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Research Article

A Note on Inclusion Intervals of Matrix Singular Values

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We establish an inclusion relation between two known inclusion intervals of matrix singular values in some special case. In addition, based on the use of positive scale vectors, a known inclusion interval of matrix singular values is also improved.

1. Introduction

The set of all n-by-n complex matrices is denoted by $\mathbb{C}^{n\times n}$. Let $A=(a_{ij})\in\mathbb{C}^{n\times n}$. Denote the Hermitian adjoint of matrix A by A^*. Then the singular values of A are the eigenvalues of $(AA^*)^{1/2}$. It is well known that matrix singular values play a very key role in theory and practice. The location of singular values is very important in numerical analysis and many other applied fields. For more review about singular values, readers may refer to [1–9] and the references therein.

Let $N=\{1,2,\ldots,n\}$. For a given matrix $A=(a_{ij})\in\mathbb{C}^{n\times n}$, we denote the deleted absolute row sums and column sums of A by

$$r_i=\sum_{j=1,\neq i}^{n}|a_{ij}|,\qquad c_i=\sum_{j=1,\neq i}^{n}|a_{ji}|,\qquad i\in N, \tag{1.1}$$

respectively. On the basis of r_i and c_i, the Geršgorin's disk theorem, Brauer's theorem and Brualdi's theorem provide some elegant inclusion regions of the eigenvalues of A (see [10–12]). Recently, some authors have made efforts to establish analogues to these theorems for matrix singular values, for example, as follows.

Theorem A (Geršgorin-type [8]). *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$. Then all singular values of A are contained in*

$$G(A) \equiv \bigcup_{i=1}^{n} B_i, \quad \textit{with } B_i = \{z \geq 0 : |z - a_i| \leq s_i\}, \tag{1.2}$$

where $s_i = \max\{r_i, c_i\}$ and $a_i = |a_{ii}|$ for each $i \in N$.

Theorem B (Brauer-type [5]). *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$. Then all singular values of A are contained in*

$$B(A) \equiv \bigcup_{i,j=1, i\neq j}^{n} \left\{z \geq 0 : |z - a_i| \left|z - a_j\right| \leq s_i s_j\right\}. \tag{1.3}$$

Let S denote a nonempty subset of N, and let $\overline{S} = N \setminus S$ denote its complement in N. For a given matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ with $n \geq 2$, define partial absolute deleted row sums and column sums as follows:

$$\begin{aligned} r_i^S(A) &= \sum_{j\in S\setminus\{i\}} \left|a_{ij}\right|, & r_i^{\overline{S}}(A) &= \sum_{j\in \overline{S}\setminus\{i\}} \left|a_{ij}\right|; \\ c_i^S(A) &= \sum_{j\in S\setminus\{i\}} \left|a_{ji}\right|, & c_i^{\overline{S}}(A) &= \sum_{j\in \overline{S}\setminus\{i\}} \left|a_{ji}\right|. \end{aligned} \tag{1.4}$$

Thus, one splits each row sum r_i and each column sum c_i from (1.1) into two parts, depending on S and $\overline{S}$, that is,

$$r_i = r_i^S(A) + r_i^{\overline{S}}(A), \qquad c_i = c_i^S(A) + c_i^{\overline{S}}(A). \tag{1.5}$$

Define, for each $i \in S$, $j \in \overline{S}$,

$$\begin{aligned} \mathcal{G}_i^S(A) &= \left\{z \geq 0 : |z - a_i| \leq s_i^S\right\}, \\ \mathcal{G}_j^{\overline{S}}(A) &= \left\{z \geq 0 : \left|z - a_j\right| \leq s_j^{\overline{S}}\right\}, \\ \mathcal{V}_{ij}^S(A) &= \left\{z \geq 0 : \left(|z - a_i| - s_i^S\right)\left(\left|z - a_j\right| - s_j^{\overline{S}}\right) \leq s_i^{\overline{S}} s_j^S\right., \end{aligned} \tag{1.6}$$

where

$$s_i^S = \max\left\{r_i^S(A), c_i^S(A)\right\}, \qquad s_i^{\overline{S}} = \max\left\{r_i^{\overline{S}}(A), c_i^{\overline{S}}(A)\right\}. \tag{1.7}$$

For convenience, we will sometimes use r_i^S (c_i^S, $r_i^{\overline{S}}$, $c_i^{\overline{S}}$) to denote $r_i^S(A)$ ($c_i^S(A)$, $r_i^{\overline{S}}(A)$, $c_i^{\overline{S}}(A)$, resp.) unless a confusion is caused.

