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Research Article

Valuation of Inflation-Linked Annuities in a Lévy Market

Sure Mataramvura

Department of Actuarial Sciences, School of Management Studies, University of Cape Town, Cape Town, Rondebosch 7701, South Africa

Correspondence should be addressed to Sure Mataramvura, sure.mataramvura@uct.ac.za

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We study the problem of pricing an inflation adjusted annuity in a forward rates market with jumps. Since the market will be incomplete, we use the minimal f^q-martingale measure Q_q which we use for computing discounted expectations. We give explicit results for Q_q together with explicit results for the price of the annuity.

1. Introduction

Inflation derivatives are a new feature in the derivatives markets with active trading starting in the 1980's. Their origin has been necessitated by the risk of inflation eroding medium to long-term maturity securities. In economies with anticipation of high inflation, it makes sense to consider the real value of future assets as opposed to the nominal value. According to the simplified Fisher's model, the real interest rate equals the nominal interest rate minus the inflation rate. It is only in cases where inflation rate is negligible that these two rates almost give the same rate of return. Pricing annuities differ from pricing options in that an annuity makes periodic or continuous payments usually up to a certain redemption date which could be predetermined or which is contingent on the life of the policy holder. An option, when it matures and if its maturity is beneficial to its holder, will make a single payment, the redemption value of the option. An option can also expire worthless. Annuities do not have this option of expiring worthless. In elementary Actuarial analysis, the premium to be paid for an annuity is the expected present value (EPV) of cash flows. As a result, we then find striking similarities between computing option prices and insurance premiums.

Motivated by the work [1] and references therein, in this paper, we find the premium of a nonlife-based annuity in a market where bond prices are driven by jump processes. We emphasize here that the inclusion of life contingency will not bring much deviation of this result because market movements are independent of the future lifetime of a policyholder. Due to incompleteness caused by the introduction of jumps, we consider and give explicitly the minimal f^q-martingale measure studied in [2].

We characterize our problem as one similar to interest rate and foreign exchange derivatives (see [3] for more). Naturally the premium of an inflation protected annuity is equal to the premium of an ordinary annuity plus some "penalty" which results from the extra benefit of the policy holder being cushioned for inflation. This paper concentrates on finding explicit results of this extra "penalty".

The paper is organized as follows: the following section looks at the mathematical preliminaries. For this part, we use the notation in [4]. We also refer the reader to the same text for more detailed analysis of Lévy processes. Section 3 looks at the assets price dynamics. We give the Consumer Price Index (CPI) as the ratio of the nominal bond to the real bond. This approach is similar to foreign exchange derivatives where we consider the nominal bond as representing the *local* currency, the real bond as representing the *foreign* currency, and then the CPI being the exchange rate of foreign currency to local currency. A high value of exchange rate indicates a weak local currency with respect to the foreign currency. In the same vein, high inflation indicates high-interest rates which may not stimulate production due to high cost of production. Of interest in Chapter 3 is our version of Fisher's equation (see [5]) which is an extended version that incorporates volatility coefficients and jump terms.

In Section 4, we give the main result which ends with the premium of a non-life annuity adjusted for inflation being given explicitly.

2. Mathematical Preliminaries

Consider a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ where $\{\mathcal{F}\}_{t\geq 0}$ is the filtration such that the Lévy-Itô processes discussed in this paper are adapted.

Given a Lévy process $X(t)$, the jump of $X(t)$ is $\Delta X(t) = X(t) - X(t^-)$. The number of jumps, which occur before or at time t and of a size contained in some Borel set U, can be counted by the measure $N(t, U)$ called the Poisson random measure. It turns out that

$$N(t, U) = \sum_{0\leq s\leq t} \chi_U(\Delta X(s)). \tag{2.1}$$

For times $0 \leq t_1 \leq t_2 < \infty$, we denote the differential form of $N(t_2, U) - N(t_1, U)$ by $N(dt, dz)$. The set function $\nu(U) = E[N(1, U)]$ is called the Lévy measure of X and we require that $\int_{\mathbb{R}} \min(1, z^2)\nu(dz) < \infty$.

The measure $\widetilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt$ is called the compensated Poisson random measure. In this paper, we will be considering jumps of sizes greater than -1. Therefore, we have the relationship (see [4]),

$$\widetilde{N}(dt, dz) = \begin{cases} N(dt, dz) - \nu(dz)dt, & \text{if } |z| < 1, \\ N(dt, dz), & \text{if } |z| \geq 1. \end{cases} \tag{2.2}$$

The Lévy-Itô process $X(t)$ that we will consider in this paper is of the form

$$dX(t) = \alpha(t, X(t))dt + \beta(t, X(t))dB(t) + \int_{\mathbb{R}} \gamma(t, X(t))\widetilde{N}(dt, dz), \tag{2.3}$$

where $B(t)$ is a standard Brownian motion and the coefficients $\alpha(\cdot,\cdot), \beta(\cdot,\cdot)$ and $\gamma(\cdot,\cdot)$ satisfy the necessary growth conditions (see [4]) for which (2.3) has a unique strong solution $X(t)$.

The unique characteristic function $\eta_X(u) = E[e^{iuX(t)}]$ is given by the Lévy-Khintchine formula (see [4] or [6] for more).

