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Research Article

Convergence and Stability in Collocation Methods of Equation $u'(t) = au(t) + bu([t])$

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This paper is concerned with the convergence, global superconvergence, local superconvergence, and stability of collocation methods for $u'(t) = au(t) + bu([t])$. The optimal convergence order and superconvergence order are obtained, and the stability regions for the collocation methods are determined. The conditions that the analytic stability region is contained in the numerical stability region are obtained, and some numerical experiments are given.

1. Introduction

This paper deals with the convergence, superconvergence, and stability of the collocation methods of the following differential equation with piecewise continuous argument (EPCA):

$$\begin{aligned} u'(t) &= au(t) + bu([t]), \quad t \in [0, T], \\ u(0) &= u_0, \end{aligned} \tag{1.1}$$

where T is an integer, $a, b \in \mathbb{R}$, $u_0 \in \mathbb{C}^d$ is a given initial value, $u(t) \in \mathbb{C}^d$ is an unknown function, and $[\cdot]$ denotes the greatest integer function. The general form of EPCA is

$$\begin{aligned} u'(t) &= f(t, u(t), u(\alpha(t))), \quad t \geq 0, \\ u(0) &= u_0, \end{aligned} \tag{1.2}$$

where the argument $\alpha(t)$ has intervals of constancy. This kind of equations has been initiated by Wiener [1, 2], Cooke and Wiener [3], and Shah and Wiener [4]. The general theory and basic results for EPCA have by now been thoroughly investigated in the book of Wiener [5].

There are some authors who have considered the stability of numerical solutions for this kind of equations (see [6–8]). Though (1.1) is a delay differential equation (see [9–11]), the delay function $t-[t]$ is discontinuous. In [12], the convergence and superconvergence of collocation methods for a differential equation with piecewise linear delays is concerned.

Definition 1.1 (see Wiener [5]). A solution of (1.1) on $[0,\infty)$ is a function $u(t)$ that satisfies the following conditions.

(1) $u(t)$ is continuous on $[0,\infty)$.

(2) The derivative $u'(t)$ exists at each point $t\in[0,\infty)$, with the possible exception of the point $[t]\in[0,\infty)$, where one-sided derivatives exist.

(3) (1.1) is satisfied on each interval $[k,k+1)\subset[0,\infty)$ with integral endpoints.

Theorem 1.2 (see Wiener [5]). *Equation* (1.1) *has on* $[0,\infty)$ *a unique solution*

$$u(t)=m_0(\{t\})b_0^{[t]}u_0, \tag{1.3}$$

where $\{t\}$ *is the fractional part of t and*

$$m_0(t):=e^{at}+\left(e^{at}-1\right)a^{-1}b, \qquad b_0:=m_0(1). \tag{1.4}$$

Equation (1.1) *is asymptotically stable (the solution of* (1.1) *tends to zero as* $t\to\infty$*), for all* u_0*, if and only if the inequalities*

$$-a\frac{e^a+1}{e^a-1}<-b<-a \tag{1.5}$$

hold.

2. Existence and Uniqueness of Collocation Methods

Let $h:=1/p$ be a given step size with integer $p\geq 1$ and let the mesh on I be defined by

$$I_h:=\{t_n:0=t_0<t_1<\cdots<t_N=T\}. \tag{2.1}$$

Accordingly, the collocation points are chosen as

$$X_h:=\{t_{n,i}=t_n+c_ih:0<c_1<\cdots<c_m\leq 1(0\leq n\leq N-1)\}, \tag{2.2}$$

where $\{c_i\}$ denotes a given set of collocation parameters.

We approximate the solution by collocation in the piecewise polynomial spaces

$$S_m^{(0)}([0,T]):=\left\{v\in C([0,T]):v|_{[t_n,t_{n+1}]}\in\pi_m\right\}, \tag{2.3}$$

where π_m denotes the set of all real polynomials of degree not exceeding m. The collocation solution u_h is the element in this space that satisfies the collocation equation

$$\begin{gathered} u_h'(t) = au_h(t) + bu_h([t]), \quad t \in X_h, \\ u_h(0) = u_0. \end{gathered} \tag{2.4}$$

Let $Y_{n,j} := u_h'(t_n + c_j h)$. Then

$$u_h'(t_n + vh) = \sum_{j=1}^{m} L_j(v) Y_{n,j}, \quad v \in (0,1], \tag{2.5}$$

where

$$L_j(v) := \prod_{i=1, i \neq j}^{m} \frac{v - c_i}{c_j - c_i}. \tag{2.6}$$

Integrating the above equality, we can get that

$$u_h(t_n + vh) = u_h(t_n) + h \sum_{j=1}^{m} \beta_j(v) Y_{n,j}, \tag{2.7}$$

where $\beta_j(v) := \int_0^v L_j(s) ds$. So

$$Y_{n,i} = au_h(t_{n,i}) + bu_h([t_{n,i}]). \tag{2.8}$$

Let $n = kp + l$, $k \in \mathbb{Z}$, $l = 0, 1, 2, \ldots, p-1$. We have

$$Y_{kp+l,i} = au_h(t_{kp+l,i}) + bu_h(t_{kp}) = a\left(u_h(t_{kp+l}) + h\sum_{j=1}^{m} a_{ij} Y_{kp+l,j} \right) + bu_h(t_{kp}), \tag{2.9}$$

where $a_{ij} := \beta_j(c_i)$.

