Research Article

A Higher-Order Hardy-Type Inequality in Anisotropic Sobolev Spaces

Paolo Secchi

Dipartimento di Matematica, Facoltà di Ingegneria, Università di Brescia, Via Valotti 9, 25133 Brescia, Italy

Correspondence should be addressed to Paolo Secchi, paolo.secchi@ing.unibs.it

We prove a higher-order inequality of Hardy type for functions in anisotropic Sobolev spaces that vanish at the boundary of the space domain. This is an important calculus tool for the study of initial-boundary-value problems of symmetric hyperbolic systems with characteristic boundary.

1. Notations and Main Result

For $n \geq 2$, let $\mathbb{R}^n_+$ denote the n-dimensional positive half-space

$$\mathbb{R}^n_+ := \left\{ x = (x_1, x'), \ x_1 > 0, \ x' := (x_2, \ldots, x_n) \in \mathbb{R}^{n-1} \right\}. \tag{1.1}$$

Let $\sigma \in C^\infty(\mathbb{R}_+)$ be a function such that $\sigma(x_1) = x_1$ close to $x_1 = 0$, and $\sigma(x_1) = 1$ for $x_1 \geq 1$. For $j = 1, 2, \ldots, n$, we set

$$Z_1 := \sigma(x_1)\partial_1, \qquad Z_j := \partial_j, \quad \text{for } j \geq 2. \tag{1.2}$$

Then, for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, the *conormal* derivative Z^α is defined by

$$Z^\alpha := Z_1^{\alpha_1} \cdots Z_n^{\alpha_n}. \tag{1.3}$$

For every positive integer m the *anisotropic Sobolev space* $H^m_*(\mathbb{R}^n_+)$ is defined as

$$H^m_*(\mathbb{R}^n_+) := \left\{ w \in L^2(\mathbb{R}^n_+) : Z^\alpha \partial_1^k w \in L^2(\mathbb{R}^n_+), \ |\alpha| + 2k \leq m \right\}. \tag{1.4}$$

In $H_*^m(\mathbb{R}_+^n)$ we introduce the norm

$$\|w\|^2_{H_*^m(\mathbb{R}_+^n)} := \sum_{|\alpha|+2k\leq m} \left\|Z^\alpha \partial_1^k w\right\|^2_{L^2(\mathbb{R}_+^n)}. \tag{1.5}$$

The space $H_*^m(\mathbb{R}_+^n)$, endowed with its norm (1.5) is a Hilbert space. We also introduce a second anisotropic Sobolev space. For every positive integer m, the space $H_{**}^m(\mathbb{R}_+^n)$ is defined as

$$H_{**}^m(\mathbb{R}_+^n) := \left\{ w \in L^2(\mathbb{R}_+^n) : Z^\alpha \partial_1^k w \in L^2(\mathbb{R}_+^n), \ |\alpha| + 2k \leq m+1, \ |\alpha| \leq m \right\}. \tag{1.6}$$

In particular, $H_{**}^1(\Omega) = H^1(\Omega)$. In $H_{**}^m(\mathbb{R}_+^n)$, we introduce the natural norm

$$\|w\|^2_{H_{**}^m(\mathbb{R}_+^n)} := \sum_{|\alpha|+2k\leq m+1, |\alpha|\leq m} \left\|Z^\alpha \partial_1^k w\right\|^2_{L^2(\mathbb{R}_+^n)}. \tag{1.7}$$

The space $H_{**}^m(\mathbb{R}_+^n)$, endowed with its norm (1.7) is a Hilbert space. For the sake of convenience we also set $H_*^0(\mathbb{R}_+^n) = H_{**}^0(\mathbb{R}_+^n) = L^2(\mathbb{R}_+^n)$. We observe that

$$H^m(\mathbb{R}_+^n) \hookrightarrow H_{**}^m(\mathbb{R}_+^n) \hookrightarrow H_*^m(\mathbb{R}_+^n) \subset H_{loc}^m(\mathbb{R}_+^n), \tag{1.8}$$

$$H_*^m(\mathbb{R}_+^n) \hookrightarrow H^{[m/2]}(\mathbb{R}_+^n), \qquad H_{**}^m(\mathbb{R}_+^n) \hookrightarrow H^{[(m+1)/2]}(\mathbb{R}_+^n), \tag{1.9}$$

where $[\cdot]$ denotes the integer part (except for $H_{loc}^m(\mathbb{R}_+^n)$, all imbeddings are continuous).

