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Research Article

Multiplicative Isometries on F-Algebras of Holomorphic Functions

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We study multiplicative isometries on the following F-algebras of holomorphic functions: Smirnov class $N_*(X)$, Privalov class $N^p(X)$, Bergman-Privalov class $AN_\alpha^p(X)$, and Zygmund F-algebra $N\log^\beta N(X)$, where X is the open unit ball $\mathbb{B}_n$ or the open unit polydisk $\mathbb{D}^n$ in $\mathbb{C}^n$.

1. Introduction

Complex-linear isometries on function spaces of holomorphic functions have been studied for almost five decades by many mathematicians. In this paper we study multiplicative isometries on certain F-algebras of holomorphic functions. Recall that an F-algebra is a topological algebra in which the topology arises from a complete metric. For a positive integer n let $\mathbb{B}_n$ denote the open unit ball in the n-dimensional complex vector space $\mathbb{C}^n$ and $\mathbb{D}^n$ the unit polydisk in $\mathbb{C}^n$. We characterize multiplicative isometries on the Smirnov class, the Privalov class, the Bergman-Privalov class and the Zygmund F-algebras on $\mathbb{B}_n$ or $\mathbb{D}^n$. Surjective multiplicative maps on the Smirnov class, and the Bergman-Privalov class have already been correspondingly characterized in [1, 2].

2. Preliminaries

In studying surjective isometries in [1, 2] we applied the Mazur-Ulam theorem for surjective maps on certain subspaces, which themselves are Banach spaces, of the given F-algebras.

Generally we do not assume surjectivity of the isometries in this paper, so instead of the Mazur-Ulam theorem we use Lemma 2.1. Recall that a normed real-linear space L is *uniformly convex* if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that the inequality $\|a+b\| \le 2-\delta$ holds for every pair of $a, b \in L$ with $\|a\| \le 1$, $\|b\| \le 1$, and $\|a-b\| \ge \varepsilon$. It is well known that Hilbert spaces and L^p-spaces for $1 < p < \infty$ are uniformly convex.

Lemma 2.1. *Let L_1 and L_2 be normed real-linear spaces with L_2 uniformly convex. Let S be an isometry from L_1 into L_2 such that $S(0) = 0$. Then S is real-linear.*

The lemma might be well known, but we give a sketch of the proof for the completeness and the benefit of the reader.

Proof of Lemma 2.1. Let a, b be arbitrary elements of L_1. Put $2r = \|a-b\|$. Then since S is an isometry, $\|S(a)-S(b)\| = 2r$ and $\|S(a)-S((a+b)/2)\| = \|S(b)-S((a+b)/2)\| = r$. We also have $\|S(a)-(S(a)+S(b))/2\| = \|S(b)-(S(a)+S(b))/2\| = r$.

Suppose that $S((a+b)/2) \ne (S(a)+S(b))/2$. Set

$$\varepsilon = \left\| S\left(\frac{a+b}{2}\right) - \frac{S(a)+S(b)}{2} \right\|. \tag{2.1}$$

Since L_2 is uniformly convex and ε is positive there exists a $\delta > 0$ such that

$$\begin{aligned} \left\| \left(S(a) - S\left(\frac{a+b}{2}\right)\right) + \left(S(a) - \frac{S(a)+S(b)}{2}\right) \right\| &\le 2r-\delta, \\ \left\| \left(S(b) - S\left(\frac{a+b}{2}\right)\right) + \left(S(b) - \frac{S(a)+S(b)}{2}\right) \right\| &\le 2r-\delta. \end{aligned} \tag{2.2}$$

Then by the triangle inequality

$$\|2S(a)-2S(b)\| \le 4r-2\delta \tag{2.3}$$

holds, which contradicts to $\|S(a)-S(b)\| = 2r$. Thus we get $S((a+b)/2) = (S(a)+S(b))/2$, from which for $b = 0$ we obtain $S(a/2) = S(a)/2$. Substituting a by $a+b$ in the last equality we get

$$\frac{S(a+b)}{2} = S\left(\frac{a+b}{2}\right) = \frac{S(a)+S(b)}{2}, \tag{2.4}$$

so that $S(a+b) = S(a)+S(b)$. A routine argument yields $S(ta) = tS(a)$, $t \in \mathbb{R}$. □

For $X \in \{\mathbb{B}_n, \mathbb{D}^n\}$, we denote by ∂X its distinguished boundary. For $X = \mathbb{B}_n$, this is the topological boundary $\partial\mathbb{B}_n$, and for the polydisk $\mathbb{D}^n$, it is the torus $\mathbb{T}^n$. Denote the normalized Lebesgue measure on ∂X by σ. A holomorphic map ψ is inner if $\lim_{r\to 1-0}\psi(rz)$ exists and lies in ∂X for almost all $z \in \partial X$ with respect to σ. We say that $\lim_{r\to 1-0}\psi(rz)$ is the boundary map of ψ and denote it by ψ^*. We say that ψ^* is measure preserving if $\sigma((\psi^*)^{-1}(E)) = \sigma(E)$ for every Borel set $E \subset \partial X$.

Now we recall definitions and some properties of the Smirnov class, the Privalov class, the Bergman-Privalov class, and the Zygmund F-algebra on $\mathbb{B}_n$ or $\mathbb{D}^n$. The space of all holomorphic functions on $X = \mathbb{B}_n$ or $\mathbb{D}^n$ is denoted by $H(X)$. For each $0 < p \leq \infty$, the Hardy space is denoted by $H^p(X)$ with the norm $\|\cdot\|_p$.

2.1. Smirnov Class $N_*(X)$

Let $X \in \{\mathbb{B}_n, \mathbb{D}^n\}$. The Nevanlinna class $N(X)$ on X is defined as the set of all holomorphic functions f on X such that

$$\sup_{0\leq r<1} \int_{\partial X} \ln(1 + |f(r\zeta)|) d\sigma(\zeta) < \infty \tag{2.5}$$

holds. It is known that every $f \in N(X)$ has a finite nontangential limit, denoted by f^*, almost everywhere on ∂X.

