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Research Article

On the Diamond Bessel Heat Kernel

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We study the heat equation in n dimensional by Diamond Bessel operator. We find the solution by method of convolution and Fourier transform in distribution theory and also obtain an interesting kernel related to the spectrum and the kernel which is called Bessel heat kernel.

1. Introduction

The operator $\diamondsuit^k$ has been first introduced by Kananthai [1], is named as the diamond operator iterated k times, and is defined by

$$\diamondsuit^k = \left(\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right)^k, \tag{1.1}$$

$p+q = n$, n is the dimension of the space $\mathbb{R}^n$ for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, and k is a nonnegative integer. The operator $\diamondsuit^k$ can be expressed in the following form:

$$\diamondsuit^k = \triangle^k \square^k = \square^k 0 \triangle^k, \tag{1.2}$$

where $\triangle^k$ is the Laplacian operator iterated k-times defined by

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}, \tag{1.3}$$

and $\square^k$ is the ultrahyperbolic operator iterated k-times defined by

$$\square^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k. \tag{1.4}$$

Kananthai [1, Theorem 1.3] has shown that the convolution $(-1)^k R_{2k}^e(x) * R_{2k}^H(x)$ is an elementary of the operator $\Diamond^k$. That is

$$\Diamond^k \left((-1)^k R_{2k}^e(x) * R_{2k}^H(x) \right) = \delta(x), \tag{1.5}$$

where $R_{2k}^e(x)$ is defined by

$$\begin{aligned} R_\alpha^e(x) &= \frac{|x|^{\alpha-n}}{W_n(\alpha)}, \\ W_n(\alpha) &= \frac{\pi^{n/2} 2^\alpha \Gamma(\alpha/2)}{\Gamma((n-\alpha)/2)}, \end{aligned} \tag{1.6}$$

α is a complex parameter, n is the dimension of $\mathbb{R}^n$, and the generalized function $R_\alpha^H(v)$ is defined by

$$R_\alpha^H(v) = \begin{cases} \dfrac{v^{(\alpha-n)/2}}{K_n(\alpha)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{1.7}$$

and the constant $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{(n-1)/2} \Gamma((2+\alpha-n)/2) \Gamma((1-\alpha)/2) \Gamma(\alpha)}{\Gamma((2+\alpha-p)/2) \Gamma((p-\alpha)/2)}. \tag{1.8}$$

The function $R_\alpha^H(v)$ is called the ultrahyperbolic kernel of Marcel Riesz and was introduced by Nozaki (see [2, page 72]).

Next, Yildirim et al. (see [3]) first introduced the Bessel diamond operator $\Diamond_B^k$ iterated k-times, defined by

$$\Diamond_B^k = \left(\left(\sum_{i=1}^{p} B_{x_i} \right)^2 - \left(\sum_{j=p+1}^{p+q} B_{x_j} \right)^2 \right)^k, \tag{1.9}$$

where $B_{x_i} = \partial^2/\partial x_i^2 + (2v_i/x_i)(\partial/\partial x_i)$, $2v_i = 2\alpha_i + 1$, $\alpha_i > -1/2$, $x_i > 0$. The operator $\lozenge_B^k$ can be expressed by $\lozenge_B^k = \triangle_B^k \Box_B^k = \Box_B^k \triangle_B^k$, where

$$\triangle_B^k = \left(\sum_{i=1}^{p} B_{x_i}\right)^k, \tag{1.10}$$

$$\Box_B^k = \left(\sum_{i=1}^{p} B_{x_i} - \sum_{j=p+1}^{p+q} B_{x_j}\right)^k. \tag{1.11}$$

And Yildirim (see [4]) have shown that the solution of the convolution form $u(x) = (-1)^k S_{2k}(x) * R_{2k}(x)$ is a unique elementary solution of $\lozenge_B^k$, that is,

$$\lozenge_B^k\left((-1)^k S_{2k}(x) * R_{2k}(x)\right) = \delta, \tag{1.12}$$

where $S_{2k}(x)$ is defined by (2.8) with $\alpha = 2k$ and $R_{2k}(x)$ is defined by (2.9) with $\gamma = 2k$.

