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*Research Article*

# An Iteration Method Generating Analytical Solutions for Blasius Problem

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We derive a new iteration method for finding solution of the generalized Blasius problem. This method results in the analytical series solutions which are consistent with the existing series solutions for some special cases.

## 1. Introduction

We consider the generalized Blasius' equation

$$y'''(x) + \alpha y(x) y''(x) = 0, \quad 0 \le x < \infty, \tag{1.1}$$

where $\alpha = 1/2$ or $\alpha = 1$, with boundary conditions

$$y(0) = 0, \qquad y'(0) = \mu, \qquad y'(\infty) = 1. \tag{1.2}$$

This problem describes the boundary layer flow over a moving plate with constant velocity $\mu$. For a special case of $\alpha = 1/2$ and $\mu = 0$, the series solution of the Blasius problem becomes

$$S(x) = \frac{\kappa}{2}x^2 - \frac{\kappa^2}{240}x^5 + \frac{11}{161280}\kappa^3 x^8 - \frac{5}{4257792}\kappa^4 x^{11} + \cdots, \tag{1.3}$$

where $\kappa = y''(0) \approx 0.3320573362$. The Blasius series, however, converges for $|x| < \rho = 5.6900380545$. In the literature [1–3], it was shown that the limitation can be overcome by Padé approximants or an Euler-accelerated series.

Lots of analytical methods such as Adomian decomposition methods [4–6], variational iteration methods [7–11], and homotopy analysis methods [12–14] have been proposed.

## 2. Derivation of an Iteration Formula

We develop a new iteration method to find the analytical series solution of the Blasius problem (1.1) subject to the boundary condition

$$y(0) = 0, \qquad y'(0) = \mu, \qquad y''(0) = \kappa, \tag{2.1}$$

where the curvature $\kappa$ of the solution is assumed to be known. It should be noted that in order to make the problem easy to be solved, we consider the one point boundary conditions in (2.1) instead of the two-point boundary conditions in (1.2).

First, for $y = y(x)$ the Blasius equation (1.1) becomes

$$\left(y'' + \alpha y y'\right)' = \alpha \left(y'\right)^2. \tag{2.2}$$

From the boundary conditions in (2.1), it follows that

$$y'' + \alpha y y' = \alpha \int_0^x \left\{y'(t)\right\}^2 dt + \kappa := A(x, y). \tag{2.3}$$

This can be represented by

$$\left(y' + \frac{\alpha}{2} y^2\right)' = A(x, y), \tag{2.4}$$

which implies

$$y' = \int_0^x A(t, y)\,dt - \frac{\alpha}{2} y^2 + \mu := B(x, y). \tag{2.5}$$

In the result, we have

$$y(x) = \int_0^x B(t, y)\,dt. \tag{2.6}$$

If we denote by $y_n$ the $n$th iterate solution and substitute it into the right hand side of (2.6), we have an iteration formula

$$y_{n+1}(x) = \int_0^x B\left(t, y_n(t)\right) dt, \quad n \geq 0, \tag{2.7}$$

where

$$B\left(x, y_n(x)\right) = \int_0^x A\left(t, y_n(t)\right) dt - \frac{\alpha}{2} y_n(x)^2 + \mu. \tag{2.8}$$

From (2.3) and (2.7), the function $A(x, y_n)$ can be represented by

$$A(x, y_n(x)) = \alpha \int_0^x \{B(t, y_{n-1}(t))\}^2 dt + \kappa, \tag{2.9}$$

for $n \geq 1$ with $A(t, y_0) \equiv \kappa$. Referring to the boundary conditions in (2.1), we may take the initial solution as

$$y_0(x) = \frac{\kappa}{2}x^2 + \mu x. \tag{2.10}$$

The proposed method can be summarized by the following algorithm.

*Algorithm A.* We have the following steps.