The Smirnov class $N_*(X)$ is defined as

$$N_*(X) = \left\{ f \in N(X) : \sup_{0\leq r<1} \int_{\partial X} \ln(1 + |f(r\zeta)|) d\sigma(\zeta) = \int_{\partial X} \ln(1 + |f^*(\zeta)|) d\sigma(\zeta) \right\}. \tag{2.6}$$

Define a metric

$$d_{N_*(X)}(f, g) = \int_{\partial X} \ln(1 + |f^*(\zeta) - g^*(\zeta)|) d\sigma(\zeta) \tag{2.7}$$

for $f, g \in N_*(X)$. With the metric $d_{N_*(X)}(\cdot, \cdot)$ the Smirnov class $N_*(X)$ becomes an F-algebra and

$$\bigcup_{q>0} H^q(X) \subset N_*(X), \tag{2.8}$$

in particular, $H^\infty(X)$ is a dense subalgebra of $N_*(X)$. The convergence in the metric is stronger than uniform convergence on compact subsets of X.

Complex-linear isometries on the Smirnov class were characterized by Stephenson in [3].

2.2. Privalov Class $N^p(X)$

Let $X \in \{\mathbb{B}_n, \mathbb{D}^n\}$. The Privalov class $N^p(X)$, $1 < p < \infty$, is defined as (for the original source see [4, 5])

$$N^p(X) = \left\{ f \in H(X) : \sup_{0\leq r<1} \int_{\partial X} (\ln(1 + |f(r\zeta)|))^p d\sigma(\zeta) < \infty \right\}. \tag{2.9}$$

It is well known that $N^p(X)$ is a subalgebra of $N_*(X)$, hence every $f \in N^p(X)$ has a finite nontangential limit almost everywhere on ∂X. Define a metric

$$d_p(f,g) = \left(\int_{\partial X} \left(\ln\left(1 + \left|f^*(\zeta) - g^*(\zeta)\right|\right)\right)^p d\sigma(\zeta)\right)^{1/p} \tag{2.10}$$

for $f, g \in N^p(X)$. With this metric $N^p(X)$ is an F-algebra (cf. [6, 7]) and

$$\bigcup_{q>0} H^q(X) \subset N^p(X) \subset N_*(X). \tag{2.11}$$

The Hardy algebra $H^\infty(X)$ is dense in $N^p(X)$. The convergence on the metric is stronger than uniform convergence on compacts of X.

Complex-linear isometries on $N^p(X)$ are investigated by Iida and Mochizuki [8] for one-dimensional case, and by Subbotin [7] for a general case.

2.3. Bergman-Privalov Class $AN^p_\alpha(X)$

Let $1 \leq p < \infty$ and $\alpha > -1$. The Bergman-Privalov class on the unit ball $\mathbb{B}_n$ and the polydisk $\mathbb{D}^n$ are defined, respectively, as

$$\begin{aligned} AN^p_\alpha(\mathbb{B}_n) &= \left\{ f \in H(\mathbb{B}_n) : \|f\|^p_{AN^p_\alpha(\mathbb{B}_n)} = \int_{\mathbb{B}_n} \left(\ln\left(1 + \left|f(z)\right|\right)\right)^p dV_{\alpha,n}(z) < \infty \right\}, \\ AN^p_\alpha(\mathbb{D}^n) &= \left\{ f \in H(\mathbb{D}^n) : \|f\|^p_{AN^p_\alpha(\mathbb{D}^n)} = \int_{\mathbb{D}^n} \left(\ln\left(1 + \left|f(z)\right|\right)\right)^p \prod_{j=1}^n dV_{\alpha,1}(z_j) < \infty \right\}, \end{aligned} \tag{2.12}$$

where $dV_{\alpha,n}(z) = c_{\alpha,n}(1 - |z|^2)^\alpha dV(z)$ for the normalized Lebesgue volume measure dV on $\mathbb{B}_n$ and $c_{\alpha,n}$ is a normalization constant, that is $V_{\alpha,n}(\mathbb{B}_n) = 1$. Let $X \in \{\mathbb{B}_n, \mathbb{D}^n\}$. In what follows $dV_\alpha(z)$ denotes $dV_{\alpha,n}(z)$ for $X = \mathbb{B}_n$ and $\prod_{j=1}^n dV_{\alpha,1}(z_j)$ for $X = \mathbb{D}^n$, respectively. The Bergman-Privalov class $AN^p_\alpha(X)$ is an F-algebra with respect to the metric

$$d_{AN^p_\alpha(X)}(f,g) = \|f - g\|_{AN^p_\alpha(X)} \tag{2.13}$$

for $f, g \in AN^p_\alpha(X)$. For some results in the case $p = 1$ see [9].

The weighted Bergman space for $q > 0$ and $\alpha > -1$ on the unit ball $\mathbb{B}_n$ and the polydisk $\mathbb{D}_n$ are defined, respectively, as

$$\begin{aligned} A^q_\alpha(\mathbb{B}_n) &= \left\{ f \in H(\mathbb{B}_n) : \|f\|^q_{A^q_\alpha(\mathbb{B}_n)} = \int_{\mathbb{B}_n} \left|f(z)\right|^q dV_{\alpha,n}(z) < \infty \right\}, \\ A^q_\alpha(\mathbb{D}^n) &= \left\{ f \in H(\mathbb{D}^n) : \|f\|^q_{A^q_\alpha(\mathbb{D}^n)} = \int_{\mathbb{D}^n} \left|f(z)\right|^q \prod_{j=1}^n dV_{\alpha,1}(z_j) < \infty \right\}. \end{aligned} \tag{2.14}$$

It is known that

$$\bigcup_{q>0} A_\alpha^q(X) \subset AN_\alpha^p(X). \tag{2.15}$$

Complex-linear isometries on the Bergman-Privalov class on the unit ball were characterized by Matsugu and Ueki in [10] and on the polydisk by Stević in [2].