Frequently we will be referring to the Lévy-Itô formula for this Lévy process. We refer the reader to ([4]) for a more general definition. We will, for the purpose of this paper, restrict ourselves to a two-dimensional version of the definition.

Definition 2.1. Let $X(t) \in \mathbb{R}^2$ be an Itô-Lévy process of the form

$$dX(t) = \alpha(t, \omega)dt + \sigma(t, \omega)dB(t) + \int_{\mathbb{R}} \gamma(t, z, \omega)\widetilde{N}(dt, dz), \tag{2.4}$$

where α: $[0, t] \times \Omega \to \mathbb{R}^2$, $\sigma : [0, T] \times \Omega \to \mathbb{R}^{2\times1}$, and $\gamma : [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R}^{2\times1}$ are adapted processes such that the integrals exist. Let $f \in C^{1,2}([0, T] \times \mathbb{R}^2; \mathbb{R})$ and put $Y(t) = f(t, X(t))$. Then

$$\begin{aligned} dY(t) = {} & \frac{\partial f}{\partial t}dt + \sum_{i=1}^{2} \frac{\partial f}{\partial x_i}(\alpha_i dt + \sigma_i dB(t)) + \frac{1}{2}\sum_{i,j=1}^{2} (\sigma\sigma^*)_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} dt \\ & + \sum_{k=1}^{2} \int_{\mathbb{R}} \left\{ f(t, X(t^-) + \gamma(t, z)) - f(t, X(t^-)) - \sum_{i=1}^{2} \gamma_i(t, z) \frac{\partial f}{\partial x_i}(X(t^-)) \right\} \nu(dz)dt \\ & + \sum_{k=1}^{2} \int_{\mathbb{R}} \{ f(t, X(t^-) + \gamma(t, z)) - f(t, X(t^-)) \} \widetilde{N}(dt, dz), \end{aligned} \tag{2.5}$$

where $\gamma \in \mathbb{R}^{2\times1}$ is column vector $\gamma = (\gamma_1\ \gamma_2)^*$. Here the $*$ represents transposition of a matrix.

The Itô-Lévy formula above shall be useful in the next chapters.

3. Asset Dynamic Models

3.1. The Bond Price Process

Suppose that the real(R) and nominal(N) forward rates are given jointly by

$$\begin{aligned} f_k(t, T) = {} & f_k(0, T) + \int_0^t \alpha_k(v, T)dv + \int_0^t \sigma_k(v, T)dB(v) \\ & + \int_0^T \gamma_k(v, T, z)\widetilde{N}_k(dv, dz), \quad k \in \{N, R\}. \end{aligned} \tag{3.1}$$

We know that the price of a zero coupon nominal (real) bond is given by

$$P_k(t,T) = \exp\left(-\int_t^T f_k(t,s)ds\right), \quad \forall t \in [0,T]. \tag{3.2}$$

Proposition 3.1. *Let $a_k(t,T) = -\int_t^T \sigma_k(t,s)dt$ and $c_k(t,T,z) = -\int_t^T \gamma_k(t,s,z)dt$. Define also $b_k(t,T) = -\int_t^T \alpha_k(t,s)dt + (1/2)a_k(t,T) - \int_{\mathbb{R}} c_k(t,T,z)\nu(dz)$. Assume that $a_k(\cdot), b_k(\cdot,\cdot)$, and $c_k(\cdot)$ saisfy the necessary regularity conditions for applying Fubini's theorem. Then $P_k(t,T)$ is given by*

$$dP_k(t,T) = P_k(t,T)\left[(r_k(t) + b_k(t,T))dt + a_k(t,T)dB(t) + \int_{\mathbb{R}} c_k(t,T,z)\widetilde{N}(dt,dz)\right]. \tag{3.3}$$

Proof. Simplifying and using the Fubini's theorem, we have

$$\begin{aligned}
\ln P_k(t,T) &= -\int_t^T f_k(t,s)ds \\
&= -\left[\int_t^T f_k(0,s)ds + \int_t^T\int_0^t \alpha_k(v,s)dv\,ds + \int_t^T\int_0^t \sigma_k(v,s)dB(v)ds \right.\\
&\qquad \left. + \int_t^t\int_0^t\int_{\mathbb{R}} \gamma_k(v,s,z)\widetilde{N}(dv,dz)dv\,ds\right] \\
&= -\int_t^t f_k(0,s)ds - \int_0^t\left[\int_t^T \alpha_k(v,s)ds\right]dv - \int_0^t\left[\int_t^T \sigma_k(v,s)ds\right]d\mathrm{B}(v) \\
&\quad - \int_0^t\left[\int_t^T\int_{\mathbb{R}} \gamma_k(v,s,z)\widetilde{N}(ds,dz)ds\right]dv \\
&= -\int_0^T f_k(0,s)ds + \int_0^t f_k(0,s)ds - \int_0^t\left[\int_v^T \alpha_k(v,s)ds\right]dv + \int_0^t\left[\int_v^t \alpha_k(v,s)ds\right]dv \\
&\quad - \int_0^t\left[\int_v^T \sigma_k(v,s)ds\right]dB(v) + \int_0^t\left[\int_v^t \sigma_k(v,s)ds\right]dB(v) \\
&\quad - \int_0^t\left[\int_v^T\int_{\mathbb{R}} \gamma_k(v,s,z)\widetilde{N}(ds,dz)ds\right]dv + \int_0^t\left[\int_v^t\int_{\mathbb{R}} \gamma_k(v,s,z)\widetilde{N}(ds,dz)ds\right]dv \\
&= \ln P_k(0,T) + \int_0^t r_k(s)ds - \int_0^t\left[\int_v^T \alpha_k(v,s)dv\right]ds - \int_0^t\left[\int_v^T \sigma_k(v,s)ds\right]dB(v) \\
&\quad - \int_0^t\left[\int_v^T\int_{\mathbb{R}} \gamma_k(v,s,z)dv\,ds\right]\widetilde{N}(ds,dz).
\end{aligned} \tag{3.4}$$