Theorem C (modified Brauer-type [7]). *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ with $n \geq 2$. Then all singular values of A are contained in*

$$\sigma(A) \subseteq \mathcal{G}\mathcal{V}^S(A) \equiv \mathcal{G}^S(A) \cup \mathcal{V}^S(A), \tag{1.8}$$

where

$$\mathcal{G}^S(A) = \left(\bigcup_{i\in S} \mathcal{G}_i^S(A)\right) \cup \left(\bigcup_{j\in \overline{S}} \mathcal{G}_j^{\overline{S}}(A)\right),$$
$$\mathcal{V}^S(A) = \bigcup_{i\in S, j\in \overline{S}} \mathcal{V}_{ij}^S(A). \tag{1.9}$$

A simple analysis shows that Theorem B improves Theorem A. On the other hand, Theorem C reduces to Theorem A if $S = \emptyset$ or $\overline{S} = \emptyset$ (see Remark 2.3 in [7]).

Now it is natural to ask whether there exists an inclusion relation between Theorem B and Theorem C or not. In this note, we establish an inclusion relation between the inclusion interval of Theorem B and that of Theorem C in a particular situation. In addition, based on the use of positive scale vectors and their intersections, the inclusion interval of matrix singular values in Theorem C is also improved.

2. Main Results

In this section, we will establish an inclusion relation between the inclusion interval of Theorem B and that of Theorem C in a particular situation. We firstly remark that Theorem B and Theorem C are incomparable, for example, as follows.

Example 2.1. Consider the following matrix:

$$A = \begin{pmatrix} 1 & 0.1 & 0.1 & 0 \\ 0 & 2 & 0 & 0.1 \\ 1 & 0 & 3 & 0.1 \\ 0 & 1 & 0 & 4 \end{pmatrix}. \tag{2.1}$$

Let $S = \{1\}$ and $\overline{S} = \{2, 3, 4\}$. Applying Theorem C, one gets

$$\begin{aligned}
\mathcal{G}_1^S(A) &= \{z \geq 0 : |z-1| \leq 0\} = \{1\}, \\
\mathcal{G}_2^{\overline{S}}(A) &= \{z \geq 0 : |z-2| \leq 1\} = [1, 3], \\
\mathcal{G}_3^{\overline{S}}(A) &= \{z \geq 0 : |z-3| \leq 0.1\} = [2.9, 3.1], \\
\mathcal{G}_4^{\overline{S}}(A) &= \{z \geq 0 : |z-4| \leq 1\} = [3, 5], \\
\mathcal{V}_{12}^S(A) &= \{z \geq 0 : (|z-1|)(|z-2|-1) \leq 0.1\} = [0.6838, 3.0488], \\
\mathcal{V}_{13}^S(A) &= \{z \geq 0 : (|z-1|)(|z-3|-0.1) \leq 1\} = [0.5707, 3.5000], \\
\mathcal{V}_{14}^S(A) &= \{z \geq 0 : (|z-1|)(|z-4|-1) \leq 0\} = \{1\} \cup [3, 5].
\end{aligned} \tag{2.2}$$

Hence, the inclusion interval of $\sigma(A)$ is $[0.5707, 5]$.

Now applying Theorem B, one gets

$$\begin{aligned}
\{z \ge 0 : |z-1|\,|z-2| \le 1.1\} &= [0.3381, 2.6619],\\
\{z \ge 0 : |z-1|\,|z-3| \le 1.1\} &= [0.5509, 3.4491],\\
\{z \ge 0 : |z-1|\,|z-4| \le 1\} &= [0.6972, 1.3820] \cup [3.6180, 4.3028],\\
\{z \ge 0 : |z-2|\,|z-3| \le 1.21\} &= [1.2917, 3.7083],\\
\{z \ge 0 : |z-2|\,|z-4| \le 1.1\} &= [1.5509, 4.4491],\\
\{z \ge 0 : |z-3|\,|z-4| \le 1.1\} &= [2.3381, 4.6619].
\end{aligned} \tag{2.3}$$

Therefore, the inclusion interval of $\sigma(A)$ is $[0.3381, 4.6619]$.

Example 2.1 shows that Theorem B and Theorem C are incomparable in the general case, but Theorem C may be better than Theorem B whenever the set S is chosen suitably, for example, as follows.