2.4. *Zygmund* F*-Algebra* $N\log^\beta N(X)$

Let $\beta > 0$ and $\varphi_\beta(t) = t(\ln(\gamma_\beta + t))^\beta$, where $\gamma_\beta = \max\{e, e^\beta\}$. Let $X \in \{\mathbb{B}_n, \mathbb{D}^n\}$. The Zygmund F-algebra $N\log^\beta N(X)$ on X is defined as

$$N\log^\beta N(X) = \left\{ f \in H(X) : \sup_{0\le r<1} \int_{\partial X} \varphi_\beta(\ln(1 + |f(r\zeta)|)) d\sigma(\zeta) < \infty \right\}. \tag{2.16}$$

It is known that

$$N\log^\beta N(X) = \left\{ f \in H(X) : \sup_{0\le r<1} \int_{\partial X} \varphi_\beta(\ln^+ |f(r\zeta)|) d\sigma(\zeta) < \infty \right\}, \tag{2.17}$$

$$\bigcup_{p>0} H^p(X) \subset N\log^\beta N(X) \subset N_*(X). \tag{2.18}$$

This implies that the finite nontangential limit f^* exists almost everywhere on ∂X, for any $f \in N\log^\beta N$. For $f, g \in N\log^\beta N$

$$d_{N\log^\beta N(X)}(f, g) = \int_{\partial X} \varphi_\beta(\ln(1 + |f^*(\zeta) - g^*(\zeta)|)) d\sigma(\zeta) \tag{2.19}$$

defines a complete metric on $N\log^\beta N(X)$ and $N\log^\beta N(X)$ is an F-algebra with this metric (cf. [11]).

Ueki [12] characterized the complex-linear isometries on the Zygmund F-algebra on the balls.

3. Main Results

In this section we formulate and prove the main results in this paper.

3.1. *Multiplicative Isometries on* $N_*(X)$

Our first result concerns the Smirnov class.

Theorem 3.1. *Let $X \in \{\mathbb{B}_n, \mathbb{D}^n\}$. Suppose that $T : N_*(X) \to N_*(X)$ is a (not necessarily linear) multiplicative isometry. Then there is an inner map φ on X whose boundary map φ^* is measure preserving and such that either of the following formulas holds:*

$$\begin{aligned} T(f) &= f \circ \varphi \quad \text{for every } f \in N_*(X), \\ T(f) &= \overline{f \circ \overline{\varphi}} \quad \text{for every } f \in N_*(X). \end{aligned} \tag{3.1}$$

Proof. First we claim that $T(1) = 1$. Since $T(1) = T(1)^2$ and $T(1)$ is a holomorphic function on the connected open set X we get $T(1) = 0$ or $T(1) = 1$. But $T(1) = 0$ is impossible because if it were $T(1) = 0$, then $0 = T(f)T(1) = T(f)$, for each $f \in N_*(X)$, which contradicts with the assumption that T is an isometry. As $T(0) = T(0)^2$ and T is injective, we obtain $T(0) = 0$. Similarly $T(-1) = -1$ is also observed by making use of $T(-1)^2 = T(1) = 1$. Then $T(i)^2 = T(i^2) = -1$ assert that $T(i) = i$ or $T(i) = -i$. If $T(i) = i$, then the first formula of the conclusion will follow and the second one will follow from $T(i) = -i$.

Next we show $T(1/2) = 1/2$. Put $r = 1/2$. Suppose that $|T(r)^*| > r$ on a set of positive measure on ∂X. Then there exists a subset E of positive measure and $\varepsilon > 0$ with $|T(r)^*| \geq (1+\varepsilon)r$ on E. Since

$$\lim_{n\to\infty} \frac{\ln(1+(1+\varepsilon)^n r^n)}{\ln(1+r^n)} = \infty, \tag{3.2}$$

there is a positive integer n_0 such that

$$\int_E \ln(1+(1+\varepsilon)^{n_0} r^{n_0})\,d\sigma > \int_{\partial X} \ln(1+r^{n_0})\,d\sigma. \tag{3.3}$$

From this and since T is a multiplicative isometry on $N_*(X)$ we have that

$$\begin{aligned} \int_{\partial X} \ln(1+r^{n_0})\,d\sigma &= \int_{\partial X} \ln\left(1+\left|T(r)^*\right|^{n_0}\right)d\sigma \\ &\geq \int_E \ln\left(1+(1+\varepsilon)^{n_0} r^{n_0}\right)d\sigma > \int_{\partial X} \ln(1+r^{n_0})\,d\sigma, \end{aligned} \tag{3.4}$$

which is a contradiction proving $|T(r)^*| \leq r$ almost everywhere on ∂X. Hence $|T(1/r)^*| \geq 1/r$ holds almost everywhere on ∂X as $T(r)T(1/r) = T(1) = 1$ almost everywhere on ∂X. Since

$$\ln\left(1+\frac{1}{r}\right) = \int_{\partial X} \ln\left(1+\frac{1}{r}\right)d\sigma = \int_{\partial X} \ln\left(1+\left|T\left(\frac{1}{r}\right)^*\right|\right)d\sigma, \tag{3.5}$$

we have that $|T(1/r)^*| = 1/r$ and $|T(r)^*| = r$ almost everywhere on ∂X.

Since $\ln(1+(1-r)) = d(r,1) = d(T(r),1)$ and

$$d(T(r),1) = \int_{\partial X} \ln\left(1+\left|1-T(r)^*\right|\right)d\sigma, \tag{3.6}$$

it is easy to check that $T(1/2)^* = 1/2$ almost everywhere on ∂X. Hence $T(1/2) = 1/2$ holds. As T is multiplicative, T is 1/2-homogeneous in the sense that $T(f/2) = T(f)/2$ holds for every $f \in N_*(X)$.