It is well known that for the heat equation

$$\frac{\partial}{\partial t} u(x,t) = c^2 \triangle u(x,t) \tag{1.13}$$

with the initial condition

$$u(x,0) = f(x), \tag{1.14}$$

where $\triangle$ is the Laplace operator and is defined by (1.3) and $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0,\infty)$, we obtain

$$u(x,t) = \frac{1}{(4c^2\pi t)^{n/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-y|^2}{4c^2 t}\right) f(y)\,dy \tag{1.15}$$

as the solution of (1.13). Now, (1.15) can be written in the classical form

$$u(x,t) = E(x,t) * f(x), \tag{1.16}$$

where

$$E(x,t) = \frac{1}{(4c^2\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4c^2 t}\right). \tag{1.17}$$

$E(x,t)$ is called *the heat kernel*, where $|x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$ and $t > 0$, see [5, pages 208-209]. Moreover, we obtain $E(x,t) \to \delta$ as $t \to 0$, where δ is the Dirac delta distribution.

Next, Saglam et al. (see [6]) have study the following equation,

$$\frac{\partial}{\partial t}u(x,t) = c^2\Box_B^k u(x,t) \tag{1.18}$$

with the initial condition

$$u(x,0) = f(x), \quad \text{for } x \in R_n^+, \tag{1.19}$$

where the operator $\Box_B^k$ is named the Bessel ultrahyperbolic operator iterated k-times and is defined by (1.4), k is a positive integer, $u(x,t)$ is an unknown function, $f(x)$ is the given generalized function, and c is a constant, and $p+q=n$ is the dimension of the $R_n^+ = \{x : x = (x_1, x_2, \dots, x_n, t),\ x_i > 0,\ i = 1, 2, 3, \dots, n\}$.

They obtain the solution in the classical convolution form

$$u(x,t) = E(x,t) * f(x), \tag{1.20}$$

where the symbol $*$ is the B-convolution in (2.3), as a solution of (1.18), which satisfies (1.19), where

$$E(x,t) = C_v \int_{\Omega^+} e^{c^2 t[(y_1^2+\cdots+y_p^2)-(y_{p+1}^2+\cdots+y_{p+q}^2)]^k} \prod_{i=1}^{n} J_{v_i-1/2}(x_i, y_i) y_i^{2v_i} dy, \tag{1.21}$$

and $\Omega^+ \subset R_n^+$ is the spectrum of $E(x,t)$ for any fixed $t>0$, and $J_{v_i-1/2}(x_i, y_i)$ is the normalized Bessel function.

Now, the purpose of this work is to study the equation

$$\frac{\partial}{\partial t}\Box_B u(x,t) = c^2 \Diamond_B u(x,t) \tag{1.22}$$

with the initial condition

$$\Box_B u(x,0) = f(x), \tag{1.23}$$

where $(x,t) = (x_1, \dots, x_n, t) \in \mathbb{R}^n \times (0, \infty)$, t is a time, c is a positive constant, $u(x,t)$ is an unknown function, and $f(x)$ is a given generalized function for $x \in \mathbb{R}^n$. We obtain

$$u(x,t) = R_2^H(x) * E(x,t) * f(x) \tag{1.24}$$

as a solution of (1.7), where

$$E(x,t) = C_v \int_{\Omega^+} e^{-c^2 t(y_1^2+y_2^2+y_3^2+\cdots+y_n^2)} \prod_{i=1}^{n} J_{v_i-1/2}(x_i, y_i) y_i^{2v_i} dy, \tag{1.25}$$

and $\Omega \subset \mathbb{R}^n$ is the spectrum of $E(x,t)$ for any fixed $t > 0$, and $R_2^H(x)$ is defined by (2.6) with $\alpha = 2$. The convolution $R_2^H(x) * E(x,t)$ is called the Diamond Bessel Heat Kernel, and all properties will be studied in details. Before proceeding, the following definitions and concepts are needed.