*Step 1.* Set initial guesses

$$\begin{aligned} y(x) &:= y_0(x), \\ B(x, y) &:= y_0'(x). \end{aligned} \tag{2.11}$$

*Step 2.* For a large integer $L > 0$, perform the iteration (2.7)–(2.9) using symbolic computations

$$\begin{gathered} l := 0, \quad \text{while } (l := l + 1) \leq L, \\ A(x, y) := \alpha \int_0^x \{B(t, y(t))\}^2 dt + \kappa, \\ B(x, y) := \int_0^x A(t, y(t))dt - \frac{\alpha}{2}y(x)^2 + \mu, \\ y(x) := \int_0^x B(t, y(t))dt. \end{gathered} \tag{2.12}$$

It should be noted that by performing this algorithm, we can also obtain the approximates $B(x, y_n)$ to the velocity $y'(x) = B(x, y)$.

## 3. Analytical Solutions

Performing the above algorithm by using the symbolic calculation software *Mathematica*, we have the successive approximate solutions below

$$\begin{aligned} y_1(x) &= \frac{\kappa x^2}{2} - \frac{\alpha k^2 x^5}{120} + \left(x - \frac{\alpha k x^4}{24}\right)\mu, \\ y_2(x) &= \frac{\kappa x^2}{2} - \frac{\alpha k^2 x^5}{120} + \frac{11\alpha^2 k^3 x^8}{40320} - \frac{\alpha^3 k^4 x^{11}}{712800} + \left(x - \frac{\alpha k x^4}{24} + \frac{11\alpha^2 k^2 x^7}{5040} - \frac{\alpha^3 k^3 x^{10}}{64800}\right)\mu \\ &\quad + \left(\frac{\alpha^2 k x^6}{240} - \frac{\alpha^3 k^2 x^9}{24192}\right)\mu^2, \end{aligned}$$

$$\begin{aligned}
y_3(x) = {} & \frac{\kappa x^2}{2} - \frac{\alpha k^2 x^5}{120} + \frac{11\alpha^2 k^3 x^8}{40320} - \frac{5\alpha^3 k^4 x^{11}}{532224} + \frac{10033\alpha^4 k^5 x^{14}}{87178291200} - \frac{5449\alpha^5 k^6 x^{17}}{3908653056000} \\
& + \frac{83\alpha^6 k^7 x^{20}}{8935557120000} - \frac{\alpha^7 k^8 x^{23}}{49080898944000} \\
& + \left( x - \frac{\alpha k x^4}{24} + \frac{11\alpha^2 k^2 x^7}{5040} - \frac{5\alpha^3 k^3 x^{10}}{48384} \right. \\
& \quad \left. + \frac{10033\alpha^4 k^4 x^{13}}{6227020800} - \frac{5449\alpha^5 k^5 x^{16}}{229920768000} + \frac{83\alpha^6 k^6 x^{19}}{446777856000} - \frac{\alpha^7 k^7 x^{22}}{2133952128000} \right) \mu \\
& + \left( \frac{\alpha^2 k x^6}{240} - \frac{43\alpha^3 k^2 x^9}{120960} + \frac{1157\alpha^4 k^3 x^{12}}{159667200} \right. \\
& \quad \left. - \frac{1147\alpha^5 k^4 x^{15}}{7925299200} + \frac{14057\alpha^6 k^5 x^{18}}{10260214272000} - \frac{197\alpha^7 k^6 x^{21}}{49145564160000} \right) \mu^2 \\
& - \left( \frac{\alpha^3 k x^8}{2688} - \frac{23\alpha^4 k^2 x^{11}}{2217600} + \frac{967\alpha^5 k^3 x^{14}}{2641766400} - \frac{1829\alpha^6 k^4 x^{17}}{414554112000} + \frac{\alpha^7 k^5 x^{20}}{66189312000} \right) \mu^3 \\
& - \left( \frac{\alpha^5 k^2 x^{13}}{3294720} - \frac{17\alpha^6 k^3 x^{16}}{3251404800} + \frac{\alpha^7 k^4 x^{19}}{47259168768} \right) \mu^4 .
\end{aligned} \tag{3.1}$$