Let $f, g \in H^1(X)$. It requires only elementary calculation applying the 1/2-homogeneity of T to check that

$$\int_{\partial X} \ln\left(1 + \left|\frac{f^*}{2^m} - \frac{g^*}{2^m}\right|\right) d\sigma = \int_{\partial X} \ln\left(1 + \left|\frac{T(f)^*}{2^m} - \frac{T(g)^*}{2^m}\right|\right) d\sigma \tag{3.7}$$

holds. Multiplying (3.7) by 2^m and then letting $m \to \infty$ we get

$$\int_{\partial X} |f^* - g^*| d\sigma = \int_{\partial X} |T(f)^* - T(g)^*| d\sigma \tag{3.8}$$

by the monotone convergence theorem, since $2^m \ln(1 + (t/2^m))$ nondecreases monotonically to t as $m \to \infty$ for any $t \geq 0$, which can be easily proved by considering the function $g_t(x) = x \ln(1 + (t/x))$. From (3.8) for $g = 0$, we obtain $T(H^1(X)) \subseteq H^1(X)$ and the restricted map $T|_{H^1(X)}$ is an isometry with respect to the metric induced by the H^1-norm $\|\cdot\|_1$.

Let the function θ on the interval $[0, \infty)$ be defined as

$$\theta(x) = \begin{cases} \dfrac{1}{2}, & x = 0 \\ \dfrac{x - \ln(1+x)}{x^2}, & x > 0. \end{cases} \tag{3.9}$$

It is easy to check that θ is positive and continuous on $[0, \infty)$ and $\lim_{x\to\infty} \theta(x) = 0$. Hence θ is bounded on $[0, \infty)$, so that

$$M_\theta := \sup_{x \geq 0} \theta(x) < \infty. \tag{3.10}$$

We claim that the inclusion $T(H^2(X)) \subseteq H^2(X)$ and $T|_{H^2(X)}$ is isometric with respect to the metric induced by the H^2-norm. For this purpose let $f, g \in H^2(X)$. Now note that since $H^2(X) \subset H^1(X)$, equality (3.7) holds and as well as the next equality

$$\int_{\partial X} \left|\frac{f^*}{2^m} - \frac{g^*}{2^m}\right| d\sigma = \int_{\partial X} \left|\frac{T(f)^*}{2^m} - \frac{T(g)^*}{2^m}\right| d\sigma. \tag{3.11}$$

By subtracting (3.7) from (3.11) and then multiplying such obtained equation by 2^m we obtain

$$\int_{\partial X} |f^* - g^*|^2 \theta\left(\left|\frac{f^*}{2^m} - \frac{g^*}{2^m}\right|\right) d\sigma = \int_{\partial X} |T(f)^* - T(g)^*|^2 \theta\left(\left|\frac{T(f)^*}{2^m} - \frac{T(g)^*}{2^m}\right|\right) d\sigma. \tag{3.12}$$

As θ is bounded the function $M_\theta |f^* - g^*|^2$ is an integrable function dominating the integrand in the left-hand side integral in (3.12). Letting $m \to \infty$ and applying the Lebesgue theorem on dominated convergence to the left-hand side and Fatou's lemma to the right-hand side (as θ is positive on $[0, \infty)$) we obtain

$$\int_{\partial X} |f^* - g^*|^2 \theta(0) d\sigma \geq \int_{\partial X} |T(f)^* - T(g)^*|^2 \theta(0) d\sigma. \tag{3.13}$$

From this and since $\theta(0) = 1/2$ we get that the function $|T(f)^* - T(g)^*|^2$ is integrable. Letting again $m \to \infty$ in (3.12) we have that

$$\int_{\partial X} |f^* - g^*|^2 d\sigma = \int_{\partial X} |T(f)^* - T(g)^*|^2 d\sigma \tag{3.14}$$

by the Lebesgue theorem on dominated convergence now applied to both integrals in (3.12). Hence $\|f - g\|_2 = \|T(f) - T(g)\|_2$ for every pair of $f, g \in H^2(X)$. For $g = 0$, we get $\|f\|_2 = \|T(f)\|_2$ and consequently $T(H^2(X)) \subseteq H^2(X)$, as claimed.

Since $H^2(X)$ is a Hilbert space, it is uniformly convex. Hence by Lemma 2.1 the restriction $T|_{H^2(X)}$ is real-linear. Since the operations of scalar multiplication and addition on N_* are continuous and $H^2(X)$ is dense in $N_*(X)$ we see that T is real-linear on $N_*(X)$.

First assume $T(i) = i$. As T is real-linear and multiplicative, T is complex-linear in this case. Then by [3, Theorem 2.2] and since $T(1) = 1$, there is an inner map ψ such that $T(f) = f \circ \psi$ for every $f \in N_*(X)$.

Now assume $T(i) = -i$. Let $\widetilde{T} : N_*(X) \to N_*(X)$ be defined as $\widetilde{T}(f) = T(\widetilde{f})$ for every $f \in N_*(X)$, where

$$\widetilde{f}(z_1, \ldots, z_n) = \overline{f(\overline{z}_1, \ldots, \overline{z}_n)} \tag{3.15}$$

for $f \in N_*(X)$. Then $\widetilde{T}$ is well defined and a complex-linear isometry from $N_*(X)$ into itself. Again by [3, Theorem 2.2] we have that there is an inner map ψ on X whose boundary map ψ^* is measure preserving such that $\widetilde{T}(f) = f \circ \psi$ for every $f \in N_*$. This implies that $T(f) = \overline{f \circ \overline{\psi}}$ for every $f \in N_*(X)$. □

Corollary 3.2 (see [1]). *Let $X \in \{\mathbb{B}_n, \mathbb{D}^n\}$. Suppose that $T : N_*(X) \to N_*(X)$ is a (not necessarily linear) surjective multiplicative isometry. Then there is a holomorphic automorphism ψ on X such that either of the following formulas holds:*

$$\begin{aligned} T(f) &= f \circ \psi \quad \text{for every } f \in N_*(X), \\ T(f) &= \overline{f \circ \overline{\psi}} \quad \text{for every } f \in N_*(X), \end{aligned} \tag{3.16}$$

where ψ is a unitary transformation for $X = \mathbb{B}_n$, while for $X = \mathbb{D}^n$, $\psi(z_1, \ldots, z_n) = (e^{i\theta_1} z_{j_1}, \ldots, e^{i\theta_n} z_{j_n})$ for some real numbers θ_j for $j = 1, \ldots, n$ and a permutation $(j_1, \ldots, j_n)$ of the integers from 1 to n.