Denote $A = (a_{ij})_{m\times m}$, $Y_n = (Y_{n,1}, Y_{n,2}, \ldots, Y_{n,m})^T$, $\beta = (\beta_1, \beta_2, \ldots, \beta_m)^T$, $e = (1, 1, \ldots, 1)^T$ and for any $x_j \in \mathbb{R}$, $\sum_{j=0}^{-1} x_j = 0$ if $k = 0$. We have

$$(I_{m\times m} - haA) Y_{kp+l} = u_h(t_{kp+l}) ae + u_h(t_{kp}) be. \tag{2.10}$$

When the solution Y_n of (2.10) has been found, the collocation solution on the interval $[t_n, t_{n+1}]$ is determined by

$$u_h(t_n + vh) = u_h(t_n) + h\beta^T(v) Y_n. \tag{2.11}$$

So we can obtain the following theorem.

Theorem 2.1. *Assume that the given functions in* (1.1) *satisfy* $a, b \in \mathbb{R}, K \in C(D)$, *where* $D := \{(t,s) : 0 \le s \le t \le T\}$. *Then there exists an* $\overline{h} > 0$ *so that for the mesh* I_h *with mesh diameter* $h > 0$ *satisfying* $h < \overline{h}$, *and each of the linear algebraic systems* (2.10) *has a unique solution* $Y_n \in \mathbb{R}^m$. *Hence the collocation of* (2.4) *defines a unique collocation solution* $u_h \in S_m^{(0)}(I_h)$ *for the initial-value problem* (1.1), *and its representation on the subinterval* $[t_n, t_{n+1}]$ *is given by* (2.11).

3. Global Convergence Results

In the following, unless otherwise specified, the derivatives of u and u_h denote the left derivatives.

Theorem 3.1. *Assume the following:*

(1) *the given functions in* (1.1) *satisfy* $a, b \in \mathbb{R}$, $K \in C^m(D)$;

(2) $u_h \in S_m^{(0)}(I_h)$ *is the collocation solution to* (1.1) *defined by* (2.10) *and* (2.11) *with* $h \in (0, \overline{h})$.

Then the estimates

$$\left\| u^{(\nu)} - u_h^{(\nu)} \right\|_\infty := \max_{t\in[0,T]} \left| u^{(\nu)}(t) - u_h^{(\nu)}(t) \right| \le C_\nu \left\| u^{(m+1)} \right\|_\infty h^m \quad (\nu = 0, 1) \tag{3.1}$$

hold for any set $X_h (k = 1, 2, \ldots)$ *of collocation points with* $0 < c_1 < \cdots < c_m \le 1$. *The constants* C_ν *dependent on the collocation parameters* $\{c_i\}$ *and but not on* h.

Proof. The collocation error $e_h := u - u_h$ satisfies the equation

$$e_h'(t) = a e_h(t) + b e_h([t]), \quad t \in X_h, \tag{3.2}$$

with $e_h(0) = 0$. Assumption (1) implies that $u \in C^{m+1}([t_n, t_{n+1}])$ (at t_n, the derivative of u denotes the right derivative and at t_{n+1}, which denotes the left derivative) and hence $u' \in C^m([t_n, t_{n+1}])$. Thus we have, using Peano's Theorem for u' on $[t_n, t_{n+1}]$,

$$u'(t_n + vh) = \sum_{j=1}^{m} L_j(v) u'(t_{n,j}) + h^m R_{m+1,n}^{(1)}(v), \quad v \in (0, 1], \tag{3.3}$$

with the Peano remainder term, and Peano kernel are given by

$$\begin{gathered} R_{m+1,n}^{(1)}(v) := \int_0^1 K_m(v,z) u^{(m+1)}(t_n + zh)\,dz, \\ K_m(v,z) := \frac{1}{(m-1)!} \left\{ (v - z)_+^{m-1} - \sum_{j=1}^{m} L_j(v) (c_j - z)_+^{m-1} \right\}, \quad v \in (0, 1]. \end{gathered} \tag{3.4}$$

Integration of (3.3) leads to

$$u(t_n + vh) = u(t_n) + h\sum_{j=1}^{m}\beta_j(v)u'(t_{n,j}) + h^{m+1}R_{m+1,n}(v), \quad v \in (0,1], \tag{3.5}$$

where

$$R_{m+1,n}(v) := \int_0^v R^{(1)}_{m+1,n}(s)ds. \tag{3.6}$$

Recalling the local representation (2.5) of the collocation solution u_h on $(t_n, t_{n+1}]$ and setting $\varepsilon_{n,j} := u'(t_{n,j}) - Y_{n,j}$, the collocation error $e_h := u - u_h$ on $(t_n, t_{n+1}]$ may be written as

$$e_h(t_n + vh) = e_h(t_n) + h\sum_{j=1}^{m}\beta_j(v)\varepsilon_{n,j} + h^{m+1}R_{m+1,n}(v), \quad v \in (0,1], \tag{3.7}$$

while

$$e'_h(t_n + vh) = \sum_{j=1}^{m}L_j(v)\varepsilon_{n,j} + h^m R^{(1)}_{m+1,n}(v), \quad v \in (0,1]. \tag{3.8}$$

Since e_h is continuous in $[0,T]$, and hence at the mesh points, we also have the relation

$$e_h(t_n) = e_h(t_{n-1} + h) = e_h(t_{n-1}) + h\sum_{j=1}^{m}b_j\varepsilon_{n-1,j} + h^{m+1}R_{m+1,n-1}(1), \quad n = 1,\ldots,N-1, \tag{3.9}$$

with $b_j := \beta_j(1)$. The fact that $e_h(0) = 0$ yields

$$e_h(t_n) = h\sum_{j=1}^{m}b_j\sum_{r=0}^{n-1}\varepsilon_{r,j} + h^{m+1}\sum_{r=0}^{n-1}R_{m+1,r}(1), \quad n = 1,\ldots,N-1. \tag{3.10}$$