For the case of $\alpha = 1/2$, we have

$$\begin{aligned}
y_1(x) = {} & \frac{\kappa x^2}{2} - \frac{k^2 x^5}{240} + \left( x - \frac{k x^4}{48} \right) \mu, \\
y_2(x) = {} & \frac{\kappa x^2}{2} - \frac{k^2 x^5}{240} + \frac{11 k^3 x^8}{161280} - \frac{k^4 x^{11}}{5702400} + \left( x - \frac{k x^4}{48} + \frac{11 k^2 x^7}{20160} - \frac{k^3 x^{10}}{518400} \right) \mu \\
& + \left( \frac{k x^6}{960} - \frac{k^2 x^9}{193536} \right) \mu^2, \\
y_3(x) = {} & \frac{\kappa x^2}{2} - \frac{k^2 x^5}{240} + \frac{11 k^3 x^8}{161280} - \frac{5 k^4 x^{11}}{4257792} + \frac{10033 k^5 x^{14}}{1394852659200} - \frac{5449 k^6 x^{17}}{125076897792000} \\
& + \frac{83 k^7 x^{20}}{571875655680000} - \frac{k^8 x^{23}}{6282355064832000} \\
& + \left( x - \frac{k x^4}{48} + \frac{11 k^2 x^7}{20160} - \frac{5 k^3 x^{10}}{387072} + \frac{10033 k^4 x^{13}}{99632332800} \right. \\
& \quad \left. - \frac{5449 k^5 x^{16}}{7357464576000} + \frac{83 k^6 x^{19}}{28593782784000} - \frac{k^7 x^{22}}{273145872384000} \right) \mu
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{kx^6}{960}-\frac{43k^2x^9}{967680}+\frac{1157k^3x^{12}}{2554675200}-\frac{1147k^4x^{15}}{253609574400}\right.\\
&\qquad\left.+\frac{14057k^5x^{18}}{656653713408000}-\frac{197k^6x^{21}}{6290632212480000}\right)\mu^2\\
&-\left(\frac{kx^8}{21504}-\frac{23k^2x^{11}}{35481600}+\frac{967k^3x^{14}}{84536524800}-\frac{1829k^4x^{17}}{26531463168000}+\frac{k^5x^{20}}{8472231936000}\right)\mu^3\\
&-\left(\frac{k^2x^{13}}{105431040}-\frac{17k^3x^{16}}{208089907200}+\frac{k^4x^{19}}{6049173602304}\right)\mu^4.
\end{aligned}
\tag{3.2}
$$

One can see that the result is consistent with the known series solution [11, 12].

In particular, when $\alpha = 1$ and $\mu = 0$, it follows that

$$
\begin{aligned}
y_1(x) &= \frac{\kappa x^2}{2}-\frac{k^2x^5}{120},\\
y_2(x) &= \frac{\kappa x^2}{2}-\frac{k^2x^5}{120}+\frac{11k^3x^8}{40320}-\frac{k^4x^{11}}{712800},\\
y_3(x) &= \frac{\kappa x^2}{2}-\frac{k^2x^5}{120}+\frac{11k^3x^8}{40320}-\frac{5k^4x^{11}}{532224}+\frac{10033k^5x^{14}}{87178291200}-\frac{5449k^6x^{17}}{3908653056000}\\
&\quad+\frac{83k^7x^{20}}{8935557120000}-\frac{k^8x^{23}}{49080898944000},\\
y_4(x) &= \frac{\kappa x^2}{2}-\frac{k^2x^5}{120}+\frac{11k^3x^8}{40320}-\frac{5k^4x^{11}}{532224}+\frac{9299k^5x^{14}}{29059430400}-\frac{2173649k^6x^{17}}{355687428096000}\\
&\quad+\frac{13722337k^7x^{20}}{115852476579840000}-\frac{27184438601k^8x^{23}}{129260083694424883200000}\\
&\quad+\frac{12320831753849k^9x^{26}}{4032914611266056355840000000}\\
&\quad+\cdots-\frac{k^{16}x^{47}}{4631724332758783424102400000000}.
\end{aligned}
\tag{3.3}
$$