Therefore, we have

$$\begin{aligned}\ln P_k(t,T) &= \ln P_k(0,T)\\&\quad+\int_0^t\left[r_k(s)-\int_v^T\alpha_k(v,s)dv-\frac{1}{2}\int_v^T\sigma_k^2(v,s)dv-\int_v^T\int_{\mathbb{R}}\gamma_k(v,s,z)dv\nu(dz)\right]ds\\&\quad-\int_0^t\int_v^T\sigma_k(v,s)dvdB(s)+\frac{1}{2}\int_0^t\int_v^T\sigma_k^2(v,s)dv\,ds\\&\quad-\int_0^t\int_v^T\int_{\mathbb{R}}\gamma_k(v,s,z)dv\,ds\widetilde{N}(ds,dz)+\int_0^t\int_v^T\int_{\mathbb{R}}(v,s,z)dv\,ds\nu(dz).\end{aligned}\tag{3.5}$$

Let $a_k(t,T)=-\int_t^T\sigma_k(t,s)dt$ and $c_k(t,T,z)=-\int_t^T\gamma_k(t,s,z)dt$. Define also

$$b_k(t,T)=-\int_t^T\alpha_k(t,s)dt+\frac{1}{2}a_k(t,T)-\int_{\mathbb{R}}c_k(t,T,z)\nu(dz)\tag{3.6}$$

and assume that $c_R(t,T,z)\neq-1$.

Then from the previous expression we have,

$$\begin{aligned}\ln P_k(t,T) &= \ln P_k(0,T)+\int_0^t(r_k(s)+b_k(s,T))ds+\int_0^t a_k(v,s)dB(s)\\&\quad-\frac{1}{2}\int_0^t a_k^2(v,s)ds+\int_0^t\int_{\mathbb{R}}c_k(v,s,z)\widetilde{N}(ds,dz)ds-\int_0^t\int_{\mathbb{R}}c_k(v,s,z)ds\nu(dz),\end{aligned}\tag{3.7}$$

$$dP_k(t,T)=P_k(t,T)\left[(r_k(t)+b_k(t,T))dt+a_k(t,T)dB(t)+\int_{\mathbb{R}}c_k(t,T,z)\widetilde{N}(dt,dz)\right].$$

□

3.2. The CPI Process

As we said before, the Consumer Price Index (CPI), $C(t)$, is taken as the ratio of the nominal bond price to the real bond price. By Lévy-Itô formula on $C(t)=P_N(t,T)/P_R(t,T)$, we have

$$dC(t)=C(t)\left[a(t)dt+\sigma(t)dB(t)+\int_{\mathbb{R}}\gamma(t,z)\widetilde{N}(dt,dz)\right],\quad C(0)=c,\tag{3.8}$$

where

$$\begin{aligned}a(t) &= r_N(t)-r_R(t)+b_N(t,T)-b_R(t,T)-a_R(t,T)(a_N(t,T)-a_R(t,T))\\&\quad+\int_{|z|<1}\left\{\frac{1+c_N(t,T,z)-(c_N(t,T,z)-c_R(t,T,z)+1)(1+c_R(t,T,z))}{1+c_R(t,T,z)}\right\}\nu(dz),\end{aligned}\tag{3.9}$$

$$\sigma(t) = a_N(t,T) - a_R(t,T), \qquad \gamma(t) = \frac{c_N(t,T,z) - c_R(t,T,z)}{1 + c_R(t,T,z)}. \tag{3.10}$$

Note that the solution of (3.8) is

$$\begin{aligned} C(t) = C(0)\exp\Bigg[&\int_0^t \left(a(s) - \frac{1}{2}\sigma^2(s)\right)ds + \int_0^t \sigma(s)dB(s) \\ &+ \int_0^t \int_{|z|<1} \{\ln(1+\gamma(s,z)) - \gamma(s,z)\}\nu(dz)ds \\ &+ \int_0^t \int_{\mathbb{R}} \ln(1+\gamma(s,z))\widetilde{N}(ds,dz)\Bigg], \quad C(0) = c \end{aligned} \tag{3.11}$$

or alternatively

$$\begin{aligned} C(t) = C(0)\exp\Bigg[&\int_0^t a^*(s)ds + \int_0^t \sigma(s)dB(s) \\ &+ \int_0^t \int_{|z|<1} \{\ln(1+\gamma(s,z)) - \gamma(s,z)\}\nu(dz)ds \\ &+ \int_0^t \int_{\mathbb{R}} \ln(1+\gamma(s,z))\widetilde{N}(ds,dz)\Bigg], \quad C(0) = c, \end{aligned} \tag{3.12}$$

where $a^*(t) = a(s) - (1/2)(\sigma^N(t))^2$.