Proof. By Theorem 3.1, T is complex-linear or conjugate linear. If T is complex-linear, then the result holds by [3, Corollary 2.3]. If T is conjugate linear, then put $\widetilde{T}(f) = T(\widetilde{f})$ for $f \in N_*(X)$, where $\widetilde{f}$ is defined as in (3.15). Then $\widetilde{T}(f) = f \circ \psi$, for every $f \in N_*(X)$, and for an inner map ψ on X whose boundary map ψ^* is measure preserving. Since $\widetilde{T}$ is a surjective isometry, the desired property of ψ again follows from [3, Corollary 2.3]. □

3.2. Multiplicative Isometries on $N^p(X)$

The next result concerns the Privalov class.

Theorem 3.3. *Let $X \in \{\mathbb{B}_n, \mathbb{D}^n\}$ and $1 < p < \infty$. Suppose that $T : N^p(X) \to N^p(X)$ is a (not necessarily linear) multiplicative isometry. Then there is an inner map ψ on X whose boundary map ψ^* is measure preserving and such that either of the following formulas holds:*

$$T(f) = f \circ \psi \quad \textit{for every } f \in N^p(X), \qquad T(f) = \overline{f \circ \overline{\psi}} \quad \textit{for every } f \in N^p(X). \tag{3.17}$$

Proof. Since T is multiplicative we see by the same way as in the proof of Theorem 3.1 that $T(0) = 0$, $T(1) = 1$ and $T(i) = i$ or $T(i) = -i$. Also we see that $T(1/2) = 1/2$. It follows by the proof of Theorem 3.1 that for every pair f and g in $H^p(X)$,

$$\int_{\partial X} \left(\ln\left(1 + \left|\frac{f^*}{2^m} - \frac{g^*}{2^m}\right|\right)\right)^p d\sigma = \int_{\partial X} \left(\ln\left(1 + \left|\frac{T(f)^*}{2^m} - \frac{T(g)^*}{2^m}\right|\right)\right)^p d\sigma \tag{3.18}$$

holds. Multiplying (3.18) by 2^{mp} and then letting $m \to \infty$ we get

$$\int_{\partial X} |f^* - g^*|^p d\sigma = \int_{\partial X} |T(f)^* - T(g)^*|^p d\sigma. \tag{3.19}$$

Thus $T(H^p(X)) \subseteq H^p(X)$. The Hardy space $H^p(X)$ can be seen as a subspace of $L^p(\partial X)$. Since $L^p(\partial X)$ is uniformly convex, so is $H^p(X)$ for $1 < p < \infty$. Then by Lemma 2.1 the operator T is real-linear on $H^p(X)$. Since $H^p(X)$ is a dense subspace of $N^p(X)$ we see that T is real-linear on $N^p(X)$. As we have already learnt that $T(i) = i$ or $T(i) = -i$, we obtain that T is complex-linear or conjugate linear on $N^p(X)$. The rest of the proof is similar to the last part of the proof of Theorem 3.1 applying [7, Theorem 1] instead of [3, Theorem 2.2]. We omit the details. □

Corollary 3.4. *Let $X \in \{\mathbb{B}_n, \mathbb{D}^n\}$ and $1 < p < \infty$. Suppose that $T : N^p(X) \to N^p(X)$ is a (not necessarily linear) surjective multiplicative isometry. Then there is a holomorphic automorphism ψ on X such that either of the following formulas holds:*

$$T(f) = f \circ \psi \quad \textit{for every } f \in N^p(X), \qquad T(f) = \overline{f \circ \overline{\psi}} \quad \textit{for every } f \in N^p(X), \tag{3.20}$$

where ψ is a unitary transformation for $X = \mathbb{B}_n$, while for $X = \mathbb{D}^n$, $\psi(z_1, \ldots, z_n) = (e^{i\theta_1} z_{j_1}, \ldots, e^{i\theta_n} z_{j_n})$ for some real numbers θ_j for $j = 1, \ldots, n$ and a permutation $(j_1, \ldots, j_n)$ of the integers from 1 to n.

Proof. By Theorem 3.3, T is complex-linear or conjugate linear. If T is complex-linear, then the result follows directly from [7, Corollary and Remark 3]. If T is conjugate linear, then put $\widetilde{T}(f) = T(\widetilde{f})$ for $f \in N^p(X)$, where $\widetilde{f}$ is defined as in (3.15). Then $\widetilde{T}$ is a complex-linear isometric surjection from $N^p(X)$ onto itself. Hence by [7, Corollary and Remark 3] there is a desired automorphism on X such that $\mathrm{T}(f) = \overline{f \circ \overline{\varphi}}$ for every $f \in N^p(X)$. □

3.3. *Multiplicative Isometries on* $AN_\alpha^p(X)$

The next result concerns the Bergman-Privalov class.

Theorem 3.5. *Let* $X \in \{\mathbb{B}_n, \mathbb{D}^n\}$, $1 \le p < \infty$ *and* $\alpha > -1$. *Suppose that* $T : AN_\alpha^p(X) \to AN_\alpha^p(X)$ *is a (not necessarily linear) multiplicative isometry. Then there is a holomorphic self-map* φ *on* X *with the property that*

$$\int_X h \circ \varphi(z) dV_\alpha(z) = \int_X h(z) dV_\alpha(z) \tag{3.21}$$

for every bounded or positive Borel function h *on* X *such that either of the following formulas holds:*

$$\begin{aligned} T(f) &= f \circ \varphi \quad \text{for every } f \in AN_\alpha^p(X), \\ T(f) &= \overline{f \circ \overline{\varphi}} \quad \text{for every } f \in AN_\alpha^p(X). \end{aligned} \tag{3.22}$$