In this case, $\kappa \approx 0.4695999883$. For comparison, we refer to another analytical solution obtained by the Adomian decomposition method as follows:

$$
\begin{aligned}
y_1^{A}(x) &= \frac{\kappa x^2}{2}-\frac{k^2x^5}{120},\\
y_2^{A}(x) &= \frac{\kappa x^2}{2}-\frac{k^2x^5}{120}+\frac{11k^3x^8}{40320},\\
y_3^{A}(x) &= \frac{\kappa x^2}{2}-\frac{k^2x^5}{120}+\frac{11k^3x^8}{40320}-\frac{5k^4x^{11}}{532224},\\
y_4^{A}(x) &= \frac{\kappa x^2}{2}-\frac{k^2x^5}{120}+\frac{11k^3x^8}{40320}-\frac{5k^4x^{11}}{532224}+\frac{9299k^5x^{14}}{29059430400},\\
y_5^{A}(x) &= \frac{\kappa x^2}{2}-\frac{k^2x^5}{120}+\frac{11k^3x^8}{40320}-\frac{5k^4x^{11}}{532224}+\frac{9299k^5x^{14}}{29059430400}-\frac{1272379k^6x^{17}}{118562476032000}.
\end{aligned}
\tag{3.4}
$$

This solution is based on

$$y_n^{\mathrm{A}}(x) = y_0(x) + \sum_{k=1}^{n} u_k(x), \quad n = 1, 2, \ldots, \tag{3.5}$$

with

$$u_{k+1}(x) = -L^{-1}(A_k(x)), \quad k = 0, 1, 2, \ldots, \tag{3.6}$$

$u_0(x) = y_0(x)$, and Adomian polynomial $A_k(x)$ generated by the formula [6]

$$A_k(x) = \frac{1}{k!}\left[\frac{d^k}{d\lambda^k}\left(\sum_{j=0}^{k}\lambda^j u_j(x)\right)\left(\sum_{j=0}^{k}\lambda^j u_j''(x)\right)\right]_{\lambda=0}, \tag{3.7}$$

where $L^{-1}$ is an inverse operator of $L = d^3/dx^3$. Comparing the formulas in (3.3) and (3.4), one can see that the presented analytical solution $y_n(x)$ has more terms than $y_n^{\mathrm{A}}(x)$ in each $n$th iteration. In other words,

$$y_n(x) = y_n^{\mathrm{A}}(x) + \sum_{j=1}^{2^n-n-1} d_j x^{3(n+j)+2}, \tag{3.8}$$

for any integer $n \geq 2$. In practice, Figure 1 depicts that the presented solutions $y_n(x)$ and their derivatives $y_n'(x)$, $n = 1, 3, 5$ approximate exact ones better than $y_n^{\mathrm{A}}(x)$ and $(y_n^{\mathrm{A}})'(x)$. Therein, we chose the initial solution $y_0(x) = (\kappa/2)x^2$ as given in (2.10) and took a numerical solution for the exact solution which is denoted by $y^*(x)$. Moreover, Table 1 includes numerical results of the errors and the CUP times spent in computations for the presented solution $y_n(x)$ compared with those of $y_n^{\mathrm{A}}(x)$. The $L_\infty$ error indicates the maximum error for the 50 nodes selected in the interval $(0, \rho)$, where $\rho$ is a radius of convergence of the series solution given in the literature [2, 15]. In fact, $\rho \approx 4.02$ for $\alpha = 1$ and $\rho \approx 5.69$ for $\alpha = 1/2$. The $L_2$ error means $\|y_n - y^*\|_2$ over the same interval.