We are now ready to establish the estimates in Theorem 3.1. Let $n = kp + l$ $(l = 0,1,\ldots,p-1)$; since the collocation error satisfies

$$e'_h(t_{kp+l,i}) = ae_h(t_{kp+l,i}) + be_h(t_{kp}), \tag{3.11}$$

it follows from (3.7) and (3.8) that

$$\begin{aligned}\varepsilon_{kp+l,i} &= e'_h(t_{kp+l,i}) = ae_h(t_{kp+l,i}) + be_h(t_{kp})\\ &= a\left(e_h(t_{kp+l}) + h\sum_{j=1}^{m}a_{ij}\varepsilon_{kp+l,j} + h^{m+1}R_{m+1,kp+l}(c_i)\right) + be_h(t_{kp}).\end{aligned} \tag{3.12}$$

Denote

$$\varepsilon_n := (\varepsilon_{n,1}, \varepsilon_{n,2}, \ldots, \varepsilon_{n,m})^T,$$
$$R_{m+1,n} := (R_{m+1,n}(c_1), R_{m+1,n}(c_2), \ldots, R_{m+1,n}(c_m))^T, \tag{3.13}$$

we can get that

$$(I_{m\times m} - haA)\varepsilon_{kp+l} = e_h(t_{kp+l})ae + e_h(t_{kp})be + ah^{m+1}R_{m+1,kp+l}. \tag{3.14}$$

According to Theorem 2.1, this linear system has a unique solution whenever $h \in (0, \overline{h})$, and hence there exists a constant $D_0 < \infty$ so that $\|(I_{m\times m} - hA_n)^{-1}\|_1 \leq D_0$ uniformly for $0 \leq n \leq N-1$. Here, for $B \in L(\mathbb{R}^m)$, $\|B\|_1$ denotes the matrix (operator) norm induced by the l_1-norm in $\mathbb{R}^m$. Denote $M_{m+1} := \|u^{(m+1)}\|_\infty$, $K_m := \max_{v\in[0,1]} \int_0^1 |K_m(v,z)|dz$, $\overline{b} := \max_{1\leq j\leq m}|b_j|$, and $\overline{\beta} := \max_{1\leq i\leq m, v\in[0,1]}\beta_i(v)$. So

$$|e_h(t_n)| \leq h\overline{b}\sum_{r=0}^{n-1}\|\varepsilon_r\|_1 + h^m T K_m M_{m+1}. \tag{3.15}$$

Equation (3.14) now leads to the estimate

$$\begin{aligned}\|\varepsilon_{kp+l}\|_1 &\leq D_0\Bigg\{ m|a|h\overline{b}\sum_{r=0}^{kp+l-1}\|\varepsilon_r\|_1 + |a|mh^mTK_mM_{m+1} \\ &\qquad + |b|mh\overline{b}\sum_{r=0}^{kp-1}\|\varepsilon_r\|_1 + |b|mh^mTK_mM_{m+1} + |a|h^{m+1}mM_{m+1}K_m\Bigg\} \\ &\leq \gamma_0 h\sum_{r=0}^{kp+l-1}\|\varepsilon_r\|_1 + \gamma_1 M_{m+1}h^m,\end{aligned} \tag{3.16}$$

with obvious meanings of γ_0 and γ_1. By using the discrete Gronwall inequality, its solution is bounded by

$$\|\varepsilon_n\|_1 \leq \gamma_1 M_{m+1}h^m \exp(\gamma_0 T) =: BM_{m+1}h^m, \tag{3.17}$$

and so (3.15) yields

$$|e_h(t_n)| \leq \left(\overline{b}B + K_m T\right)M_{m+1}h^m. \tag{3.18}$$

Denote

$$\Lambda_m := \max_{1\leq j\leq m, v\in[0,1]} L_j(v), \tag{3.19}$$

we have

$$\begin{aligned}|e_h(t_n+vh)| &\le \left(\bar{b}B+K_mT\right)M_{m+1}h^m+h\bar{\beta}BM_{m+1}h^m+h^{m+1}M_{m+1}K_m =: C_0M_{m+1}h^m,\\ \left|e_h'(t_n+vh)\right| &\le \Lambda_m\|\varepsilon_n\|_1+h^mK_mM_{m+1}\le \Lambda_mBM_{m+1}h^m+h^mK_mM_{m+1} =: C_1M_{m+1}h^m.\end{aligned} \tag{3.20}$$

This concludes the proof of Theorem 3.1. □

4. Global Superconvergence Results

Theorem 4.1. *Assume that the assumptions* (2) *of Theorem 3.1 hold, and let* (1) *be replaced by* $a, b \in C^d(I)$ *and* $K \in C^d(D)$, *with* $d \ge m+1$. *If the* m *collocation parameters* $\{c_i\}$ *are subject to the orthogonality condition*

$$J_0 := \int_0^1 \prod_{i=1}^{m}(s-c_i)ds = 0, \tag{4.1}$$

then the corresponding collocation solution $u_h \in S_m^{(0)}(I_h)$ *satisfies, for* $h \in (0,\bar{h})$,

$$\|u-u_h\|_\infty \le C_2h^{m+1}, \tag{4.2}$$

with C_2 *depending on the collocation parameters and on* $\|u^{(m+2)}\|_\infty$ *but not on* h. *The exponent* $m+1$ *cannot, in general, be replaced by* $m+2$. *For the derivative* u_h', *we attain only* $\|u'-u_h'\|_\infty = O(h^m)$.

Proof. Let

$$\delta_h(t) := -u_h'(t)+au_h(t)+bu_h([t]), \quad t \in I, \tag{4.3}$$

denote the defect (or: residual) associated with the collocation solution $u_h \in S_m^{(0)}(I_h)$ to the initial-value problem (1.1). by definition of the collocation solution the defect δ_h vanishes on the set X_h as follwos:

$$\delta_h(t) = 0 \quad \forall t \in X_h. \tag{4.4}$$

Moreover, the uniform convergence of u_h and u_h' established in Theorem 3.1 implies the uniform boundedness (as $h \to 0$) of δ_h on I, as well as that of its derivatives of order not exceeding d (here the derivatives refer to the left derivatives).