Example 2.2. Take $S = \{1,2\}$ and $\overline{S} = \{3,4\}$ in Example 2.1. Applying Theorem C, one gets

$$\begin{aligned}
\mathcal{G}_1^S(A) &= \{z \ge 0 : |z-1| \le 0.1\} = [0.9, 1.1],\\
\mathcal{G}_2^S(A) &= \{z \ge 0 : |z-2| \le 0.1\} = [1.9, 2.1],\\
\mathcal{G}_3^{\overline{S}}(A) &= \{z \ge 0 : |z-3| \le 0.1\} = [2.9, 3.1],\\
\mathcal{G}_4^{\overline{S}}(A) &= \{z \ge 0 : |z-4| \le 0.1\} = [3.9, 4.1],\\
\mathcal{V}_{13}^S(A) &= \{z \ge 0 : (|z-1|-0.1)(|z-3|-0.1) \le 1\} = [0.4858, 3.5142],\\
\mathcal{V}_{23}^S(A) &= \{z \ge 0 : (|z-2|-0.1)(|z-3|-0.1) \le 1\} = [1.2820, 3.7180],\\
\mathcal{V}_{14}^S(A) &= \{z \ge 0 : (|z-1|-0.1)(|z-4|-0.1) \le 1\} = [0.5972, 1.5202] \cup [3.4798, 4.4028],\\
\mathcal{V}_{24}^S(A) &= \{z \ge 0 : (|z-2|-0.1)(|z-4|-0.1) \le 1\} = [1.4858, 4.5142].
\end{aligned} \tag{2.4}$$

Hence, the inclusion interval of $\sigma(A)$ is $[0.4858, 4.5142]$. However, applying Theorem B, we get that the inclusion interval of $\sigma(A)$ is $[0.3381, 4.6619]$ (see Example 2.1).

Example 2.2 shows that Theorem C is an improvement on Theorem B in some cases, but Theorem C is complex in calculation. In order to simplify our calculations, we may consider the following special case that the set S is a singleton, that is, $S_i = \{i\}$ for some $i \in N$. In this case, the associated sets from (1.6) may be defined as the following sets:

$$\begin{aligned}
\mathcal{G}_i^{S_i}(A) &= \{z \ge 0 : |z-a_i| \le 0\},\\
\mathcal{G}_j^{\overline{S}_i}(A) &= \left\{z \ge 0 : \left|z-a_j\right| \le s_j^{\overline{S}_i}\right\},
\end{aligned} \tag{2.5}$$

$$\mathcal{V}_{ij}^{S_i}(A) = \left\{z \ge 0 : (|z-a_i|)\left(\left|z-a_j\right| - s_j^{\overline{S}_i}\right) \le s_i \max\{\left|a_{ij}\right|, \left|a_{ji}\right|\}\right\}. \tag{2.6}$$

By a simple analysis, $\mathcal{G}_i^{S_i}(A)$ and $\mathcal{G}_j^{\overline{S}_i}(A)$ are necessarily contained in $\mathcal{V}_{ij}^{S_i}(A)$ for any $j \neq i$, we can simply write from (1.8) that, for any $i \in N$,

$$\sigma(A) \subseteq \mathcal{V}^{S_i}(A) \equiv \bigcup_{j \in N \setminus \{i\}} \mathcal{V}_{ij}^{S_i}(A). \tag{2.7}$$

This shows that $\mathcal{V}^{S_i}(A)$ is determined by $(n-1)$ sets $\mathcal{V}_{ij}^{S_i}(A)$. The associated Geršgorin-type set $G(A)$ from (1.2) is determined by n sets B_i $(i \in N)$ and the associated Brauer-type set $B(A)$ from (1.3) is determined by $n(n-1)/2$ sets. The following corollary is an immediate consequence of Theorem C.

Corollary 2.3. *Let $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ with $n \geq 2$. Then all singular values of A are contained in*

$$\sigma(A) \subseteq \mathcal{V}(A) \equiv \bigcap_{i \in N} \mathcal{V}^{S_i}(A). \tag{2.8}$$

Proof. From (2.7), we get the required result. □

Notice that $\mathcal{V}^{S_1}(A) = \mathcal{V}^{S_2}(A) = B(A)$ whenever $n = 2$. Next, we will assume that $n \geq 3$. It is interesting to establish their relations between $\mathcal{V}^{S_i}(A)$ and $G(A)$, as well as between $\mathcal{V}(A)$ and $B(A)$.

Definition 2.4 (see [9]). $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ *is called a matrix with property $\mathcal{AS}$ (absolute symmetry) if $|a_{ij}| = |a_{ji}|$ for any $i, j \in N$.*

Note that a matrix A with property $\mathcal{AS}$ is said as A with property B in [9].