2. Preliminaries

The shift operator according to the law remarks that this shift operator connected to the Bessel differential operator (see [3, 5, 7, 8]):

$$\begin{aligned} T_x^y \varphi(x) = C_v^* \int_0^\pi \cdots \int_0^\pi \varphi\left(\sqrt{x_1^2 + y_1^2 - 2x_1 y_1 \cos\theta_1}, \ldots, \sqrt{x_n^2 + y_n^2 - 2x_n y_n \cos\theta_n}\right) \\ \times \left(\prod_{i=1}^n \sin^{2v_i - 1}\theta_i\right) d\theta_1 \cdots d\theta_n, \end{aligned} \tag{2.1}$$

where $x, y \in R_n^+, C_v^* = \prod_{i=1}^n \Gamma(v_i + 1)/\Gamma(1/2)\Gamma(v_i)$. We remark that this shift operator is closely connected to the Bessel differential operator (see [3, 5, 7, 8]),

$$\begin{gathered} \frac{d^2U}{dx^2} + \frac{2v}{x}\frac{dU}{dx} = \frac{d^2U}{dy^2} + \frac{2v}{y}\frac{dU}{dy}, \\ U(x,0) = f(x), \qquad U_y(x,0) = 0. \end{gathered} \tag{2.2}$$

The convolution operator determined by the T_x^y is as follows:

$$(f * \varphi)(y) = \int_{R_n^+} f(y) T_x^y \varphi(x) \left(\prod_{i=1}^n y_i^{2v_i}\right) dy. \tag{2.3}$$

Convolution (2.3) is known as a B-convolution. We note the following properties of the B-convolution and the generalized shift operator.

(1) $T_x^y \cdot 1 = 1$,

(2) $T_x^0 \cdot f(x) = f(x)$,

(3) If $f(x), g(x) \in C(R_n^+)$, $g(x)$ is a bounded function for all $x > 0$, and $\int_0^\infty |f(x)| (\prod_{i=1}^n x_i^{2v_i}) dx < \infty$, then $\int_{R_n^+} T_x^y f(x) g(y) (\prod_{i=1}^n y_i^{2v_i}) dy = \int_{R_n^+} f(y) T_x^y g(x) (\prod_{i=1}^n y_i^{2v_i}) dy$.

(4) From (3), we have the following equality for $g(x) = 1$: $\int_{R_n^+} T_x^y f(x) (\prod_{i=1}^n y_i^{2v_i}) dy = \int_{R_n^+} f(y) (\prod_{i=1}^n y_i^{2v_i}) dy$.

(5) $(f * g)(x) = (g * f)(x)$.

The Fourier-Bessel transformation and its inverse transformation are defined as follows:

$$(\mathcal{F}_B f)(x) = C_v \int_{R_n^+} f(y) \left(\prod_{i=1}^{n} J_{v_i - 1/2}(x_i, y_i) y_i^{2v_i} \right) dy, \tag{2.4}$$

$$\left(\mathcal{F}_B^{-1} f \right)(x) = (\mathcal{F}_B f)(-x), \qquad C_v = \left(\prod_{i=1}^{n} 2^{v_i - 1/2} \Gamma\left(v_i + \frac{1}{2} \right) \right)^{-1}, \tag{2.5}$$

where $J_{v_i-1/2}(x_i, y_i)$ is the normalized Bessel function which is the eigenfunction of the Bessel differential operator. The following equalities for Fourier-Bessel transformation are true (see [3, 5, 7, 8]):

$$\mathcal{F}_B \delta(x) = 1, \tag{2.6}$$

$$\mathcal{F}_B (f * g)(x) = \mathcal{F}_B f(x) \cdot \mathcal{F}_B g(x). \tag{2.7}$$

Definition 2.1. Let $x = (x_1, x_2, \ldots, x_n)$, $\nu = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{R}_n^+$. For any complex number α, we define the function $S_\alpha(x)$ by

$$S_\alpha(x) = \frac{2^{n+2|\nu|-2\alpha} \Gamma((n + 2|\nu| - \alpha)/2) |x|^{\alpha - n - 2|\nu|}}{\prod_{i=1}^{n} 2^{\nu_i - 1/2} \Gamma(\nu_i + 1/2)}. \tag{2.8}$$

Definition 2.2. Let $x = (x_1, x_2, \ldots, x_n), \nu = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{R}_n^+$, and denote by $V = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2$ the nondegenerated quadratic form. Denote the interior of the forward cone by $\Gamma_+ = \{x \in \mathbb{R}_n^+ : x_1 > 0, x_2 > 0, \ldots, x_n > 0, V > 0\}$. The function $R_\beta(x)$ is defined by

$$R_\gamma(x) = \frac{V^{(\gamma - n - 2|\nu|)/2}}{K_n(\gamma)}, \tag{2.9}$$

where

$$K_n(\gamma) = \frac{\pi^{(n+2|\nu|-1)/2} \Gamma((2 + \gamma - n - 2|\nu|)/2) \Gamma((1 - \gamma)/2) \Gamma(\gamma)}{\Gamma((2 + \gamma - p - 2|\nu|)/2) \Gamma((p - 2|\nu| - \gamma)/2)}, \tag{2.10}$$

and γ is a complex number.