It following from (4.3) that the collocation error $e_h = u-u_h$ satisfies the equation

$$\delta_h(t) = e_h'(t)-ae_h(t)-be_h([t]), \quad t \in I. \tag{4.5}$$

By Theorem 3.1, there exists a constant D, such that

$$\|\delta_h(t)\|_\infty \le Dh^mM_{m+1}, \tag{4.6}$$

and this holds for any choice of the $\{c_i\}$. On the other hand, the collocation error e_h solves the initial-value problem

$$e_h'(t) = ae_h(t) + be_h([t]) + \delta_h(t), \quad t \in I, \; e_h(0) = 0. \tag{4.7}$$

For $t \in [k, k+1]$, whose solution is given by

$$e_h(t) = \left[r(t,k) + \int_k^t br(t,s)ds \right] e_h(k) + \int_k^t r(t,s)\delta_h(s)ds, \quad t \in I. \tag{4.8}$$

The function $r = r(t,s)$ denotes the "resolvent" (or: resolvent kernel) of (1.1) as follows:

$$r(t,s) := e^{a(t-s)}, \quad \text{with } r \in C^{m+1}(D). \tag{4.9}$$

If $k = 0$, let $t = t_l + vh$, $v \in [0,1]$, and $0 \le l \le p-1$; we have

$$\begin{aligned} e_h(t_l + vh) &= \int_0^{t_l+vh} r(t_l + vh, s)\delta_h(s)ds \\ &= \sum_{j=0}^{l-1} \int_{t_j}^{t_{j+1}} r(t_l + vh, s)\delta_h(s)ds + \int_{t_l}^{t_l+hv} r(t_l + vh, s)\delta_h(s)ds \\ &= h\sum_{j=0}^{l-1} \int_0^1 r\left(t_l + vh, t_j + hs\right)\delta_h\left(t_j + hs\right)ds + h\int_0^v r(t_l + vh, t_l + hs)\delta_h(t_l + hs)ds. \end{aligned} \tag{4.10}$$

Suppose now that each of the integrals over $[0,1]$ is approximated by the interpolatory m-point quadrature formula with abscissas $\{c_i\}$, then

$$\int_0^1 r\left(t_l + vh, t_j + hs\right)\delta_h\left(t_j + hs\right)ds = \sum_{i=1}^{m} b_j r\left(t_l + vh, t_j + hc_i\right)\delta_h\left(t_j + hc_i\right) + E_j(v), \quad v \in [0,1]. \tag{4.11}$$

Here, terms $E_j(v)$ denote the quadrature errors induced by these quadrature approximations. By assumption (4.1) each of these quadrature formulas has degree of precision m, and thus the Peano Theorem for quadrature implies that the quadrature errors can be bounded by

$$\left|E_j(v)\right| \le Qh^{m+1}, \quad v \in [0,1], \tag{4.12}$$

because the defect δ_h is in C^{m+1} on each subinterval $[t_n, t_{n+1}]$. Due to the special choice of the quadrature abscissas, we have $\sum_{i=1}^{m} b_j r(t_l + vh, t_j + hc_i)\delta_h(t_j + hc_i) = 0$, because $\delta_h(t) = 0$ whenever $t \in X_h$. Hence

$$e_h(t_l + vh) = h\sum_{j=0}^{l-1} E_j(v) + h\int_0^v r(t_l + vh, t_l + hs)\delta_h(t_l + hs)ds, \quad v \in [0,1]. \tag{4.13}$$

This leads to the estimate

$$|e_h(t_l + vh)| \le h\sum_{j=0}^{l-1} Qh^{m+1} + hr_0\|\delta_h\|_\infty \le QTh^{m+1} + Dr_0h^{m+1}M_{m+1} =: \overline{C}_0h^{m+1}, \tag{4.14}$$

for $0 \le l \le p-1$ and $v \in [0,1]$, with $r_0 := \max_{t\in I}\int_0^T |r(t,s)|ds$.

We assume for $t \in [k-1, k]$

$$\left|e_h\left(t_{(k-1)p+l} + vh\right)\right| \le \overline{C}_{k-1}h^{m+1}, \quad v \in [0,1],\ 0 \le l \le p-1. \tag{4.15}$$

Then for $t \in [k, k+1]$, let $t = t_{kp+l} + vh,\ v \in [0,1]$, and $0 \le l \le p-1$; we have

$$\begin{aligned} e_h\left(t_{kp+l} + vh\right) &= \left[r\left(t_{kp+l} + vh, k\right) + \int_k^{t_{kp+l}+vh} br\left(t_{kp+l} + vh, s\right)ds\right]e_h(k) \\ &\quad + \int_k^{t_{kp+l}+vh} r\left(t_{kp+l} + vh, s\right)\delta_h(s)ds \\ &= \left[r\left(t_{kp+l} + vh, k\right) + \int_k^{t_{kp+l}+vh} br\left(t_{kp+l} + vh, s\right)ds\right]e_h(k) \\ &\quad + h\sum_{j=kp}^{kp+l-1}\int_0^1 r\left(t_{kp+l} + vh, t_j + hs\right)\delta_h\left(t_j + hs\right)ds \\ &\quad + h\int_0^v r\left(t_{kp+l} + vh, t_{kp+l} + hs\right)\delta_h\left(t_{kp+l} + hs\right)ds. \end{aligned} \tag{4.16}$$

Similarly to the case of $t \in [0,1]$, we have

$$\left|e_h\left(\mathrm{t}_{kp+l} + vh\right)\right| \le (r_0 + r_0|b|)\overline{C}_{k-1}h^{m+1} + pQh^{m+2} + r_0DM_{m+1}h^{m+1} =: \overline{C}_kh^{m+1}. \tag{4.17}$$

This completes the proof. □

5. The Local Superconvergence Results on I_h

Theorem 5.1. *Assume the following:*

(a) $a, b \in C^{m+\kappa}(I)$ *and* $K \in C^{m+\kappa}(D)$*, for some* κ *with* $1 \leq \kappa \leq m$ *and value as specified in* (b) *below,*

(b) *The* m *distinct collocation parameters* $\{c_i\}$ *are chosen so that the general orthogonality condition*

$$J_\nu := \int_0^1 s^\nu \prod_{i=1}^{m} (s - c_i) ds = 0, \quad \nu = 0, \ldots, \kappa - 1 \tag{5.1}$$

holds, with $J_\kappa \neq 0$.