Definition 3.2 (Fisher's inflation model). The rate of inflation $I(t)$ at time t is the drift process of the CPI, so that

$$\begin{aligned} I(t) = r_N(t) - r_R(t) + b_N(t,T) - b_R(t,\mathrm{T}) - a_R(t,T)(a_N(t,T) - a_R(t,T)) \\ + \int_{|z|<1} \left\{ \frac{1 + c_N(t,T,z) - (c_N(t,T,z) - c_R(t,T,z) + 1)(1 + c_R(t,T,z))}{1 + c_R(t,T,z)} \right\}\nu(dz), \end{aligned} \tag{3.13}$$

which is essentially our version of the Fisher equation.

In particular, if the volatility coefficients are zero and there are no jumps in the bond price dynamics, then the Fisher's equation says that the real interest rate is approximately equal to the nominal rate less inflation rate. High inflation coupled with low nominal rates may result in negative returns and no incentives for depositors to keep their money in the bank. This will also affect pensioners whose future payments in the form of annuities will be eroded by inflation. Keeping inflation low is one of the main tasks of the central bank of any country since hyperinflation brings with itself also economic and social chaos.

In our case, we see that volatility and jumps can further reduce the real rate to far lower than the original rate given by Fisher. It is thus crucial to offer annuities that are inflation adjusted in that case.

Theorem 3.3 (Girsanov Theorem; see [4]). *Suppose that there exists a process $u(t)$ and $\theta(t,z)$ which are $\mathcal{F}_t$-adapted and such that*

$$a_k(t,T)u(t) + \int_{\mathbb{R}} c_k(t,T,z)\theta(t,z)\nu(dz) = b_k(t,T), \tag{3.14}$$

and such that the process

$$\begin{aligned} Z(t) := \exp\Bigg[&-\int_0^t u(s)dB(s) - \frac{1}{2}\int_0^t u^2(s)ds + \int_0^t\int_{\mathbb{R}} \ln(1-\theta(s,z))\widetilde{N}(ds,dz) \\ &+ \int_0^t\int_{\mathbb{R}} \{\ln(1-\theta(s,z)) + \theta(s,z)\}\nu(dz)ds \Bigg], \quad 0 \le t \le T \end{aligned} \tag{3.15}$$

is well defined and satisfies $E[Z(T)] = 1$ where expectation is taken with respect to the original probability measure P. Define the probability measure $Q = Q^{(u(t),\,\theta(t,z))}$ on $\mathcal{F}_T$ by $dQ(\omega) = Z(T)dP(\omega)$. Then

(1) *$P_k(t,T)$ is a local martingale with respect to Q,*

(2) *the process $\widetilde{B}(t)$ given by $d\widetilde{B}(t) = u(t)dt + dB(t)$ is a Q-Brownian motion,*

(3) *the random measure measure $\widetilde{N}^Q(dt,dz)$ given by $\widetilde{N}^Q(dt,dz) = \theta(t,z)\nu(dz)dt + \widetilde{N}(dt,dz)$ is a Q compensated Poisson random measure in the sense that $M(t) = \int_0^t\int_{\mathbb{R}} \gamma_k(s,z)\widetilde{N}^Q(ds,dz); 0 \le t \le T$ is a local Q-martingale provided the condition $\int_0^T\int_{\mathbb{R}} \gamma_k(s,z)\theta(s,z)\nu(dz)ds < \infty$ holds a.s.*

For a proof, we just consider the Itô-Lévy formula on $Y_k(t) = Z(t)P_k(t)$ and proceed as in ([4]).

We have to observe here that for $k \in \{R,N\}$, (3.14) will result in infinitely many solutions $(u(t),\theta(t,z))$ and as such the market of nominal (real) bonds is incomplete. We do have an infinite number of equivalent martingale measures Q. Let $\mathcal{M}^e(P_k)$ denote the set of all equivalent martingale measure Q for $P_k(t)$.

Observe that, with respect to Q, we have

$$dP_k(t,T) = P_k(t,T)\left[r_k(t)dt + a_k(t,T)d\widetilde{B}(t) + \int_{\mathbb{R}} c_k(t,T,z)\widetilde{N}^Q(dt,dz) \right]. \tag{3.16}$$