Proof. We can prove the theorem in a way similar to that in the proofs of Theorem 3.1 for $p = 1$ and Theorem 3.3 for $1 < p < \infty$. For the case of $p = 1$, instead of using the Hardy spaces $H^1(X)$ and $H^2(X)$ we make use of the weighted Bergman spaces $A_\alpha^1(X)$ and $A_\alpha^2(X)$. For the case of $1 < p < \infty$, instead of using the Hardy space $H^p(X)$ we make use of the weighted Bergman space $A_\alpha^p(X)$. We also apply [10, Theorem 1] for $X = \mathbb{B}_n$ and [2, Theorem 2] for $X = \mathbb{D}^n$ to represent complex-linear isometries instead of [3, Theorem 2.2]. □

Corollary 3.6 (see [2]). *Let* $X \in \{\mathbb{B}_n, \mathbb{D}^n\}$, $1 \le p < \infty$ *and* $\alpha > -1$. *Suppose that* $T : AN_\alpha^p(X) \to AN_\alpha^p(X)$ *is a (not necessarily linear) surjective multiplicative isometry. Then there is a holomorphic automorphism* φ *on* X *such that either of the following formulas holds:*

$$\begin{aligned} T(f) &= f \circ \varphi \quad \text{for every } f \in AN_\alpha^p(X), \\ T(f) &= \overline{f \circ \overline{\varphi}} \quad \text{for every } f \in AN_\alpha^p(X), \end{aligned} \tag{3.23}$$

where φ *is a unitary transformation for* $X = \mathbb{B}_n$, *while for* $X = \mathbb{D}^n$, $\varphi(z_1, \dots, z_n) = (e^{i\theta_1} z_{j_1}, \dots, e^{i\theta_n} z_{j_n})$ *for some real numbers* θ_j *for* $j = 1, \dots, n$ *and a permutation* $(j_1, \dots, j_n)$ *of the integers from 1 to* n.

Proof. By Theorem 3.5, T is complex-linear or conjugate linear. Suppose that T is complex-linear. If $X = \mathbb{B}_n$, then the conclusion follows by [10, Theorem 2], while for $X = \mathbb{D}^n$ the conclusion follows similar to the corresponding part of the proof of [2, Theorem 3]. If T is conjugate linear, then the conclusion follows from the similar argument in the proof of Corollary 3.2. □

3.4. Isometries on $N\log^{\beta}N(X)$

In [12] Ueki characterized complex-linear isometries on the Zygmund F-algebra on $\mathbb{B}_n$. For $\mathbb{D}^n$ the following result is proved similar to [12, Theorem 1]. Hence it is omitted.

Theorem 3.7. *Let $\beta > 0$. If T is a complex-linear isometry of $N\log^{\beta}N(\mathbb{D}^n)$ into itself, then there exist an inner function Ψ and an inner map ψ on $\mathbb{D}^n$ whose boundary map ψ^* is measure preserving on $\mathbb{T}^n$ such that*

$$T(f) = \Psi C_{\psi}(f) = \Psi(f \circ \psi) \quad \text{for every } f \in N\log^{\beta}N(\mathbb{D}^n). \tag{3.24}$$

Conversely, for given such Ψ and ψ, the weighted composition operator ΨC_{ψ} is an injective linear isometry of $N\log^{\beta}N(\mathbb{D}^n)$.

For the surjective isometries the result is as follows.

Corollary 3.8. *An isometry T of $N\log^{\beta}N(\mathbb{D}^n)$ is surjective if and only if $T = aC_{\mathcal{U}}$ where $a \in \mathbb{C}$ with $|a| = 1$ and $\mathcal{U}(z_1, \ldots, z_n) = (e^{i\theta_1}z_{j_1}, \ldots, e^{i\theta_n}z_{j_n})$ for some real numbers θ_j, $j = 1, \ldots, n$ and a permutation $(j_1, \ldots, j_n)$ of the integers from 1 to n.*

To prove Corollary 3.8 we need the next auxiliary result.

Lemma 3.9. *For any function $f \in N(\mathbb{D}^n)$, $f \in N\log^{\beta}N(\mathbb{D}^n)$ if and only if $\varphi_{\beta}(\ln^+|f^*|) \in L^1(\mathbb{T}^n)$ and*

$$\varphi_{\beta}(\ln^+|f(z)|) \le \int_{\mathbb{T}^n} P(z,\zeta)\varphi_{\beta}(\ln^+|f^*(\zeta)|)d\sigma(\zeta) \quad \text{for } z \in \mathbb{D}^n, \tag{3.25}$$

where $P(z,\zeta)$ denotes the Poisson kernel for $\mathbb{D}^n$;

$$P(z,\zeta) = P_{r_1}(\theta_1 - \phi_1) \cdots P_{r_n}(\theta_n - \phi_n) \tag{3.26}$$

for $z = (r_1e^{i\theta_1}, \ldots, r_ne^{i\theta_n})$, $\zeta = (e^{i\phi_1}, \ldots, e^{i\phi_n})$ and

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2} \tag{3.27}$$

is the Poisson kernel for the unit disk $\mathbb{D}$.

Proof. If $f \in N\log^{\beta}N(\mathbb{D}^n)$, then Fatou's lemma shows that $\varphi_{\beta}(\ln^+|f^*|) \in L^1(\mathbb{T}^n)$. The inclusion (2.18) implies $f \in N_*(\mathbb{D}^n)$, and so we see that $\ln^+|f|$ has the least n-harmonic majorant. Since the least n-harmonic majorant of $\ln^+|f|$ is the Poisson integral $P[\ln^+|f^*|]$, we obtain the following inequality:

$$\ln^+|f(z)| \le \int_{\mathbb{T}^n} P(z,\zeta)\ln^+|f^*(\zeta)|d\sigma(\zeta) \quad \text{for } z \in \mathbb{D}^n. \tag{3.28}$$

Note that $\varphi_\beta(t)$ is strictly increasing and convex on $[0,\infty)$, and the measures $d\mu_z(\zeta) = P(z,\zeta)d\sigma(\zeta)$ are normalized on $\mathbb{T}^n$, which follows from the well-known equality

$$\int_{\mathbb{T}^n} P(z,\zeta)d\sigma(\zeta) = 1. \tag{3.29}$$

Applying Jensen's inequality to (3.28), we obtain the desired inequality (3.25).