Then, for all meshes I_h *with* $h \in (0, \overline{h})$*, the collocation solution* $u_h \in S_m^{(0)}(I_h)$ *corresponding to the collocation points* X_h *based on these* $\{c_i\}$ *satisfies*

$$\max\{|u(t) - u_h(t)| : t \in I_h\} \leq C_3 h^{m+\kappa}, \tag{5.2}$$

where C_3 *depends on the collocation parameters and on* $\|u^{(m+\kappa+1)}\|_\infty$ *but not on* h.

Proof. If $k = 0$, for $t = t_l$ $(0 \leq l \leq p - 1)$

$$\begin{aligned} e_h(t_l) &= \int_0^{t_l} r(t_l, s)\delta_h(s) ds = \sum_{j=0}^{l-1} \int_{t_j}^{t_{j+1}} r(t_l + vh, s)\delta_h(s) ds \\ &= h\sum_{j=0}^{l-1} \int_0^1 r(t_l + vh, t_j + hs)\delta_h(t_j + hs) ds \\ &= h\sum_{j=0}^{l-1} \left(\sum_{i=1}^{m} b_j r(t_l + vh, t_j + hc_i)\delta_h(t_j + hc_i) + E_j(v) \right), \end{aligned} \tag{5.3}$$

with

$$\left|E_j(v)\right| \leq Ch^{m+\kappa}, \tag{5.4}$$

so

$$|e_h(t_l)| \leq Ch^{m+\kappa}. \tag{5.5}$$

By the induction method similarly to the proof of Theorem 4.1, the assertion of Theorem 5.1 follows. □

Table 1: The absolute values of absolute errors of u_h for example (7.1) with $m = 2$.

N	Gauss	Radau IIA	Lobatto IIIA	$(1/4, 1/2)$	$(1/3, 2/3)$	$(2/3, 1)$
2^1	$1.2551e-03$	$3.1476e-04$	$1.3124e-03$	$2.9655e+02$	$1.3113e-03$	$9.3953e-06$
2^2	$4.5928e-06$	$1.4601e-06$	$8.5094e-04$	$1.2874e-03$	$6.4778e-05$	$8.1731e-10$
2^3	$2.4209e-10$	$7.8934e-11$	$8.1731e-10$	$8.5023e-07$	$1.0324e-08$	$9.3978e-10$
2^4	$6.1162e-12$	$2.8697e-11$	$7.7079e-11$	$5.7012e-10$	$1.3715e-10$	$1.8633e-10$
2^5	$3.4588e-13$	$4.5808e-12$	$3.9746e-11$	$4.1164e-11$	$2.0163e-11$	$4.3362e-11$
2^6	$2.1210e-14$	$6.2301e-13$	$1.2117e-11$	$7.6587e-12$	$4.4835e-12$	$1.1275e-11$
Ratio	$1.6307e+01$	$7.3526e+00$	$3.2802e+00$	$5.3747e+00$	$4.4973e+00$	$3.8460e+00$

Table 2: The absolute values of absolute errors of u_h for example (7.1) with $m = 3$.

N	Gauss	Radau IIA	Lobatto IIIA	$(1/3, 1/2, 2/3)$	$(1/4, 1/3, 1/2)$	$(1/2, 2/3, 1)$
2^1	$3.1476e-04$	$9.9709e-05$	$1.2551e-03$	$1.2301e-03$	$4.9431e+04$	$4.8903e-06$
2^2	$3.2248e-11$	$1.6050e-08$	$4.5928e-06$	$2.6790e-06$	$9.5944e-04$	$3.2687e-09$
2^3	$4.7508e-12$	$2.0843e-11$	$2.4209e-10$	$6.7725e-11$	$7.9577e-11$	$1.1031e-10$
2^4	$6.3943e-14$	$5.8750e-13$	$6.1162e-12$	$6.7093e-12$	$2.9944e-11$	$1.6213e-11$
2^5	$9.5085e-16$	$1.9290e-14$	$3.4588e-13$	$4.1702e-13$	$3.1783e-12$	$2.5836e-12$
2^6	$1.0192e-17$	$6.3187e-16$	$2.1210e-14$	$2.5846e-14$	$3.3739e-13$	$3.7792e-13$
Ratio	$9.3298e+01$	$3.0528e+01$	$1.6307e+01$	$1.6135e+01$	9.4201	$6.8363e+00$

6. Numerical Stability

In this section, we will discuss the stability of the collocation methods. We introduce the set H consisting of all pairs $(a, b) \in \mathbb{R}^2$ which satisfy the condition

$$H := \left\{ (a, b) : -a\frac{e^a + 1}{e^a - 1} < b < -a \right\}, \tag{6.1}$$

and divide the region into three parts:

$$\begin{aligned} H_0 &:= \{(a, b) : (a, b) \in H,\ a = 0\}, \\ H_1 &:= \{(a, b) : (a, b) \in H,\ a < 0\}, \\ H_2 &:= \{(a, b) : (a, b) \in H,\ a > 0\}. \end{aligned} \tag{6.2}$$

By (2.9) and (2.10), we can obtain that

$$u(t_{kp+l+1}) = R(x)u(t_{kp+l}) + \alpha(x, y)u(t_{kp}), \quad l = 0, 1, \ldots, p-1, \tag{6.3}$$

where $x := ha, y := hb$, $R(x) := 1 + b^T x(I - Ax)^{-1}e$, and $\alpha(x, y) := y(1 + xb^T(I - Ax)^{-1}e) = yb^T(I - Ax)^{-1}e$.