The anisotropic spaces H_*^m, H_{**}^m are the natural function spaces for the study of initial-boundary-value problems of symmetric hyperbolic systems with characteristic boundary, see [1–6]. In fact, for such problems, the full regularity (i.e., solvability in the usual Sobolev spaces H^m) cannot be expected generally because of the possible loss of derivatives in the normal direction to the characteristic boundary, see [7, 8]. The introduction of the anisotropic Sobolev spaces H_*^m, H_{**}^m is motivated by the observation that the one-order gain of normal differentiation should be compensated by two-order loss of conormal differentiation.

The equations of ideal magnetohydrodynamics provide an important example of ill-posedness in Sobolev spaces H^m, see [7]. Application to MHD of H_*^m and H_{**}^m spaces may be found in [9–13]. For an extensive study of such spaces we refer the reader to [2, 3, 14, 15] and references therein. Function spaces of this type have also been considered in [16, 17].

The purpose of this note is the proof of the following Theorems 1.1 and 1.2. These results are an important calculus tool in the use of the anisotropic spaces H_*^m, H_{**}^m, and accordingly for the study of initial-boundary-value problems of symmetric hyperbolic systems with characteristic boundary. Typically, in such problems one has to deal with terms of the form $A\partial_1 U$, where A is a real $d \times d$ matrix-valued function, and U is a vector function with d components. The matrix A admits the decomposition

$$A = A_1 + A_2, \quad A_1 := \begin{pmatrix} A^{I,I} & 0 \\ 0 & 0 \end{pmatrix}, \quad A_{2|x_1=0} = 0, \tag{1.10}$$

with $A^{I,I}$ invertible in a neighborhood of the boundary $\{x_1 = 0\}$. Hence, one may write

$$A_2\partial_1 U = HZ_1 U, \tag{1.11}$$

where $H(x) = \sigma(x_1)^{-1}A_2(x)$, and looks for an estimate of HZ_1U in H_*^m, H_{**}^m, as sharp as possible. Given suitable estimates for the product of functions, the problem is then the estimate of H in H_*^m and H_{**}^m. This motivates the following results.

Theorem 1.1. *Let $m \geq 2$. Let $u \in H_*^m(\mathbb{R}_+^n) \cap H_0^1(\mathbb{R}_+^n)$ be a function, and let H be defined by*

$$H(x_1, x') = \frac{u(x_1, x')}{\sigma(x_1)}. \tag{1.12}$$

Then

$$\|H\|_{H_*^{m-2}(\mathbb{R}_+^n)} \leq C\|u\|H_*^m(\mathbb{R}_+^n). \tag{1.13}$$

Proof. For all integers $m \geq 1$, the space $C_{(0)}^\infty(\mathbb{R}_+^n)$ ($C_{(0)}^\infty(\mathbb{R}_+^n)$ denotes the set of restriction to $\mathbb{R}_+^n$ of functions in $C_0^\infty(\mathbb{R}_+^n)$) is dense in $H_*^m(\mathbb{R}_+^n)$, see [4]. Hence, without loss of generality, we may assume that u is supported in a small neighborhood of $x_1 = 0$ where $\sigma(x_1) = x_1$. For the proof of the theorem we use an induction argument somehow inspired from [18].

The case $m = 2$ follows from the classical Hardy inequality, see [19]. Given any $x' \in \mathbb{R}^{n-1}$, the Hardy inequality yields

$$\int_0^\infty \left|\frac{u(x_1, x')}{x_1}\right|^2 dx_1 \leq 4\int_0^\infty |\partial_1 u(x_1, x')|^2 dx_1, \quad \forall u \in H_0^1(\mathbb{R}_+^n). \tag{1.14}$$

Integrating in x' and using (1.9) with $m = 2$ we get

$$\left\|\frac{u}{x_1}\right\|_{L^2(\mathbb{R}_+^n)} \leq 2\|u\|_{H^1(\mathbb{R}_+^n)} \leq C\|u\|_{H_*^2(\mathbb{R}_+^n)}. \tag{1.15}$$