Then with respect to Q, we have $C(t)$ given by

$$dC(t) = C(t)\left[\alpha(t)dt + \sigma(t)d\widetilde{B}(t) + \int_{\mathbb{R}} \gamma(t,z)\widetilde{N}^Q(dt,dz) \right], \quad C(0) = c, \tag{3.17}$$

where

$$\alpha(t) = r_N(t) - r_R(t) - a_R(t,T)(a_N(t,T) - a_R(t,T)) \\ + \int_{|z|<1} \left\{ \frac{1 + c_N(t,T,z) - (c_N(t,T,z) - c_R(t,T,z) + 1)(1 + c_R(t,T,z))}{1 + c_R(t,T,z)} \right\} \nu(dz) \tag{3.18}$$

and as before $\sigma(t) = a_N(t,T) - a_R(t,T)$ and

$$\gamma(t) = \frac{c_N(t,T,z) - c_R(t,T,z)}{1 + c_R(t,T,z)}. \tag{3.19}$$

Consequently, the solution of (3.17) is

$$C(t) = C(0)\exp\left[\int_0^t \left(\alpha(s) - \frac{1}{2}\sigma^2(s)\right)ds + \int_0^t \sigma(s)d\widetilde{B}(s) \right. \\ + \int_0^t \int_{|z|<1} \{\ln(1 + \gamma(s,z)) - \gamma(s,z)\}\nu(dz)ds \\ \left. + \int_0^t \int_{\mathbb{R}} \ln(1 + \gamma(s,z))\widetilde{N}^Q(ds,dz)\right], \quad C(0) = c \tag{3.20}$$

or alternatively

$$C(t) = C(0)\exp\left[\int_0^t \alpha^*(s)ds + \int_0^t \sigma(s)d\widetilde{B}(s) + \int_0^t \int_{|z|<1} \{\ln(1 + \gamma(s,z)) - \gamma(s,z)\}\nu(dz)ds \right. \\ \left. + \int_0^t \int_{\mathbb{R}} \ln(1 + \gamma(s,z))\widetilde{N}^Q(ds,dz)\right], \quad C(0) = c, \tag{3.21}$$

where $\alpha^*(t) = \alpha(s) - (1/2)(\sigma^N(t))^2$.

4. Pricing an Inflation Adjusted Annuity

Pricing in incomplete markets requires one to choose one of the equivalent martingale measures in $\mathcal{M}^e(P_k)$. The most popular measures are minimal martingale measure which reduces the squared variance between the payoff of a contingent claim and the terminal value of a portfolio of stocks and bond chosen to hedge the claim. Alternatively the minimal relative entropy measure has been widely suggested as one of the best measures "closest" in some sense to the original probability P. In this paper, we will consider the minimal f^q-martingale measure.

Definition 4.1 ((Minimal f^q-martingale measure) (see [2])). Let $I = (-\infty, 0) \bigcup (1, \infty)$ and let $q \in I$ be arbitrary but fixed. For $f^q(z) = z^q$, the minimal f^q-martingale measure (qMMM) Q_q is the equivalent martingale measure which minimizes the f^q-divergence

$$f^q(Q \setminus P) := E_P\left[f^q(Z)\right] = E_P\left[Z(T)^q\right]. \tag{4.1}$$

Corollary 4.2 (see [2]). *Let* $\mathcal{M}^q := \{Q^{(u,\theta)} \in \mathcal{M}^e(P_k) \mid f^q(Q \setminus P) < \infty\}$ *and let* $\overline{\mathcal{M}} = \{Q^{(u(t),\theta(t,z))} \in \mathcal{M}^e(P_k) \mid (u(t), \theta(t,z))$ *time independent and deterministic*$\}$*, then*

$$f^q(Q_q \setminus P) = \inf_{Q \in \overline{\mathcal{M}}} f^q(Q \setminus P). \tag{4.2}$$

The above corollary tells us that $Q_q^{(u(t),\, \theta(t,z))}$ consists only of deterministic and time independent pair (u, θ) which minimizes the f^q-divergence.

Theorem 4.3. *Let* $\omega_1 = \int_{\mathbb{R}} \nu(dz)$ *and* $\omega_2 = \int_{\mathbb{R}} c_k(t, T, z)\nu(dz)$. *Assume that the Girsanov parameters of Theorem 3.3 are time-independent and constant, that is, assume that* $u(t) = u$ *and* $\theta(t, z) = \theta$. *Then the minimal* f^q*-martingale measure* Q_q *is given by the optimal Girsanov pair* $(\overline{u}, \overline{\theta})$ *where* $\overline{\theta}$ *is a solution of the equation*

$$\omega_1 a_k^2(t, T)(1-\theta)^{q-1} - \omega_2^2(q-1)\theta + (q-1)b_k\omega_2 + \omega_1 a_k^2(t, T) = 0\ , \tag{4.3}$$

and $\overline{u}$ *and* $\overline{\theta}$ *are related by*

$$a_k(t, T)\overline{u} + \overline{\theta} \int_{\mathbb{R}} c_k(t, T, z)\nu(dz) = b_k(t, T). \tag{4.4}$$

Proof. *First observe that* (4.4) *is actually* (3.14) *with* $u(t)$ *replaced with* $\overline{u}$ *and* $\theta(t, z)$ *replaced with* $\overline{\theta}$. *Next, we need also the following proposition which can be found as an exercise in [4]:* □

Proposition 4.4. *Suppose that* $\gamma(s, z)$ *is deterministic and satisfies some regularity conditions. Then*

$$E_P\left[\exp\left(\int_0^t \int_{\mathbb{R}} \gamma(s, z)\widetilde{N}(ds, dz)\right)\right] = \exp\left(\int_0^t \int_{\mathbb{R}} \left\{e^{\gamma(s,z)} - 1 - \gamma(s, z)\right\}\nu(dz)ds\right). \tag{4.5}$$

The proof is trivial. We now go back to the proof of Theorem 4.3.