Conversely we put $z = r\eta(0 \le r < 1, \eta \in \mathbb{T}^n)$ in (3.25). By integrating with respect to η and applying Fubini's theorem, we have that

$$\int_{\mathbb{T}^n} \varphi_\beta(\ln^+|f(r\eta)|)d\sigma(\eta) \le \int_{\mathbb{T}^n} \varphi_\beta(\ln^+|f^*(\zeta)|)d\sigma(\zeta)\int_{\mathbb{T}^n} P(r\eta,\zeta)d\sigma(\eta). \tag{3.30}$$

By the symmetric property $P(r\eta,\zeta) = P(r\zeta,\eta)$ and the normalization property of the Poisson kernel, we obtain that

$$\sup_{0\le r<1}\int_{\mathbb{T}^n} \varphi_\beta(\ln^+|f(r\eta)|)d\sigma(\eta) \le \int_{\mathbb{T}^n} \varphi_\beta(\ln^+|f^*(\zeta)|)d\sigma(\zeta). \tag{3.31}$$

Hence the condition $\varphi_\beta(\ln^+|f^*|) \in L^1(\mathbb{T}^n)$ implies that $f \in N\log^\beta N(\mathbb{D}^n)$. □

Now we give a proof of Corollary 3.8.

Proof of Corollary 3.8. Suppose that T is surjective. Then Theorem 3.7 gives that $T = \Psi C_\psi$. A standard argument shows that ψ is an automorphism of $\mathbb{D}^n$. So there are conformal maps φ_j $(j = 1,\dots,n)$ of $\mathbb{D}$ onto $\mathbb{D}$ and there is a permutation $(j_1,\dots,j_n)$ of the integers from 1 to n such that

$$\psi(z_1,\dots,z_n) = (\varphi_1(z_{j_1}),\dots,\varphi_n(z_{j_n})). \tag{3.32}$$

The mean value theorem shows that

$$\int_{\mathbb{T}^n} \varphi_k(\zeta_{j_k})d\sigma(\zeta) = \int_{\mathbb{T}} \varphi_k(\zeta_{j_k})d\sigma_1(\zeta_{j_k}) = \varphi_k(0) \tag{3.33}$$

for each $k \in \{1,\dots,n\}$. Here $d\sigma_1$ denotes the one-dimensional normalized Lebesgue measure on the unit circle $\mathbb{T}$.

On the other hand, the measure-preserving property of ψ^* gives that

$$\int_{\mathbb{T}^n} \varphi_k(\zeta_{j_k})d\sigma(\zeta) = \int_{\mathbb{T}^n} \langle\psi^*(\zeta), e_k\rangle d\sigma(\zeta) = \int_{\mathbb{T}^n} \langle\zeta, e_k\rangle d\sigma(\zeta) = \int_{\mathbb{T}^n} \zeta_k d\sigma(\zeta) = 0. \tag{3.34}$$

By (3.33) and (3.34) we see that ψ fixes the origin, and so each φ_k is the rotation transform.

Next we prove that Ψ is a unimodular constant. If $f \in N\log^{\beta} N(\mathbb{D}^n)$ is such that $1 = T(f) = \Psi C_{\psi}(f)$, then $1/\Psi = f \circ \psi \in N\log^{\beta} N(\mathbb{D}^n)$. Inequality (3.25) in Lemma 3.9 gives that

$$\varphi_{\beta}\left(\ln^{+}\frac{1}{|\Psi(z)|}\right) \le \int_{\mathbb{T}^n} P(z,\zeta)\varphi_{\beta}\left(\ln^{+}\frac{1}{|\Psi^{*}(\zeta)|}\right)d\sigma(\zeta) = 0, \tag{3.35}$$

and so we have $1/|\Psi| \le 1$ on $\mathbb{D}^n$. Since Ψ is inner, Ψ is a unimodular constant. □

Now we show results on multiplicative isometries on the Zygmund F-algebras on $\mathbb{B}_n$ and $\mathbb{D}^n$.

Theorem 3.10. *Let $X \in \{\mathbb{B}_n, \mathbb{D}^n\}$. Suppose that $T : N\log^{\beta} N(X) \to N\log^{\beta} N(X)$ is a (not necessarily linear) multiplicative isometry. Then there exists an inner map ψ on X whose boundary map ψ^{*} is measure preserving on ∂X, such that either of the following formulas holds:*

$$\begin{aligned} T(f) &= f \circ \psi \quad \text{for every } f \in N\log^{\beta} N(X), \\ T(f) &= \overline{f \circ \overline{\psi}} \quad \text{for every } f \in N\log^{\beta} N(X). \end{aligned} \tag{3.36}$$

Note that multiplicative isometries of the Privalov class and the Zygmund F-algebra have the same form as multiplicative isometries of the Smirnov class.

Proof of Theorem 3.10. As T is multiplicative we obtain $T(1) = 1$, $T(0) = 0$, $T(-1) = -1$ and $T(i) = i$ or $T(i) = -i$. Since

$$\lim_{n\to\infty} \frac{\varphi_{\beta}(((1+\varepsilon)/2)^n)}{\varphi_{\beta}((1/2^n))} = \infty \tag{3.37}$$

holds for every $\varepsilon > 0$, the equation $T(1/2) = 1/2$ is proved similarly as in Theorem 3.1.