Let $U_k := (u_{kp}, u_{kp+1}, \ldots, u_{kp+p})^T$ and $B := \prod_{i=1}^{p} B_i$. It is easy to see

$$U_k = BU_{k+1}, \quad k = 1, 2, \ldots, \tag{6.4}$$

Table 3: The absolute values of absolute errors of u_h for example (7.2) with $m = 2$.

N	Gauss	Radau IIA	Lobatto IIIA	(1/4, 1/2)	(1/3, 2/3)	(2/3, 1)
2^1	$9.0789e-01$	$9.1680e-01$	$9.1647e-01$	$9.0070e-01$	$9.1460e-01$	$9.1700e-01$
2^2	$2.3033e-01$	$8.6133e-01$	$4.2429e-01$	$5.4055e-01$	$7.4023e-01$	$9.1661e-01$
2^3	$8.9360e-03$	$1.2191e-01$	$8.3085e-02$	$1.3822e-01$	$1.5563e-01$	$7.6499e-01$
2^4	$5.0981e-04$	$9.9479e-03$	$5.9089e-02$	$3.4017e-02$	$2.8752e-02$	$1.5140e-01$
2^5	$3.1272e-05$	$1.0927e-03$	$1.8134e-02$	$8.7800e-03$	$6.5674e-03$	$2.5303e-02$
2^6	$1.9458e-06$	$1.2993e-04$	$4.7169e-03$	$2.2713e-03$	$1.6041e-03$	$5.3924e-03$
Ratio	$1.6071e+01$	$8.4098e+00$	$3.8445e+00$	$3.8656e+00$	$4.0940e+00$	$4.6924e+00$

Table 4: The absolute values of absolute errors of u_h for example (7.2) with $m = 3$.

N	Gauss	Radau IIA	Lobatto IIIA	(1/3, 1/2, 2/3)	(1/4, 1/3, 1/2)	(1/2, 2/3, 1)
2^1	$4.8459e-02$	$9.0087e-01$	$9.0789e-01$	$9.0508e-01$	$7.4280e-02$	$9.1693e-01$
2^2	$7.0515e-03$	$5.0901e-02$	$2.3033e-01$	$8.2874e-02$	$2.5303e-02$	$8.3086e-02$
2^3	$9.4311e-05$	$1.3131e-03$	$8.9360e-03$	$9.9825e-03$	$1.2122e-02$	$6.3787e-02$
2^4	$1.4090e-06$	$3.5203e-05$	$5.0981e-04$	$6.1535e-04$	$2.2659e-03$	$7.4707e-03$
2^5	$2.1766e-08$	$1.0294e-06$	$3.1272e-05$	$3.8124e-05$	$3.4308e-04$	$7.9289e-04$
2^6	$3.3592e-10$	$3.1217e-08$	$1.9458e-06$	$2.3768e-06$	$4.7157e-05$	$9.0681e-05$
Ratio	$6.4794e+01$	$3.2977e+01$	$1.6071e+01$	$1.6040e+01$	$7.2752e+00$	$8.7437e+00$

where

$$B = \begin{pmatrix} 0 & \cdots & 0 & b_{1,p+1} \\ 0 & \cdots & 0 & b_{2,p+1} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & b_{p+1,p+1} \end{pmatrix},$$

$$b_{i,p+1} = \begin{cases} 1 + \left(1 + \dfrac{a}{b}\right)\left[R(x)^{i-1} - 1\right], & a \neq 0, \\ 1 + (i-1)hb, & a = 0. \end{cases} \quad i = 1, 2, \ldots, p+1. \tag{6.5}$$

Let $\varphi(x) := b^T(I - xA)^{-1}e$. Then there exists $\delta > 0$ such that

$$\varphi(x) > 0 \quad \forall x \text{ with } |x| \leq \delta, \tag{6.6}$$

since $\varphi(0) = 1$ and $\varphi(x)$ is continuous in a neighborhood of zero. In the rest of the paper we define

$$M := \begin{cases} 1, & a \leq 0, \\ \dfrac{a}{\delta}, & a > 0. \end{cases} \tag{6.7}$$

Definition 6.1 (see [6]). Process (2.11) for (1.1) is called asymptotically stable at (a, b) if and only if for all $m \geq M$ and $h = 1/m$.

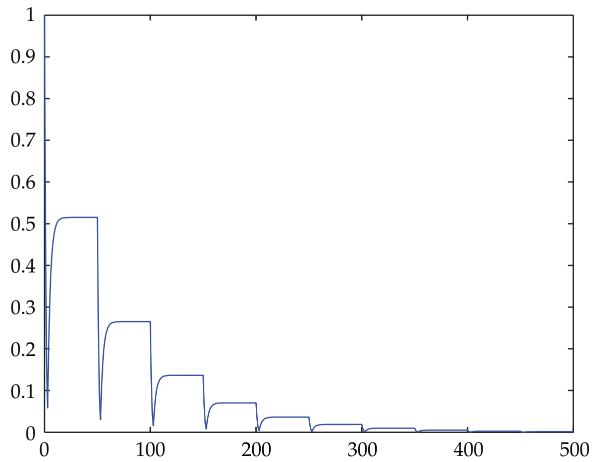

Figure 1: The Gauss collocation method with $m = 2$ and $p = 50$ for (7.1).