Let us now assume that inequality (1.13) holds for a given $m \geq 2$, and suppose that $u \in H_*^{m+1}(\mathbb{R}_+^n) \cap H_0^1(\mathbb{R}_+^n)$. A simple computation shows that for $k \in \mathbb{N}$,

$$\partial_1^k\left(\frac{u}{x_1}\right) = \frac{f}{x_1^{k+1}}, \tag{1.16}$$

with

$$f = \sum_{h=0}^k \binom{k}{h} \partial_1^{k-h} u\, h!(-1)^h x_1^{k-h}. \tag{1.17}$$

From its definition, we see that $f = 0$ for $x_1 = 0$. Next, we obtain the identity

$$\partial_1 f = \sum_{h=0}^k \binom{k}{h} \partial_1^{k+1-h} u\, h!(-1)^h x_1^{k-h} + \sum_{h=0}^{k-1} \binom{k}{h} \partial_1^{k-h} u\, h!(-1)^h x_1^{k-h-1}(k-h)$$

$$= \partial_1^{k+1} u\, x_1^k + \sum_{h=1}^k \binom{k}{h} \partial_1^{k+1-h} u\, h!(-1)^h x_1^{k-h} + \sum_{h=0}^{k-1} \binom{k}{h+1} \partial_1^{k-h} u\,(h+1)!(-1)^h x_1^{k-h-1} \tag{1.18}$$

$$= \partial_1^{k+1} u\, x_1^k.$$

We deduce from (1.18) that

$$f(x_1, x') = \int_0^{x_1} \partial_1^{k+1} u(y_1, x') y_1^k dy_1, \tag{1.19}$$

which by substitution in (1.16) yields the identity

$$\partial_1^k \left(\frac{u}{x_1} \right)(x_1, x') = \frac{\int_0^{x_1} \partial_1^{k+1} u(y_1, x') y_1^k dy_1}{x_1^{k+1}}. \tag{1.20}$$

Given any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, with $\alpha_1 = 0$, we also get

$$Z^\alpha \partial_1^k \left(\frac{u}{x_1} \right)(x) = \frac{\int_0^{x_1} Z^\alpha \partial_1^{k+1} u(y_1, x') y_1^k dy_1}{x_1^{k+1}}, \tag{1.21}$$

from which it readily follows that

$$\left| Z^\alpha \partial_1^k \left(\frac{u}{x_1} \right)(x) \right| \leq \frac{\int_0^{x_1} \left| Z^\alpha \partial_1^{k+1} u(y_1, x') \right| dy_1}{x_1}. \tag{1.22}$$

Setting

$$g(x_1, x') = \int_0^{x_1} \left| Z^\alpha \partial_1^{k+1} u(y_1, x') \right| dy_1 \tag{1.23}$$

the Hardy inequality yields

$$\int_0^\infty \left| \frac{g}{x_1}(x_1, x') \right|^2 dx_1 \leq 4 \int_0^\infty |\partial_1 g(x_1, x')|^2 dx_1. \tag{1.24}$$

From (1.22) and (1.24) we deduce

$$\left\| Z^\alpha \partial_1^k \left(\frac{u}{x_1} \right) \right\|_{L^2(\mathbb{R}_+^n)}^2 \leq 4 \left\| Z^\alpha \partial_1^{k+1} u \right\|_{L^2(\mathbb{R}_+^n)}^2. \tag{1.25}$$

It follows that

$$\left\| Z^\alpha \partial_1^k \left(\frac{u}{x_1} \right) \right\|_{L^2(\mathbb{R}_+^n)} \leq C \|u\|_{H_*^{m+1}(\mathbb{R}_+^n)} \tag{1.26}$$

for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, with $\alpha_1 = 0$, and $k \in \mathbb{N}$ such that $|\alpha| + 2k \leq m - 1$.

In order to treat the case $\alpha_1 \geq 1$, we use an induction argument. We first invert the position of conormal and normal derivatives in the norm (1.5) to get

$$\left\| \frac{u}{x_1} \right\|^2_{H^{m-1}_*(\mathbb{R}^n_+)} \leq C \sum_{|\alpha|+2k \leq m-1} \left\| \partial_1^k Z^\alpha \left(\frac{u}{x_1} \right) \right\|^2_{L^2(\mathbb{R}^n_+)} + C \left\| \frac{u}{x_1} \right\|_{H^{m-2}_*(\mathbb{R}^n_+)}, \tag{1.27}$$

where the last term comes from the control of the commutator. Then, from the inductive assumption

$$\left\| \frac{u}{x_1} \right\|^2_{H^{m-1}_*(\mathbb{R}^n_+)} \leq C \sum_{|\alpha|+2k \leq m-1} \left\| \partial_1^k Z^\alpha \left(\frac{u}{x_1} \right) \right\|^2_{L^2(\mathbb{R}^n_+)} + C \|u\|_{H^m_*(\mathbb{R}^n_+)}. \tag{1.28}$$