Assume that $u(t) = u$ and $\theta(t,z) = \theta$, then due to independence of the Brownian motion and the poisson random measure, one has

$$
\begin{aligned}
E_P[Z^q(T)] &= E_P\left[e^{-qu_k B(T)}\right] \cdot E_P\left[\exp\left(\int_0^T \int_{\mathbb{R}} q \ln(1-\theta)\widetilde{N}(ds,dz)\right)\right], \\
&\exp\left[-\frac{1}{2}qu^2 T + \int_0^T \int_{\mathbb{R}} q\{\ln(1-\theta) + \theta\}\nu(dz)ds\right] \\
&= \exp\left[\frac{T}{2}u^2 q(q-1) + T\int_{\mathbb{R}} \{(1-\theta)^q - 1 - q\theta\}\nu(dz)\right].
\end{aligned}
\tag{4.6}
$$

Therefore Q_q is found by solving the following deterministic optimization problem, which is a particular case of Problem $\mathcal{P}_q$ in [2]:

$$
\begin{aligned}
&\text{minimize} \quad k_q(u,\theta) = \frac{q(q-1)}{2}u^2 + \int_{\mathbb{R}} \{(1-\theta)^q - 1 - q\theta\}\nu(dz), \\
&\text{subject to} \quad a_k(t,T)u + \theta\int_{\mathbb{R}} c_k(t,T,z)\nu(dz) = b_k(t,T).
\end{aligned}
\tag{4.7}
$$

It was proved in [2] that a general solution exists under certain conditions. We conjecture here that the necessary conditions for a solution exist. We the strive to find the optimal solution explicitly.

Let $\omega_1 = \int_{\mathbb{R}} \nu(dz)$ and $\omega_2 = \int_{\mathbb{R}} c_k(t,T,z)\nu(dz)$. Thus $(\overline{u},\overline{\theta})$ is a solution of

$$
\begin{aligned}
&\text{minimize} \quad e_{(u,\theta)} \frac{q(q-1)}{2}u^2 + \omega_1\left[(1-\theta)^q - 1 - q\theta\right] \\
&\text{subject to} \quad a_k(t,T)u + \theta\omega_2 = b_k(t,T).
\end{aligned}
\tag{4.8}
$$

Define the Lagrangian

$$
\mathcal{L}(\lambda,u,\theta) = \frac{q(q-1)}{2}u^2 + \omega_1\left[(1-\theta)^q - 1 - q\theta\right] + \lambda\left[a_k(t,T)u + \theta\omega_2 - b_k(t,T)\right],
\tag{4.9}
$$

then one gets the following:

$$
\begin{aligned}
a_k(t,T)u + \theta\omega_2 &= b_k(t,T), \\
q(q-1)u + \lambda a_k(t,T) &= 0, \\
\omega_1 q(1-\theta)^{q-1} + q\omega_1 &= \lambda\omega_2.
\end{aligned}
\tag{4.10}
$$

The result comes from solving these equations.

As a special case, we now let $\widetilde{B}^q(t)$ given by $d\widetilde{B}^q(t) = \overline{u}dt + dB(t)$ denote the Q_q-Brownian motion while $\widetilde{N}^{Q_q}(dt,dz)$ given by $\widetilde{N}^{Q_q}(dt,dz) = \overline{\theta}\nu(dz)dt + N(dt,dz)$ denote the Q_q compensated Poisson random measure.

In that case, with respect to Q_q, we have

$$dC(t) = C(t)\left[\alpha(t)d\widetilde{B}^q(t) + \int_{\mathbb{R}} \gamma(t,z)\widetilde{N}^{Q_Q}(dt,dz)\right], \quad C(0) = c. \tag{4.11}$$

An inflation protected annuity pays if inflation (as measured by the CPI) exceeds a certain threshold K. The payout at any time t will be given by a payout function $\rho(t)$ given by

$$\rho(t) = fN\left[1 + \max\left\{\left(\frac{C(t)}{C(0)} - 1\right) - K, 0\right\}\right], \tag{4.12}$$

where f is the *nominal* annuity and N is the *notional*. Assume that this payment is for a fixed period $[0,T]$ where T is a predetermined limiting age for the policy holder. In most cases T is a random variable with a given distribution. For nonlife policies, this may not be necessary. Assume further that the nominal interest rate r_N is constant over time. Then, the premium to be paid for such an annuity at any time t is given by

$$\pi_0(c) = E_{Q_q}\left[\int_0^T \rho(t)e^{-r_N(T-t)}dt\right]. \tag{4.13}$$

In order to simplify this value, we may want to consider the fact that f and N are constant (or at best time dependent) and as such we are interested in the value $\pi_0^*(c)$ given by

$$\pi_0^*(c) = E_{Q_q}\left[\int_0^T \rho^*(t)e^{-r_N(T-t)}dt\right], \tag{4.14}$$

where $\rho^*(t) = \max\{(C(t)/C(0) - 1) - K, 0\}$. The following result follows

Theorem 4.5. *Let the CPI be given by* (3.8) *and* ρ *given by* (4.12).
Let $\Lambda(x) = \max\{(x/c - 1) - K, 0\}$ *for some constant* c.