By putting $p = 1$ in $R_{2k}(x)$ and taking into account Legendre's duplication formula for $\Gamma(z)$:

$$\Gamma(2z) = 2^{2z-1} \pi^{-1/2} \Gamma(z) \Gamma\left(z + \frac{1}{2} \right), \tag{2.11}$$

we obtain

$$I_\gamma(x) = \frac{\upsilon^{(\gamma-n-2|\nu|)/2}}{M_n(\gamma)} \tag{2.12}$$

and $\upsilon = x_1^2 - x_2^2 - \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2$, where

$$M_n(2k) = \pi^{(n+2|\nu|-1)/2}2^{2k-1}\Gamma\left(\frac{2+2k-n-2|\nu|}{2}\right)\Gamma(k). \tag{2.13}$$

Definition 2.3. The spectrum of the kernel $E(x,t)$ of (1.21) is the bounded support of the Fourier Bessel transform $\mathcal{F}_B E(y,t)$ for any fixed $t > 0$.

Definition 2.4. Let $x = (x_1, x_2, \ldots, x_n)$ be a point in $\mathbb{R}_n^+$, and denote by

$$\Gamma_+ = \left\{x \in \mathbb{R}_n^+ : x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2 > 0,\ \xi_1 > 0\right\} \tag{2.14}$$

the set of an interior of the forward cone, and $\overline{\Gamma}_+$ denotes the closure of Γ_+.

Let Ω^+ be spectrum of $E(x,t)$ defined by (1.21) for any fixed $t > 0$ and $\Omega \subset \overline{\Gamma}_+$. Let $F_B E(y,t)$ be the Fourier Bessel transform of $E(x,t)$, which is defined by

$$\mathcal{F}_B E(y,t) = \begin{cases} e^{-c^2 t(y_1^2+y_2^2+y_3^2+\cdots+y_p^2)}, & \text{for } \xi \in \Omega_+, \\ 0, & \text{for } \xi \notin \Omega_+. \end{cases} \tag{2.15}$$

Lemma 2.5. *Given the equation $\triangle_B^k u(x) = \delta(x)$ for $x \in \mathbb{R}_n^+$, where $\triangle_B^k$ is defined by* (1.10)*. Then,*

$$u(x) = (-1)^k S_{2k}(x), \tag{2.16}$$

where $S_{2k}(x)$ is defined by (2.3)*, with $\alpha = 2k$. We obtain that $(-1)^k R_{2k}^e(x)$ is an elementary solution of the operator $\triangle_B^k$. That is*

$$\triangle_B^k (-1)^k S_{2k}(x) = \delta(x). \tag{2.17}$$

Proof. (See [3, page 379]). □

Lemma 2.6. *Given the equation $\Box_B^k u(x) = \delta(x)$ for $x \in \mathbb{R}_n^+$, where $\Box_B^k$ is defined by* (1.11)*. Then,*

$$u(x) = R_{2k}(x), \tag{2.18}$$

where $R_{2k}(x)$ is defined by (2.4)*, with $\gamma = 2k$. We obtain that $R_{2k}(x)$ is an elementary solution of the operator $\Box_B^k$. That is*

$$\Box_B^k R_{2k}(x) = \delta(x). \tag{2.19}$$

Proof (see [3, Page 379]). From (2.8) with $k = 1$, we obtain $u(x) = R_2^H(x)$ as an elementary solution of the equation

$$\Box u(x) = \delta. \tag{2.20}$$

Now, from (2.17),

$$R_2^H(v) = \begin{cases} \dfrac{v^{(2-n-2|\nu|)/2}}{K_n(2)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{2.21}$$