Let $f, g \in H^1(X)$. Then we can prove that

$$\int_{\partial X} 2^m \varphi_{\beta}\left(\ln\left(1 + \left|\frac{f^{*}}{2^m} - \frac{g^{*}}{2^m}\right|\right)\right)d\sigma = \int_{\partial X} 2^m \varphi_{\beta}\left(\ln\left(1 + \left|\frac{T(f)^{*}}{2^m} - \frac{T(g)^{*}}{2^m}\right|\right)\right)d\sigma, \tag{3.38}$$

following the lines of the corresponding part of the proof in Theorem 3.1. By some calculation we see that

$$\varphi_{\beta}(\ln(1+x)) \le (\ln \gamma_{\beta})^{\beta} x \tag{3.39}$$

holds for every $x \ge 0$. Hence we get

$$2^m \varphi_{\beta}\left(\ln\left(1 + \left|\frac{f^{*}}{2^m} - \frac{g^{*}}{2^m}\right|\right)\right) \le (\ln \gamma_{\beta})^{\beta} |f^{*} - g^{*}|, \tag{3.40}$$

almost everywhere on ∂X and $(\ln\gamma_\beta)^\beta|f^* - g^*|$ is an integrable function dominating $2^m\varphi_\beta(\ln(1+|(f^*/2^m)-(g^*/2^m)|))$. We get

$$\lim_{m\to\infty}\int_{\partial X}2^m\varphi_\beta\left(\ln\left(1+\left|\frac{f^*}{2^m}-\frac{g^*}{2^m}\right|\right)\right)d\sigma = (\ln\gamma_\beta)^\beta\int_{\partial X}|f^*-g^*|d\sigma \tag{3.41}$$

by the Lebesgue dominated convergence theorem since

$$\lim_{m\to\infty}2^m\varphi_\beta\left(\ln\left(1+\left|\frac{f^*}{2^m}-\frac{g^*}{2^m}\right|\right)\right) = (\ln\gamma_\beta)^\beta|f^*-g^*|. \tag{3.42}$$

On the other hand, applying Fatou's lemma we get

$$\begin{aligned}
&(\ln\gamma_\beta)^\beta\int_{\partial X}\left|T(f)^*-T(g)^*\right|d\sigma\\
&\quad\le \liminf_{m\to\infty}\int_{\partial X}2^m\varphi_\beta\left(\ln\left(1+\left|\frac{T(f)^*}{2^m}-\frac{T(g)^*}{2^m}\right|\right)\right)d\sigma\\
&\quad= \liminf_{m\to\infty}\int_{\partial X}2^m\varphi_\beta\left(\ln\left(1+\left|\frac{f^*}{2^m}-\frac{g^*}{2^m}\right|\right)\right)d\sigma\\
&\quad= (\ln\gamma_\beta)^\beta\int_{\partial X}|f^*-g^*|d\sigma<\infty,
\end{aligned} \tag{3.43}$$

from which for $g=0$ we get $T(H^1(X))\subseteq H^1(X)$. Since

$$2^m\varphi_\beta\left(\ln\left(1+\left|\frac{T(f)^*}{2^m}-\frac{T(g)^*}{2^m}\right|\right)\right)\le(\ln\gamma_\beta)^\beta\left|T(f)^*-T(g)^*\right| \tag{3.44}$$

follows from (3.40), the function $(\ln\gamma_\beta)^\beta|T(f)^*-T(g)^*|$ is an integrable function dominating $2^m\varphi_\beta\left(\ln\left(1+\left|\frac{T(f)^*}{2^m}-\frac{T(g)^*}{2^m}\right|\right)\right)$. Hence

$$(\ln\gamma_\beta)^\beta\int_{\partial X}\left|T(f)^*-T(g)^*\right|d\sigma = \lim_{m\to\infty}\int_{\partial X}2^m\varphi_\beta\left(\ln\left(1+\left|\frac{T(f)^*}{2^m}-\frac{T(g)^*}{2^m}\right|\right)\right)d\sigma \tag{3.45}$$

holds by the Lebesgue dominated convergence theorem. Consequently

$$\int_{\partial X}|f^*-g^*|d\sigma=\int_{\partial X}\left|T(f)^*-T(g)^*\right|d\sigma \tag{3.46}$$

holds. As f and g are arbitrary elements of $H^1(X)$ we obtain that $T|_{H^1(X)}$ is isometric on $H^1(X)$ with respect to the metric induced by the H^1-norm.

We also obtain that there exists a bounded positive continuous function θ_1 on $[0, \infty)$ such that $\theta_1(0) \neq 0$ and

$$x^2\theta_1(x) = \{\ln \gamma_\beta\}^\beta x - \varphi_\beta(\ln(1+x)). \tag{3.47}$$

Applying this equality we obtain that $T(H^2(X)) \subseteq H^2(X)$ and $T|_{H^2(X)}$ is a real-linear isometry on $H^2(X)$, hence T is a complex-linear (if $T(i) = i$) or conjugate linear isometry (if $T(i) = -i$) on $N\log^\beta N(X)$, similar as in the proof of Theorem 3.1. The rest of the proof is similar to the last part of the proof of Theorem 3.1 applying [12, Theorem 1] for $X = \mathbb{B}_n$ and Theorem 3.7 for $X = \mathbb{D}^n$ instead of [3, Theorem 2.2]. We omit the details. □

Corollary 3.11. *Let $X \in \{\mathbb{B}_n, \mathbb{D}^n\}$. Suppose that $T : N\log^\beta N(X) \to N\log^\beta N(X)$ is a (not necessarily linear) surjective multiplicative isometry. Then there exists a holomorphic automorphism ψ on X such that either of the following formulas holds:*

$$\begin{aligned} T(f) &= f \circ \psi \quad \text{for every } f \in N\log^\beta N(X), \\ T(f) &= \overline{f \circ \overline{\psi}} \quad \text{for every } f \in N\log^\beta N(X), \end{aligned} \tag{3.48}$$

where ψ is a unitary transformation for $X = \mathbb{B}_n$, while for $X = \mathbb{D}^n$, $\psi(z_1, \ldots, z_n) = (e^{i\theta_1}z_{j_1}, \ldots, e^{i\theta_n}z_{j_n})$ for some real numbers θ_j, $j = 1, \ldots, n$ and a permutation $(j_1, \ldots, j_n)$ of the integers from 1 to n.

Note that surjective multiplicative isometries of the Privalov class, the Bergman-Privalov class, and the Zygmund F-algebra have the same form as surjective multiplicative isometries of the Smirnov class.

Proof of Corollary 3.11. By Theorem 3.10, T is complex-linear or conjugate linear. Suppose that T is complex-linear. Applying [12, Corollary 1] for $X = \mathbb{B}_n$ and Corollary 3.8 for $X = \mathbb{D}^n$ the result follows in this case. If T is conjugate linear, then the result follows by similar arguments as in the proof of Corollary 3.2. □

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