By the numerical performance, we can surmise the convergence of the algorithm proposed in this work, and the rate of convergence is better than that of the Adomian decomposition method though it spends more CPU time as shown above. In addition, for example, for the case of $\alpha = 1$ and $\mu = 0$, we may guess that the presented method has the same radius of convergence, $\rho = 4.0234644935$ as the well known Blasius' series [2] as follows:

$$y(x) = \sum_{k=0}^{\infty}(-1)^k p_k \frac{\kappa^{k+1}}{(3k+2)!} x^{3k+2}, \tag{3.9}$$

where $p_0 = 1$ and

$$p_k = \sum_{j=0}^{k-1}\binom{3k-1}{3j} p_j p_{k-j-1}, \quad k \geq 1. \tag{3.10}$$

Theoretical convergence analysis with extended application of the presented method to more general problems is left for further works.

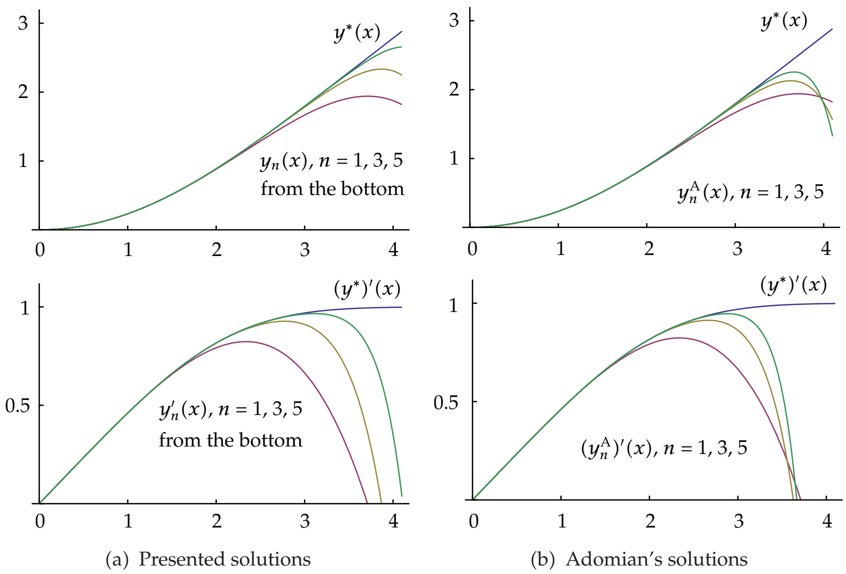


**Figure 1:** Graphs of the presented solutions $y_n(x)$ and their derivatives $y_n'(x)$ in (a) and those of Adomian's solutions $y_n^A(x)$ and their derivatives $(y_n^A)'(x)$ in (b).

**Table 1:** Comparison of the numerical results obtained by the presented method and the Adomian's decomposition method for $\alpha = 1, 1/2$ with $\mu = 0$.

| | $n$ | Presented solution, $y_n(x)$ | | | Adomian's solution, $y_n^A(x)$ | | |
|---|---|---|---|---|---|---|---|
| | | $L_\infty$ error | $L_2$ error | CPU time (sec) | $L_\infty$ error | $L_2$ error | CPU time (sec) |
| $\alpha = 1$ | 2 | 0.690 | 0.318 | 0.14 | 0.995 | 0.445 | 0.09 |
| | 4 | 0.300 | 0.110 | 1.03 | 1.080 | 0.382 | 0.37 |
| | 6 | 0.078 | 0.025 | 6.37 | 1.110 | 0.335 | 1.48 |
| | 8 | 0.013 | 0.004 | 26.5 | 1.118 | 0.299 | 5.74 |
| $\alpha = 1/2$ | 2 | 0.976 | 0.534 | 0.13 | 1.407 | 0.748 | 0.11 |
| | 4 | 0.424 | 0.185 | 1.01 | 1.527 | 0.643 | 0.41 |
| | 6 | 0.111 | 0.041 | 6.71 | 1.570 | 0.564 | 1.55 |
| | 8 | 0.019 | 0.006 | 27.4 | 1.581 | 0.504 | 5.93 |

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