We can compute $K_n(2)$ from (2.7) as

$$K_n(2) = \frac{\pi^{(n+2|\nu|-1)/2}\Gamma((4-n-2|\nu|)/2)\Gamma(-1/2)\Gamma(2)}{\Gamma((4-p-2|\nu|)/2)\Gamma((p+2|\nu|-2)/2)}. \tag{2.22}$$

By using the formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \tag{2.23}$$

we obtain $\Gamma(-1/2) = -2\sqrt{\pi}$, $\Gamma(2) = 1$ and

$$\begin{aligned} \Gamma\left(\frac{p+2|\nu|-2}{2}\right)\Gamma\left(\frac{4-p-2|\nu|}{2}\right) &= \Gamma\left(\frac{p+2|\nu|-2}{2}\right)\Gamma\left(1-\frac{p+2|\nu|-2}{2}\right) \\ &= \frac{\pi}{\sin(\pi(p+2|\nu|-2)/2)}. \end{aligned} \tag{2.24}$$

Then, we obtain

$$\begin{aligned} K_n(2) &= \frac{\pi^{(n+2|\nu|-1)/2}\Gamma((4-n)/2)(-2\sqrt{\pi})\sin(\pi((p+2|\nu|-2)/2))}{\pi} \\ &= -2\pi^{(n+2|\nu|-2)/2}\Gamma\left(\frac{4-n-2|\nu|}{2}\right)\sin\left(\pi\left(\frac{p+2|\nu|-2}{2}\right)\right). \end{aligned} \tag{2.25}$$

Thus,

$$R_2^H(v) = \begin{cases} \dfrac{v^{(2-n-2|\nu|)/2}}{-2\pi^{(n+2|\nu|-2)/2}\Gamma((4-n-2|\nu|)/2)\sin(\pi((p+2|\nu|-2)/2))}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{2.26}$$

where $v = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2$. □

Lemma 2.7. *Let $S_\alpha(x)$ and $R_\beta(x)$ be the functions defined by* (2.8) *and* (2.9)*, respectively. Then*

$$S_\alpha(x) * S_\beta(x) = S_{\alpha+\beta}(x),$$
$$R_\beta(x) * R_\alpha(x) = R_{\beta+\alpha}(x), \tag{2.27}$$

where α and β are a positive even number.

Proof. (See [3, pages 171–190]). □

Lemma 2.8 (Fourier Bessel transform of $\square_B^k$ operator). *Consider*

$$\mathcal{F}_B \square_B^k u(x) = (-1)^k V_1^k(x) \mathcal{F}_B u(x), \tag{2.28}$$

where

$$V_1^k(x) = \left(\sum_{i=1}^{p} x_i^2 - \sum_{j=p+1}^{p+q} x_j^2 \right)^k. \tag{2.29}$$

Proof. (See [4]). □

Lemma 2.9 (Fourier Bessel transform of $\triangle_B^k$ operator). *Consider*

$$\mathcal{F}_B \triangle_B^k u(x) = (-1)^k |x|^{2k} \mathcal{F}_B u(x), \tag{2.30}$$

where

$$|x|^{2k} = \left(x_1^2 + x_2^2 + \cdots + x_n^2 \right)^k. \tag{2.31}$$

Proof. (see [4, 9]). □

Lemma 2.10. *For $t, v > 0$, and $x, y \in \mathbb{R}^n$, we have*

$$\int_0^\infty e^{-c^2 x^2 t} x^{2v} dx = \frac{\Gamma(v)}{2c^{2v+1} t^{v+1/2}},$$
$$\int_0^\infty e^{-c^2 x^2 t} J_{v-1/2}(xy) x^{2v} dx = \frac{\Gamma(v+1/2)}{2(c^2 t)^{v+1/2}} e^{-y^2/4c^2 t}, \tag{2.32}$$

where c is a positive constant.