(a) *Suppose that*

$$\gamma(t,z) = 0. \tag{4.15}$$

Then the price of an inflation protected annuity is

$$\pi_0(c) = \frac{fN}{r_N}\left[1 - e^{-r_N T}\right] + fN \cdot v(t, C(t)) , \tag{4.16}$$

where

$$
\begin{aligned}
v(t,x) &= \frac{x}{c}\exp\left[-r_N(T-t)+\int_t^T\left(\alpha^*(s)+\frac{1}{2}\sigma^2(s)\right)ds\right]N(d_1)-(1+K)e^{-r_N(T-t)}N(d_2),\\
d_1 &= \frac{\ln(x/c(1+K))+\int_t^T(\alpha^*(s)+\sigma^2(s))ds}{\left(\int_t^T\sigma^2(s)ds\right)^{1/2}},\\
d_2 &= d_1-\left(\int_t^T\sigma^2(s)ds\right)^{1/2}\\
&= \frac{\ln(x/c(1+K))+\int_t^T\alpha^*(s)ds}{\left(\int_t^T\sigma^2(s)ds\right)^{1/2}}.
\end{aligned}
\tag{4.17}
$$

(b) *If on the other hand*

$$
\begin{gathered}
\gamma(t,z)\neq 0,\\
\alpha(t)>\frac{1}{2}\sigma^2(t)+\frac{1}{T}\ln x
\end{gathered}
\tag{4.18}
$$

then

$$
\pi_0(c)=\frac{fN}{r_N}\left[1-e^{-r_NT}\right]+fN\int_0^T Ax^{n^*}e^{-t\varepsilon^*(n^*)}dt,
\tag{4.19}
$$

where, A is a constant,

$$
\varepsilon^*(n)=n\alpha(t)+\frac{1}{2}\sigma^2(t)n(n-1)-r_N+\int_{|z|<1}\left\{\sum_{k=2}^{n}\binom{n}{k}\gamma^k(t,z)\right\}\nu(dz),
\tag{4.20}
$$

and n^* is a solution of the equation

$$
\begin{aligned}
&\frac{1}{2}\sigma^2(t)n^2+n\left(T\alpha(t)-\ln x-T\frac{1}{2}\sigma^2(t)\right)+T\int_{|z|<1}\left\{\sum_{k=2}^{n}\binom{n}{k}\gamma^k(t,z)\right\}\nu(dz)\\
&\quad+\ln\left(\frac{\Lambda(x)}{f(0)}\right)-r_N=0.
\end{aligned}
\tag{4.21}
$$

Note that the second expression of (4.16) and (4.19) represents the "penalty" charged to the investor for protection against inflation. We also assume that payments are not linked to an inflation index with a time lag, which is a common scenario in real life. We also should point out here that the assumption on part (a) of the theorem that $\gamma(t,z) = 0$ will not result in a unique equivalent martingale measure because (3.14) depends on $c_k(t,T,z)$ and $\gamma(t,z) = 0 \not\Rightarrow c_k(t,T,z)$. However, $C(t)$ given by (4.11) will be a Q_q-geometric Brownian motion.

Proof. (a) By Fubini's Theorem on expectations, it is enough to consider

$$v(t,C(t)) = E_{Q_q}\left[e^{-r_N(T-t)}\left(\left(\frac{C(t)}{C(0)} - 1\right) - K\right)^+ \mid \mathcal{F}_t\right]. \tag{4.22}$$

Note that $v(0,C(t)) = v(t,0) = 0$ for all $t \in [0,T]$ and

$$v(T,C(T)) = \left(\left(\frac{C(T)}{C(0)} - 1\right) - K\right)^+. \tag{4.23}$$

If $\gamma(t,z) = 0$, then

$$v\big(t,C(t) + C(t)\gamma(t,z)\big) = v(t,C(t)). \tag{4.24}$$

Now with respect to Q_q, we have

$$C(T) = c(t)\exp\left[\int_t^T \alpha^* ds + \int_0^T \sigma(s)d\widetilde{B}^q(s)\right], \tag{4.25}$$

Therefore, by direct integration or otherwise, (4.22) easily simplifies to

$$\begin{aligned} v(t,C(t)) = {} & \frac{C(t)}{c}\exp\left[-r_N(T-t) + \int_t^T\left(\alpha^*(s) + \frac{1}{2}\sigma^2(s)\right)ds\right]N(d_1) \\ & - (1+K)e^{-r_N(T-t)}N(d_2) \end{aligned} \tag{4.26}$$

with

$$\begin{aligned} d_1 &= \frac{\ln(C(t)/C(0)(1+K)) + \int_t^T(\alpha^*(s) + \sigma^2(s))ds}{\left(\int_t^T \sigma^2(s)ds\right)^{1/2}}, \\ d_2 &= d_1 - \left(\int_t^T \sigma^2(s)ds\right)^{1/2} = \frac{\ln(C(t)/C(0)(1+K)) + \int_t^T \alpha^*(s)ds}{\left(\int_t^T \sigma^2(s)ds\right)^{1/2}}. \end{aligned} \tag{4.27}$$