Theorem 2.5. *Let $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ with $n \geq 3$. If A is a matrix with property $\mathcal{AS}$, then for each $i \in N$*

$$\mathcal{V}^{S_i}(A) \subseteq G(A), \qquad \mathcal{V}(A) \subseteq B(A). \tag{2.9}$$

Proof. Fix some $i \in N$ and consider any $z \in \mathcal{V}^{S_i}(A)$. Then from (2.7), there exists a $j \in N \setminus \{i\}$ such that $z \in \mathcal{V}_{ij}^{S_i}(A)$, that is, from (2.6),

$$(|z - a_i|)\left(|z - a_j| - s_j^{\overline{S}_i}\right) \leq s_i^{\overline{S}_i} \max\{|a_{ij}|, |a_{ji}|\} = s_i \cdot |a_{ij}|, \tag{2.10}$$

where the last equality holds as A has the property $\mathcal{AS}$ (i.e., $|a_{ij}| = |a_{ji}|$ for any $i, j \in N$).

Now assume that $z \notin G(A)$, then $|z - a_k| > s_k$ for each $k \in N$, implying that $|z - a_i| > s_i \geq 0$ and $|z - a_j| > s_j \geq 0$ for above $i, j \in N$. Thus, the left part of (2.10) satisfies

$$(|z - a_i|)\left(|z - a_j| - s_j^{\overline{S}_i}\right) > s_i\left(s_j - s_j^{\overline{S}_i}\right) = s_i \cdot |a_{ij}|, \tag{2.11}$$

which contradicts the inequality (2.10). Hence, $z \in \mathcal{V}^{S_i}(A)$ implies $z \in G(A)$, that is, $\mathcal{V}^{S_i}(A) \subseteq G(A)$.

Next, we will show that $\mathcal{U}(A) \subseteq B(A)$. Since $\mathcal{U}^{S_i}(A) \subseteq G(A)$ for any $i \in N$, then, from (2.8), we get $\mathcal{U}(A) \subseteq G(A)$. Now consider any $z \in \mathcal{U}(A)$, so that $z \in \mathcal{U}^{S_i}(A)$ for each $i \in N$. Hence, for each $i \in N$, there exists a $j \in N \setminus \{i\}$ such that $z \in \mathcal{U}_{ij}^{S_i}(A)$, that is, the inequality (2.10) holds. Since $\mathcal{U}(A) \subseteq G(A)$, there exists a $k \in N$ such that $|z - a_k| \leq s_k$. For this index k, there exists a $l \in N \setminus \{k\}$ such that $z \in \mathcal{U}_{kl}^{S_k}(A)$, that is,

$$(|z - a_k|)\left(|z - a_l| - s_l^{\overline{S}_k}\right) \leq s_k^{\overline{S}_k} \max\{|a_{kl}|, |a_{lk}|\} = s_k \cdot |a_{kl}|. \tag{2.12}$$

Hence,

$$|z - a_k|\,|z - a_l| \leq |z - a_k| s_l^{\overline{S}_k} + s_k \cdot |a_{kl}| \leq s_k\left(s_l^{\overline{S}_k} + |a_{kl}|\right) = s_k s_l, \tag{2.13}$$

which implies $z \in B(A)$. Since this is true for any $z \in \mathcal{U}(A)$. Then $\mathcal{U}(A) \subseteq B(A)$. This completes our proof. □

Remark that the condition “the matrix A has the property $\mathcal{AS}$” is necessary in Theorem 2.5, for example, as follows.

Example 2.6. Consider the following matrix:

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}. \tag{2.14}$$

Let $S_i = \{1\}$, $S_i = \{2\}$, and $S_i = \{3\}$. From (2.7), we get that the inclusion intervals of $\sigma(A)$ are $[0, 4.5616]$, $[0, 4.7321]$ and $[0, 4.6180]$, respectively. Hence, applying Corollary 2.3, we have $\sigma(A) \subseteq [0, 4.5616]$. However, applying Theorem A and Theorem B, we get $\sigma(A) \subseteq G(A) = B(A) = [0, 4]$, which implies Theorem 2.5 is failling if the condition “the matrix A has the property $\mathcal{AS}$” is omitted.