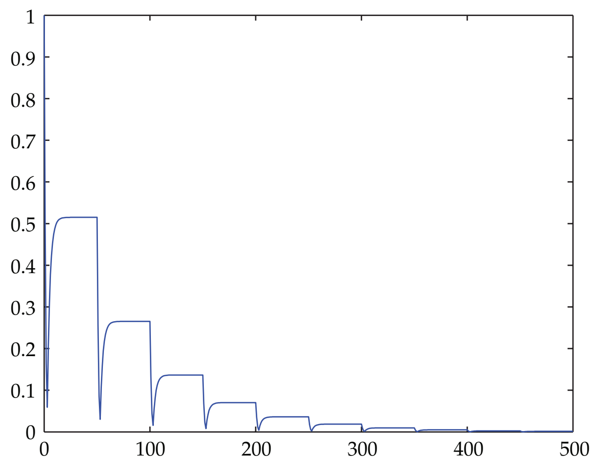

Figure 2: The Radau IIA collocation method with $m = 2$ and $p = 100$ for (7.1).

(i) $(I - xA)$ is invertible.

(ii) for any given u_i $(1 \leq i \leq m)$ relation (6.4) defines U_k $(k = 1, 2, \ldots)$ that satisfy $U_k \to 0$ for $k \to \infty$.

Definition 6.2 (see [6]). The set of all pairs (a, b) at which the process (2.11) for (1.1) is asymptotically stable is called asymptotical stability region denoted by S.

Theorem 6.3 (see [6]). *Suppose that the collocation method is A_0-stable and the stability function is given by the (r, s)-Padé approximation to the exponential e^x. Then $H_1 \subseteq S$ if and only if r is even.*

Theorem 6.4 (see [6]). *Suppose that the stability function of the collocation method is given by the (r, s)-Padé approximation to the exponential e^z. Then $H_2 \subseteq S$ if and only if s is even.*

Theorem 6.5 (see [6]). *For all the collocation methods, we have $H_0 \subseteq S$.*

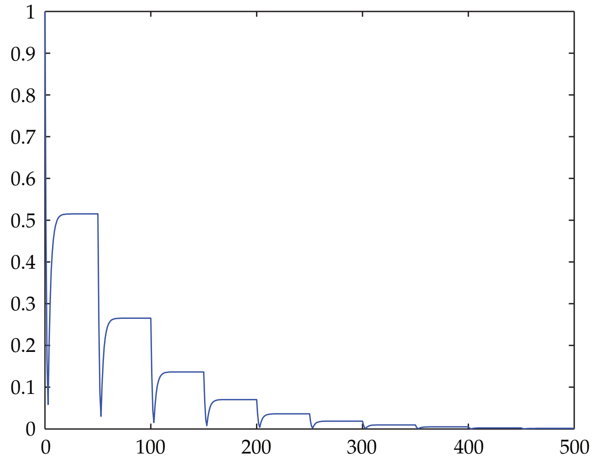

Figure 3: The Gauss collocation method with $m = 3$ and $p = 50$ for (7.1).

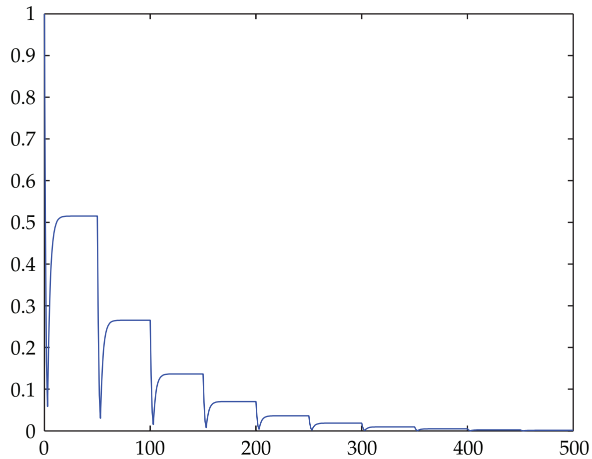

Figure 4: The Radau IIA collocation method with $m = 3$ and $p = 1000$ for (7.1).

Using the above theorems we can formulate the following result.

Theorem 6.6 (see [6]). *Suppose that the collocation method is A_0-stable and the stability function is given by the (r, s)-Padé approximation to the exponential e^x. Then $H_0 \subseteq S$ and $H \subseteq S$ if and only if both r and s are even,*

$$\begin{aligned} H_1 \subseteq S &\quad \text{iff } r, \\ H_2 \subseteq S &\quad \text{iff } s \text{ is even.} \end{aligned} \tag{6.8}$$

Corollary 6.7. *For the A-stable higher order collocation methods, it is easy to see from Theorem 6.6.*

(i) *For the ν-stage Gauss-Legendre method, $H \subseteq S$ if and only if ν is even.*

(ii) *For the ν-stage Lobatto IIIA method, $H \subseteq S$ if and only if ν is old.*

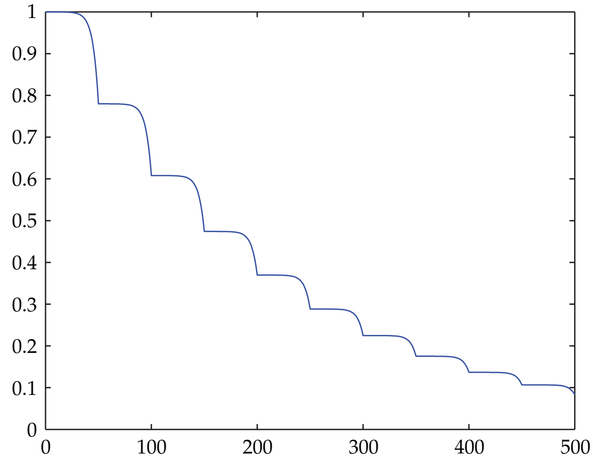

Figure 5: The Gauss collocation method with $m = 2$ and $p = 50$ for (7.2).