Proof. (See [6, Lemma 3.1.1] and [10]). □

Lemma 2.11. *Let the operator L be defined by*

$$L = \frac{\partial}{\partial t} - c^2 \triangle_B, \tag{2.33}$$

where $\triangle_B$ is the Laplace Bessel operator defined by

$$\triangle_B = B_{x_1} + B_{x_2} + B_{x_3} + \cdots + B_{x_n},$$
$$B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i}\frac{\partial}{\partial x_i} \tag{2.34}$$

$p + q = n$ is the dimension $\mathbb{R}_n^+$, k is a positive integer, $(x_1, x_2, \ldots + x_n) \in \mathbb{R}_n^+$, and c is a positive constant. Then,

$$E(x,t) = C_v \int_{\Omega^+} e^{-c^2 t(y_1^2+y_2^2+\cdots+y_n^2)} \prod_{i=1}^{n} J_{v_i-1/2}(x_i, y_i) y_i^{2v_i} dy \tag{2.35}$$

is the elementary solution of (2.15) *in the spectrum $\Omega^+ \subset R_n^+$ for $t > 0$.*

Proof. Let $LE(x,t) = \delta(x,t)$, where $E(x,t)$ is the elementary solution of L and δ is the Dirac-delta distribution. Thus,

$$\frac{\partial}{\partial t}E(x,t) - c^2(B_{x_1} + B_{x_2} + B_{x_3} + \cdots + B_{x_n})E(x,t) = \delta(x)\delta(t). \tag{2.36}$$

Applying the Fourier Bessel transform, which is defined by (2.4) to the both sides of the above equation and using Lemma 2.7 by considering $\digamma_B \delta(x) = 1$, we obtain

$$\frac{\partial}{\partial t}\digamma_B E(x,t) + c^2\left(x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2\right)\digamma_B E(x,t) = \delta(t). \tag{2.37}$$

Thus, we get

$$\digamma_B E(x,t) = H(t) e^{-c^2 t(x_1^2+x_2^2+x_3^2+\cdots+x_n^2)}, \tag{2.38}$$

where $H(t)$ is the Heaviside function, because $H(t) = 1$ holds for $t \geq 0$.

Therefore,

$$\digamma_B E(x,t) = e^{-c^2 t(x_1^2+x_2^2+x_3^2+\cdots+x_n^2)}, \tag{2.39}$$

which has been already given by (2.7). Thus, from (2.5), we have

$$E(x,t) = C_v \int_{\mathbb{R}_n^+} e^{-c^2 t(y_1^2+y_2^2+y_3^2+\cdots+y_n^2)} \prod_{i=1}^{n} J_{v_i-1/2}(x_i, y_i) y_i^{2v_i} dy, \tag{2.40}$$

where Ω^+ is the spectrum of $E(x,t)$. Thus, we obtain

$$E(x,t) = C_v \int_{\Omega^+} e^{-c^2 t(y_1^2+y_2^2+y_3^2+\cdots+y_p^2)} \prod_{i=1}^{n} J_{v_i-1/2}(x_i, y_i) y_i^{2v_i} dy \tag{2.41}$$

as an elementary solution of (2.15) in the spectrum $\Omega^+ \subset R_n^+$ for $t > 0$. □

Definition 2.12. We can extend $E(x,t)$ to $\mathbb{R}^n \times \mathbb{R}$ by setting

$$E(x,t) = \begin{cases} \dfrac{1}{(2\pi)^n}\displaystyle\int_{\Omega} \exp\left(-c^2 t\left(\xi_i^2 + \xi_2^2 + \xi_3^2 + \cdots + \xi_n^2\right)\right), & \text{for } t > 0, \\ 0, & \text{for } t \leq 0. \end{cases} \tag{2.42}$$

Lemma 2.13. *Let us consider the equation*

$$\frac{\partial}{\partial t} u(x,t) - c^2 \triangle_B u(x,t) = 0 \tag{2.43}$$

with the initial condition

$$u(x,0) = f(x), \tag{2.44}$$

where $\triangle_B$ is the operator iterated k-times and is defined by

$$\begin{gathered} \triangle_B = \left(B_{x_1} + B_{x_2} + B_{x_3} + \cdots + B\right), \\ B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i}\frac{\partial}{\partial x_i}, \end{gathered} \tag{2.45}$$

$p + q = n$ is the dimension $\mathbb{R}_n^+$, k is a positive integer, $u(x,t)$ is an unknown function for $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}_n^+ \times (0,\infty)$, $f(x)$ is the given generalized function, and c is a positive constant. Then,

$$u(x,t) = E(x,t) * f(x) \tag{2.46}$$

is a solution of (2.43), *which satisfies* (2.44), *where $E(x,t)$ is given by* (2.35).