(b) If $\gamma(t,z) \neq 0$, consider the discounted value function $Y(t) = e^{-r_N t}v(t,C(t))$, then by the one-dimensional Lévy-Itô formula, we have

$$\begin{aligned}dY(t) = e^{-r_N}\Bigg[\Bigg(&v_t(t,C(t)) + \alpha C(t)v_c(t,C(t)) - rv(t,C(t)) + \frac{1}{2}\sigma^2(t)c^2(t)v_{cc}(t,C(t))\\&+\int_{|z|<1}\{v(t,C(t)(1+\gamma(t,z))) - v(t,C(t)) - C(t)\gamma(t,z)v_c(t,C(t))\}\nu(dz)\Bigg)dt\\&+\int_{\mathbb{R}}\{v(t,C(t)(1+\gamma(t,z)))\}\widetilde{N}^{Q_q}(dt,dz) + \sigma C(t)v_c(t,C(t))d\widetilde{B}^q(t)\Bigg].\end{aligned} \tag{4.28}$$

The discounted value function is a Q_q-martingale if the dt term equals zero, or alternatively, if

$$\begin{aligned}&v_t(t,x) + \alpha x v_x(t,x) + \frac{1}{2}\sigma^2(t)x^2v_{xx}(t,x)\\&\quad+\int_{|z|<1}\{v(t,x(1+\gamma(t,z))) - v(t,x) - x\gamma(t,z)v_x(t,x)\}\nu(dz) = r_Nv(t,x).\end{aligned} \tag{4.29}$$

Then $v(t,x)$ is a solution of the following boundary value problem:

$$\begin{aligned}&v_t(t,x) + \alpha x v_x(t,x) + \frac{1}{2}\sigma^2(t)x^2v_{xx}(t,x)\\&\quad+\int_{|z|<1}\{v(t,x(1+\gamma(t,z))) - v(t,x) - x\gamma(t,z)v_x(t,x)\}\nu(dz) = r_Nv(t,x),\\&v(T,0) = 0,\\&v(T,x) = \max\left\{\left(\frac{x}{c}-1\right) - K, 0\right\},\\&v(0,x) = 0,\end{aligned} \tag{4.30}$$

where $x = C(t)$.

We assume a solution of the form $v(t,x) = f(t)x^n$ then by substitution into

$$\begin{aligned}&v_t(t,x) + \alpha x v_x(t,x) + \frac{1}{2}\sigma^2(t)x^2v_{xx}(t,x)\\&\quad+\int_{|z|<1}\{v(t,x(1+\gamma(t,z))) - v(t,x) - x\gamma(t,z)v_x(t,x)\}\nu(dz) = r_Nv(t,x)\end{aligned} \tag{4.31}$$

we get $f'(t) + \varepsilon^*(n)f(t) = 0$ where

$$\varepsilon^*(n) = n\alpha(t) + \frac{1}{2}\sigma^2(t)n(n-1) - r_N + \int_{|z|<1}\left\{\sum_{k=2}^{n}\binom{n}{k}\gamma^k(t,z)\right\}\nu(dz). \tag{4.32}$$

Therefore $v(t,x) = f(0)x^n e^{-t\varepsilon^*(n)}$ and if we let $\Lambda(x) = f(0)x^n e^{-T\varepsilon^*(n)}$, then the terminal conditions imply

$$\begin{aligned} &\frac{1}{2}\sigma^2(t)n^2 + n\left(T\alpha(t) - \ln x - T\frac{1}{2}\sigma^2(t)\right) + T\int_{|z|<1}\left\{\sum_{k=2}^{n}\binom{n}{k}\gamma^k(t,z)\right\}\nu(dz) \\ &\quad + \ln\left(\frac{\Lambda(x)}{f(0)}\right) - r_N = 0. \end{aligned} \tag{4.33}$$

Suppose that $Ta(t) - \ln x - T\ (1/2)\sigma^2(t) > 0$, then there exists a unique solution n^* to this equation.

Then $v(t,x) = f(0)x^{n^*}e^{-t\varepsilon^*(n^*)}$. □

Remark 4.6. Note that if $\alpha(t) = \alpha$ and $\sigma(t) = \sigma$ are constants, then one can explicitly simplify the integral of Theorem 4.5 since for any $y > n$, then $\varepsilon(y) > 0$ and $\int_0^T v(t,x) = Ax^n/\varepsilon(n)(1 - e^{-T\varepsilon(n)})$ which increases with term to maturity. Therefore, inflation adjusted annuities of this nature have capital losses in that the price charged at issue or purchase is more than redemption value. This applies to the case of course where we consider the price as a function of maturity time T.

5. Conclusions

Indexation helps to adjust future earnings in line with the cost of living at that time. Usually the income is indexed by reference to the values of inflation with some time lag. This is done to allow for publishing the RPI values. In that case proper adjustment must be made to the premium. In this paper we have managed to get explicit results to the price of inflation indexed annuities without time lag. We looked at non-life-based annuities and conjecture that there is not much value added by considering life based annuities.

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