Let us consider the estimate

$$\sum_{|\alpha|+2k \leq m-1} \left\| \partial_1^k Z^\alpha \left(\frac{u}{x_1} \right) \right\|_{L^2(\mathbb{R}^n_+)} \leq C \|u\|_{H^{m+1}_*(\mathbb{R}^n_+)}. \tag{1.29}$$

Notice that (1.29) holds true if $\alpha_1 = 0$, because of (1.26). Assume that (1.29) is true for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and $k \in \mathbb{N}$ such that $|\alpha| + 2k \leq m - 1$ and $0 \leq \alpha_1 \leq \beta_1 - 1$, for some $1 \leq \beta_1 \leq m - 1$. We have

$$\begin{aligned}
&\sum_{|\alpha|+2k \leq m-1, 1 \leq \alpha_1 \leq \beta_1} \left\| \partial_1^k Z^\alpha \left(\frac{u}{x_1} \right) \right\|^2_{L^2(\mathbb{R}^n_+)} \\
&= \sum_{|\alpha|+2k \leq m-1, 1 \leq \alpha_1 \leq \beta_1} \left\| \partial_1^k Z^{\alpha'} Z^{\alpha_1-1} x_1 \partial_1 \left(\frac{u}{x_1} \right) \right\|^2_{L^2(\mathbb{R}^n_+)} \\
&= \sum_{|\alpha|+2k \leq m-1, 1 \leq \alpha_1 \leq \beta_1} \left\| \partial_1^k Z^{\alpha'} Z^{\alpha_1-1} \left(\partial_1 u - \frac{u}{x_1} \right) \right\|^2_{L^2(\mathbb{R}^n_+)} \\
&\leq C \sum_{|\alpha|+2k \leq m-1, 1 \leq \alpha_1 \leq \beta_1} \left(\left\| \partial_1^{k+1} Z^{\alpha'} Z^{\alpha_1-1} u \right\|^2_{L^2(\mathbb{R}^n_+)} + \left\| \partial_1^k Z^{\alpha'} Z^{\alpha_1-1} \left(\frac{u}{x_1} \right) \right\|^2_{L^2(\mathbb{R}^n_+)} \right) \\
&\leq C \|u\|^2_{H^m_*(\mathbb{R}^n_+)}
\end{aligned} \tag{1.30}$$

because for the first term we have $|\alpha| - 1 + 2(k + 1) \leq m$, and for the second term we can apply estimate (1.13), true for m by inductive assumption. Hence (1.29) is true also for $\alpha_1 = \beta_1$. We deduce that (1.29) holds for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, and $k \in \mathbb{N}$ such that $|\alpha| + 2k \leq m - 1$.

Therefore, from (1.28) and (1.29) we get

$$\left\| \frac{u}{x_1} \right\|_{H^{m-1}_*(\mathbb{R}^n_+)} \leq C \|u\|_{H^{m+1}_*(\mathbb{R}^n_+)}. \tag{1.31}$$

The proof of Theorem 1.1 is complete. $\qquad\square$

In the second anisotropic space $H^m_{**}(\Omega)$ we have the following results.

Theorem 1.2. *Let $u \in H_{**}^m(\mathbb{R}_+^n) \cap H_0^1(\mathbb{R}_+^n)$, for $m \geq 1$, and let H be the function defined in (1.12).*

(1) *If $m = 1$, then*

$$\|H\|_{L^2(\mathbb{R}_+^n)} \leq C\|u\|_{H^1(\mathbb{R}_+^n)} \leq C\|u\|_{H_{**}^1(\mathbb{R}_+^n)}. \tag{1.32}$$

(2) *If $m = 2$, then*

$$\|H\|_{H_*^1(\mathbb{R}_+^n)} \leq C\|u\|_{H_{**}^2(\mathbb{R}_+^n)}. \tag{1.33}$$

(3) *If $m \geq 3$, then*

$$\|H\|_{H_{**}^{m-2}(\mathbb{R}_+^n)} \leq C\|u\|_{H_{**}^m(\mathbb{R}_+^n)}. \tag{1.34}$$

Proof. The proof of (1.32) follows by direct application of Hardy's inequality; then (1.33) follows by applying (1.32) to Zu. In case of $m \geq 3$ the proof is similar to that of Theorem 1.1, hence we omit the details. □

Acknowledgment

References

[1] O. Guès, "Probleme mixte hyperbolique quasi-lineaire caracteristique," *Comm. Partial Differential Equations*, vol. 15, no. 5, pp. 595–645, 1990.