Proof. Taking the Fourier Bessel transform, the both sides of (2.43), for $x \in \mathbb{R}_n^+$ and using Lemma 2.9, we obtain

$$\frac{\partial}{\partial t} \mathcal{F}_B u(x,t) = -c^2\left(x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2\right)\mathcal{F}_B u(x,t). \tag{2.47}$$

Thus, we consider the initial condition (2.44), then we have the following equality for (2.47):

$$u(x,t) = f(x) * \mathcal{F}_B^{-1} e^{-c^2 t\left(x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2\right)}. \tag{2.48}$$

Here, if we use (2.4) and (2.5), then we have

$$\begin{aligned} u(x,t) &= f(x) * \mathcal{F}_B^{-1} e^{-c^2 t (y_1^2+y_2^2+y_3^2+\cdots+y_n^2)} \\ &= \int_{\mathbb{R}_n^+} \mathcal{F}_B^{-1} e^{-c^2 t (y_1^2+\cdots+y_n^2)} T_x^y f(x) \left(\prod_{i=1}^n y_i^{2v_i} \right) dy \\ &= \int_{\mathbb{R}_n^+} \left(C_v \int_{\mathbb{R}_n^+} e^{-c^2 t V^k(z)} \prod_{i=1}^n J_{v_i-1/2}(y_i, z_i) z_i^{2v_i} dz \right) T_x^y f(x) \left(\prod_{i=1}^n y_i^{2v_i} \right) dy, \end{aligned} \tag{2.49}$$

where $V(z) = (z_1^2 + z_2^2 + z_3^2 + \cdots + z_n^2)$. Set

$$E(x,t) = C_v \int_{\mathbb{R}_n^+} e^{-c^2 t (y_1^2+y_2^2+y_3^2+\cdots+y_n^2)} \prod_{i=1}^n J_{v_i-1/2}(x_i, y_i) y_i^{2v_i} dy. \tag{2.50}$$

We choose $\Omega^+ \subset \mathbb{R}_n^+$, to be the spectrum of $E(x,t)$, and, by (2.35), we have

$$\begin{aligned} E(x,t) &= C_v \int_{\mathbb{R}_n^+} e^{-c^2 t (y_1^2+y_2^2+y_3^2+\cdots+y_n^2)} \prod_{i=1}^n J_{v_i-1/2}(x_i, y_i) y_i^{2v_i} dy \\ &= C_v \int_{\Omega^+} e^{-c^2 t (y_1^2+y_2^2+y_3^2+\cdots+y_n^2)} \prod_{i=1}^n J_{v_i-1/2}(x_i, y_i) y_i^{2v_i} dy. \end{aligned} \tag{2.51}$$

Thus, (2.49) can be written in the convolution form

$$u(x,t) = E(x,t) * f(x). \tag{2.52}$$

Moreover, since $E(x,t)$ exists, we can see that

$$\begin{aligned} \lim_{t\to 0} E(x,t) &= C_v \int_{\Omega^+} \prod_{i=1}^n J_{v_i-1/2}(x_i, y_i) y_i^{2v_i} dy \\ &= C_v \int_{\mathbb{R}_n^+} \prod_{i=1}^n J_{v_i-1/2}(x_i, y_i) y_i^{2v_i} dy \\ &= \delta(x), \quad \text{for } x \in \mathbb{R}_n^+. \end{aligned} \tag{2.53}$$

Thus, for the solution $u(x,t) = E(x,t) * f(x)$ of (2.43), we have

$$\lim_{t\to 0} u(x,t) = u(x,0) = \delta * f(x) = f(x), \tag{2.54}$$

which satisfies (2.44). This completes the proof. □

3. Main Results

Theorem 3.1. *Given the equation*

$$\frac{\partial}{\partial t}(\Box_B u(x,t)) = c^2 \Diamond_B u(x,t) \tag{3.1}$$

with the initial condition

$$\Box_B u(x,0) = f(x), \tag{3.2}$$

where $f(x)$ is a given generalized function for $\mathbb{R}^n$, $\Box_B$ is an ultrahyperbolic Bessel operator, and $\Diamond_B$ is the Diamond Bessel operator defined by (1.11) *and* (1.9)*, respectively. Then, we obtain*

$$u(x,t) = R_2^H(x) * E(x,t) * f(x) \tag{3.3}$$

as a solution of (3.1)*, where $R_2^H(x)$ is given by* (2.9) *with $\gamma = 2$ and $E(x,t)$ is given by* (2.35)*.*