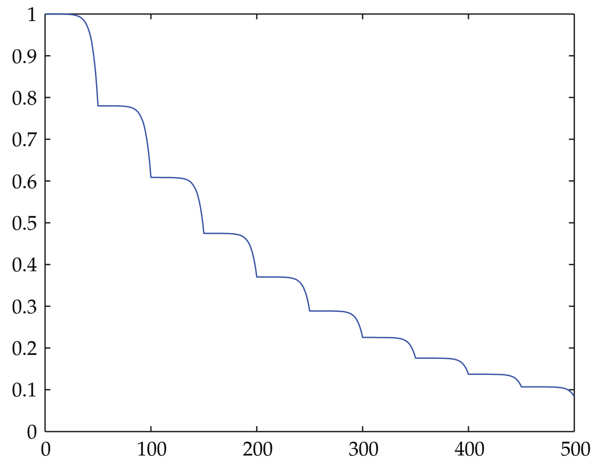

Figure 6: The Radau IIA collocation method with $m = 2$ and $p = 1000$ for (7.2).

(iii) *For the* ν*-stage Radau IIA method,* $H_1 \subseteq S$ *if and only if* ν *is old and* $H_2 \subseteq S$ *if and only if* ν *is even.*

7. Numerical Experiments

In order to give a numerical illustration to the conclusions in the paper, we consider the following two problems ([6]):

$$u_1'(t) = -20u_1(t) - 10.3u_1([t]), \quad u_1(0) = 1, \tag{7.1}$$

$$u_2'(t) = 10u_2(t) - 10.0001u_2([t]), \quad u_2(0) = 1. \tag{7.2}$$

It can be checked that $(-20, -10.3) \in H_1$ and $(10, -10.0001) \in H_2$.

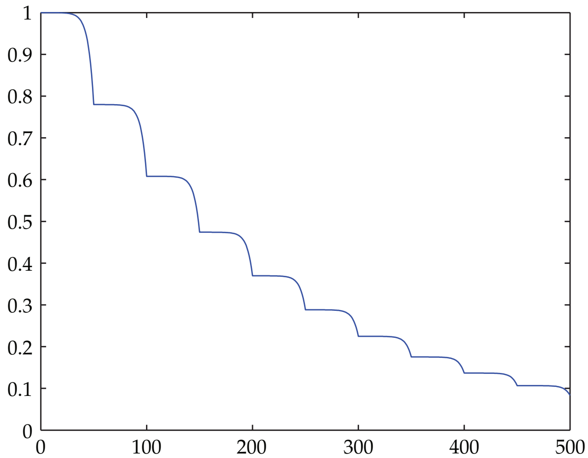

Figure 7: The Gauss collocation method with $m = 3$ and $p = 50$ for (7.2).

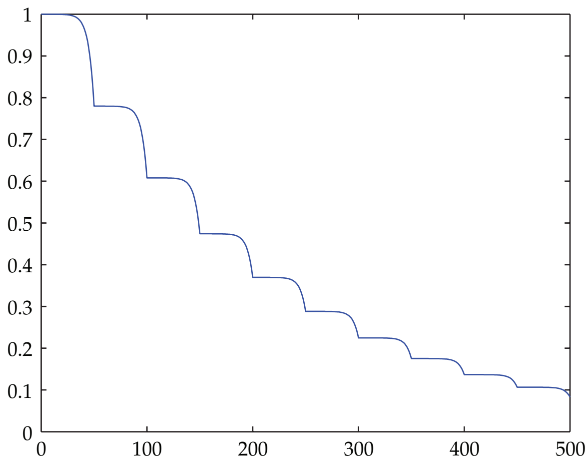

Figure 8: The Radau IIA collocation method with $m = 3$ and $p = 1000$ for (7.2).

For illustrating the convergence and superconvergence orders in this paper, we choose $m = 2$ and $m = 3$ and use the Gauss collocation parameters: $c_1 = (3 - \sqrt{3})/6$, $c_2 = (3 + \sqrt{3})/6$, the Radau IIA collocation parameters: $c_1 = 1/3$, $c_2 = 1$, the Lobatto IIIA collocation parameters: $c_1 = 0$, $c_2 = 1$, and three sets of random collocation parameters: $c_1 = 1/4$, $c_2 = 1/2$; $c_1 = 1/3$, $c_2 = 2/3$; $c_1 = 2/3$, $c_2 = 1$, respectively, for $m = 2$; and we use the Gauss collocation parameters: $c_1 = (5 - \sqrt{15})/10$, $c_2 = 1/2$, and $c_3 = (5 + \sqrt{15})/10$, the Radau IIA collocation parameters: $c_1 = (4 - \sqrt{6})/10$, $c_2 = (4 + \sqrt{6})/10$, and $c_3 = 1$, the Lobatto IIIA collocation parameters: $c_1 = 0$, $c_2 = 1/2$, and $c_3 = 1$, and three sets of random collocation parameters: $c_1 = 1/3$, $c_2 = 1/2$, $c_3 = 2/3$; $c_1 = 1/4$, $c_2 = 1/3$, $c_3 = 1/2$; $c_1 = 1/2$, $c_2 = 2/3$, $c_3 = 1$, respectively, for $m = 3$. In Tables 1, 2, 3, and 4 we list the absolute values of the absolute errors of $ut = 10$ for the six collocation parameters and for $m = 2$ and $m = 3$, respectively, and the ratios of the absolute values of the errors of $N = 100$ over that of $N = 200$.

From the above tables, we can see that the convergence orders are consistent with our theoretical analysis.

In Figures 1, 2, 3, 4, 5, 6, 7, and 8, we draw the absolute values of the numerical solution of collocation methods. It is easy to see that the numerical solution is asymptotically stable.

Acknowledgments

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