, vol. 4(53), no. 2, pp. 63–71, 2011.
[2] S. S. Miller and P. T. Mocanu, *Differential Subordinations: Theory and Applications*, vol. 225 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 2000.
[3] V. Pescar and V. D. Breaz, *The Univalence of Integral Operators, Monograph*, Marin Drinov Academic Publishing House, Sofia, Bulgaria, 2008.
[4] D. Breaz, N. Breaz, and H. M. Srivastava, "An extension of the univalent condition for a family of integral operators," *Applied Mathematics Letters*, vol. 22, no. 1, pp. 41–44, 2009.
[5] H. M. Srivastava, E. Deniz, and H. Orhan, "Some general univalence criteria for a family of integral operators," *Applied Mathematics and Computation*, vol. 215, no. 10, pp. 3696–3701, 2010.
[6] D. Breaz and N. Breaz, "Two integral operators," *Studia Universitatis Babeş-Bolyai, Mathematica*, vol. 47, no. 3, pp. 13–19, 2002.
[7] D. Breaz, S. Owa, and N. Breaz, "A new integral univalent operator," *Acta Universitatis Apulensis, Mathematics, Informatics*, no. 16, pp. 11–16, 2008.
[8] N. N. Pascu, "An improvement of Becker's univalence criterion," in *Proceedings of the Commemorative Session: Simion Stoïlow (Braşov, 1987)*, pp. 43–48, University of Braşov, Braşov, Romania, 1987.

[9] O. Mayer, The Functions Theory of One Variable Complex, Bucuresti, Romania, 1981.
[10] P. L. Duren, *Univalent Functions—A Series of Comprehensive Studies in Mathematics*, vol. 259 of *Grundlehren der Mathematischen Wissenschaften*, Springer, New York, NY, USA, 1983.
[11] P. T. Mocanu, T. Bulboacă, and G. Ş. Sălăgean, *Teoria geometrică a funcţiilor univalente*, Casa Cărţii de Ştiinţă, Cluj-Napoca, Romania, 2nd edition, 2006.