Proof. Equation (3.1) can be written in the form

$$\frac{\partial}{\partial t}(\Box_B u(x,t)) = c^2 \triangle_B(\Box_B u(x,t)). \tag{3.4}$$

Let $w(x,t) = \Box_B u(x,t)$. Thus, the above equation can be written as

$$\frac{\partial}{\partial t}(w(x,t)) = c^2(\triangle_B w(x,t)). \tag{3.5}$$

We can solve the above equation by applying the n-dimensional Fourier Bessel transform to both sides of (3.5). By Lemma 2.7, we obtain

$$w(x,t) = \Box_B u(x,t) = E(x,t) * f(x). \tag{3.6}$$

By convolving both sides of (3.6) by function $R_2^H(x)$, we obtain

$$R_2^H(x) * \Box_B u(x,t) = R_2^H(x) * E(x,t) * f(x). \tag{3.7}$$

By properties of convolution,

$$\Box R_2^H(x) * u(x,t) = R_2^H(x) * E(x,t) * f(x). \tag{3.8}$$

By Lemma 2.5, we obtain

$$\delta * u(x,t) = R_2^H(x) * E(x,t) * f(x), \tag{3.9}$$

or

$$u(x,t) = R_2^H(x) * E(x,t) * f(x) \tag{3.10}$$

is a solution of (3.1). As shown in (3.5) and by the continuity of convolution,

$$\lim_{t\to 0} \square_B u(x,t) = \lim_{t\to 0} \left(E(x,t) * f(x)\right) = \delta * f(x) = f(x). \tag{3.11}$$

□

Theorem 3.2 (The properties of the Diamond Bessel Heat Kernel $R_2^H(x) * E(x,t)$). *(1) $R_2^H(x) * E(x,t)$ exists and is tempered distribution.*

(2) $R_2^H(x) * E(x,t) \in \mathbb{C}^\infty$ the space of continuous function and infinitely differentiable.

(3) $\lim_{t\to 0}(R_2^H(x) * E(x,t)) = R_2^H(x)$.

(4) $(\partial/\partial t)\square_B(R_2^H(x) * E(x,t)) - c^2\lozenge_B(R_2^H(x) * E(x,t)) = 0$.

Proof. (1) Since $E(x,t)$ and $R_2^H(x)$ are tempered distribution with compact support. Thus, $R_2^H(x) * E(x,t)$ exists and is a tempered distribution.

(2) We have

$$\frac{\partial^n}{\partial x^n}\left(R_2^H(x) * E(x,t)\right) = R_2^H(x) * \frac{\partial^n}{\partial x^n}E(x,t), \tag{3.12}$$

since $E(x,t)$ is infinitely differentiable and $R_2^H(x) * E(x,t) \in \mathbb{C}^\infty$.

(3) By the continuity of the convolution,

$$R_2^H(x) * E(x,t) \longrightarrow R_2^H(x) * \delta \quad \text{as } t \longrightarrow 0. \tag{3.13}$$

Thus,

$$\lim_{t\to 0}\left(R_2^H(x) * E(x,t)\right) = R_2^H(x). \tag{3.14}$$

(4) Since

$$\begin{aligned}\square_B\left(R_2^H(x) * E(x,t)\right) &= \square_B R_2^H(x) * E(x,t) = \delta * E(x,t) = E(x,t),\\ \lozenge_B\left(R_2^H(x) * E(x,t)\right) &= \triangle_B\left(\square_B R_2^H(x) * E(x,t)\right) = \triangle_B(\delta * E(x,t)) = \triangle_B E(x,t).\end{aligned} \tag{3.15}$$

thus

$$\begin{aligned}&\frac{\partial}{\partial t}\square_B\left(R_2^H(x) * E(x,t)\right) - c^2\lozenge_B\left(R_2^H(x) * E(x,t)\right)\\ &\quad= \frac{\partial}{\partial t}E(x,t) - c^2\triangle_B E(x,t) = \left(\frac{\partial}{\partial t} - c^2\triangle_B\right)E(x,t) = 0,\end{aligned} \tag{3.16}$$

where $E(x,t)$ is defined by (2.35). □

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