[2] A. Morando and P. Secchi, "Regularity of weakly well posed hyperbolic mixed problems with characteristic boundary," *Journal of Hyperbolic Differential Equations*, vol. 8, no. 1, pp. 37–99, 2011.

[3] A. Morando, P. Secchi, and P. Trebeschi, "Regularity of solutions to characteristic initial-boundary value problems for symmetrizable systems," *Journal of Hyperbolic Differential Equations*, vol. 6, no. 4, pp. 753–808, 2009.

[4] M. Ohno, Y. Shizuta, and T. Yanagisawa, "The initial-boundary value problem for linear symmetric hyperbolic systems with boundary characteristic of constant multiplicity," *Kyoto Journal of Mathematics*, vol. 35, no. 2, pp. 143–210, 1995.

[5] P. Secchi, "The initial-boundary value problem for linear symmetric hyperbolic systems with characteristic boundary of constant multiplicity," *Differential Integral Equations*, vol. 9, no. 4, pp. 671–700, 1996.

[6] P. Secchi, "Well-posedness of characteristic symmetric hyperbolic systems," *Archive for Rational Mechanics and Analysis*, vol. 134, no. 2, pp. 155–197, 1996.

[7] M. Ohno and T. Shirota, "On the initial-boundary-value problem for the linearized equations of magnetohydrodynamics," *Archive for Rational Mechanics and Analysis*, vol. 144, no. 3, pp. 259–299, 1998.

[8] M. Tsuji, "Regularity of solutions of hyperbolic mixed problems with characteristic boundary," *Proceedings of the Japan Academy*, vol. 48, pp. 719–724, 1972.

[9] P. Secchi, "Well-posedness for a mixed problem for the equations of ideal Magneto-Hydrodynamics," *Archiv der Mathematik*, vol. 64, no. 3, pp. 237–245, 1995.

[10] P. Secchi, "An initial boundary value problem in ideal Magneto-Hydrodynamics," *Nonlinear Differential Equations and Applications*, vol. 9, no. 4, pp. 441–458, 2002.

[11] P. Secchi and Y. Trakhinin, "Well-posednessof the linearized plasma-vacuum interface problem.," Submitted.

[12] Y. Trakhinin, "The existence of current-vortex sheets in ideal compressible magnetohydrodynamics," *Archive for Rational Mechanics and Analysis*, vol. 191, no. 2, pp. 245–310, 2009.

[13] T. Yanagisawa and A. Matsumura, "The fixed boundary value problems for the equations of ideal Magneto-Hydrodynamics with a perfectly conducting wall condition," *Communications in Mathematical Physics*, vol. 136, no. 1, pp. 119–140, 1991.

[14] M. Ohno, Y. Shizuta, and T. Yanagisawa, "The trace theorem on anisotropic Sobolev spaces," *Tohoku Mathematical Journal*, vol. 46, no. 3, pp. 393–401, 1994.

[15] P. Secchi, "Some properties of anisotropic Sobolev spaces," *Archiv der Mathematik*, vol. 75, no. 3, pp. 207–216, 2000.

[16] S. Alinhac, "Existence d'ondes de raréfaction pour des systèmes quasi-linéaires hyperboliques multidimensionnels," *Communications in Partial Differential Equations*, vol. 14, no. 2, pp. 173–230, 1989.

[17] J. Francheteau and G. Métivier, "Existence de chocs faibles pour des systèmes quasi-linéaires hyperboliques multidimen- sionnels," in *Astérisque*, vol. 268, pp. 1–198, 2000.

[18] D. Coutand and S. Shkoller, "Well-posedness in smooth function spaces for moving-boundary 1-D compressible euler equations in physical vacuum," *Communications on Pure and Applied Mathematics*, vol. 64, no. 3, pp. 328–366, 2011.

[19] G. H. Hardy and J. E. Littlewood, *Inequalities. Cambridge Mathematical Library*, Cambridge, UK, Cambridge University Press, 1988.