In the following, we will give a new inclusion interval for matrix singular values, which improves that of Theorem C. The proof of this result is based on the use of scaling techniques. It is well known that scaling techniques pay important roles in improving inclusion intervals for matrix singular values. For example, using positive scale vectors and their intersections, Qi [8] and Li et al. [6] obtained two new inclusion intervals (see Theorem 4 in [8] and Theorem 2.2 in [6], resp.), which improve these of Theorems A and B, respectively. Recently, Tian et al. [9], using this techniques, also obtained a new inclusion interval (see Theorem 2.4 in [9]), which is an improvement on these of Theorem 2.2 in [6] and Theorem B.

Theorem 2.7. *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ with $n \geq 2$ and $k = (k_1, k_2, \ldots, k_n)^T$ be any vector with positive components. Then Theorem C remains true if one replaces the definition of $s_i^S(A)$ and $s_i^{\overline{S}}(A)$ by*

$$S_i^S(A) = \max\left\{R_i^S, C_i^S\right\}, \qquad S_i^{\overline{S}}(A) = \max\left\{R_i^{\overline{S}}, C_i^{\overline{S}}\right\}, \tag{2.15}$$

where

$$R_i^S = \frac{1}{k_i}\sum_{j\in S\setminus\{i\}}|a_{ij}|k_j, \qquad R_i^{\overline{S}} = \frac{1}{k_i}\sum_{j\in \overline{S}\setminus\{i\}}|a_{ij}|k_j;$$
$$C_i^S = \frac{1}{k_i}\sum_{j\in S\setminus\{i\}}|a_{ji}|k_j, \qquad C_i^{\overline{S}} = \frac{1}{k_i}\sum_{j\in \overline{S}\setminus\{i\}}|a_{ji}|k_j. \tag{2.16}$$

Proof. Suppose that σ is any singular value of A. Then there exist two nonzero vectors $x = (x_1, x_2, \ldots, x_n)^T$ and $y = (y_1, y_2, \ldots, y_n)^T$ such that

$$Ax = \sigma y, \qquad A^* y = \sigma x, \tag{2.17}$$

(see Problem 5 of Section 7.3 in [11]).

The fundamental equation (2.17) implies that, for each $i \in N$,

$$\sigma x_i - \overline{a}_{ii} y_i = \sum_{j\in S\setminus\{i\}} \overline{a}_{ji} y_j + \sum_{j\in \overline{S}\setminus\{i\}} \overline{a}_{ji} y_j,$$
$$\sigma y_i - a_{ii} x_i = \sum_{j\in S\setminus\{i\}} a_{ij} x_j + \sum_{j\in \overline{S}\setminus\{i\}} a_{ij} x_j. \tag{2.18}$$

Let $x_i = k_i \widehat{x}_i$, $y_i = k_i \widehat{y}_i$ for each $i \in N$. Then our fundamental equation (2.18) and become into, for each $i \in N$,

$$\sigma \widehat{x}_i - \overline{a}_{ii} \widehat{y}_i = \frac{1}{k_i}\sum_{j\in S\setminus\{i\}} \overline{a}_{ji} k_j \widehat{y}_j + \frac{1}{k_i}\sum_{j\in \overline{S}\setminus\{i\}} \overline{a}_{ji} k_j \widehat{y}_j,$$
$$\sigma \widehat{y}_i - a_{ii} \widehat{x}_i = \frac{1}{k_i}\sum_{j\in S\setminus\{i\}} a_{ij} k_j \widehat{x}_j + \frac{1}{k_i}\sum_{j\in \overline{S}\setminus\{i\}} a_{ij} k_j \widehat{x}_j. \tag{2.19}$$

Denote $z_i = \max\{|\widehat{x}_i|, |\widehat{y}_i|\}$ for each $i \in N$. Now using the similar technique as the proof of Theorem 2.2 in [7], one gets the required result. □

Remarks. Write the inclusion intervals in Theorem 2.7 as $\mathfrak{G}\mathfrak{V}^S(A)$. Since $k = (k_1, k_2, \ldots, k_n)^T$ is any vector with positive components, then all singular values of A are contained in

$$\sigma(A) \subseteq \bigcap_{k>0} \mathfrak{G}\mathfrak{V}^S(A). \tag{2.20}$$

Obviously, Theorem 2.7 reduces to Theorem C whenever $k = (1, 1, \ldots, 1)^T$, which implies that

$$\bigcap_{k>0} \mathfrak{G}\mathfrak{V}^S(A) \subseteq \mathcal{G}\mathcal{V}^S(A). \tag{2.21}$$

Hence, the inclusion interval (2.20) is an improvement on that of (1.8).

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