Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 867217, 9 pages
doi:10.1155/2011/867217

Research Article

A Note on the Generalized q-Bernoulli Measures with Weight α

T. Kim,[1] S. H. Lee,[1] D. V. Dolgy,[2] and C. S. Ryoo[3]

[1] *Division of General Education, Kwangwoon University, Seoul 139-701, Republic of Korea*
[2] *Hanrimwon, Kwangwoon University, Seoul 139-701, Republic of Korea*
[3] *Department of Mathematics, Hannam University, Daejeon 306-791, Republic of Korea*

Correspondence should be addressed to T. Kim, tkkim@kw.ac.kr

Received 21 April 2011; Accepted 16 May 2011

Academic Editor: Gabriel Turinici

Copyright © 2011 T. Kim et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We discuss a new concept of the q-extension of Bernoulli measure. From those measures, we derive some interesting properties on the generalized q-Bernoulli numbers with weight α attached to χ.

1. Introduction

Let p be a fixed prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = 1/p$ (see [1–14]).

When we talk of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. Throughout this paper we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$, and we use the notation of q-number as

$$[x]_q = \frac{1-q^x}{1-q}, \tag{1.1}$$

(see [1–14]). Thus, we note that $\lim_{q\to 1}[x]_q = x$.

In [2], Carlitz defined a set of numbers $\xi_k = \xi_k(q)$ inductively by

$$\xi_0 = 1, \qquad (q\xi+1)^k - \xi_k = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases} \tag{1.2}$$

with the usual convention of replacing ξ^k by ξ_k.

These numbers are q-extension of ordinary Bernoulli numbers B_k. But they do not remain finite when $q = 1$. So he modified (1.2) as follows:

$$\beta_{0,q} = 1, \qquad q(q\beta + 1)^k - \beta_{k,q} = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases} \tag{1.3}$$

with the usual convention of replacing β^k by $\beta_{k,q}$.

The numbers $\beta_{k,q}$ are called the k-th Carlitz q-Bernoulli numbers.

In [1], Carlitz also considered the extended Carlitz's q-Bernoulli numbers as follows:

$$\beta_{0,q}^h = \frac{h}{[h]_q}, \qquad q^h\left(q\beta^h + 1\right)^k - \beta_{k,q}^h = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases} \tag{1.4}$$

with the usual convention of replacing $(\beta^h)^k$ by $\beta_{k,q}^h$.

Recently, Kim considered q-Bernoulli numbers, which are different extended Carlitz's q-Bernoulli numbers, as follows: for $\alpha \in \mathbb{N}$ and $n \in \mathbb{Z}_+$,

$$\widetilde{\beta}_{0,q}^{(\alpha)} = 1, \quad q\left(q^\alpha \widetilde{\beta}^{(\alpha)} + 1\right)^n - \widetilde{\beta}_{n,q}^{(\alpha)} = \begin{cases} \dfrac{\alpha}{[\alpha]_q}, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases} \tag{1.5}$$

with the usual convention of replacing $(\widetilde{\beta}^{(\alpha)})^k$ by $\widetilde{\beta}_{k,q}^{(\alpha)}$ (see [3]).

The numbers $\widetilde{\beta}_{k,q}^{(\alpha)}$ are called the k-th q-Bernoulli numbers with weight α.

For fixed $d \in \mathbb{Z}_+$ with $(p, d) = 1$, we set

$$\begin{gathered} X = X_d = \varprojlim_N \left(\frac{\mathbb{Z}}{dp^N\mathbb{Z}}\right), \qquad X_1 = \mathbb{Z}_p, \\ X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p), \\ a + dp^N\mathbb{Z}_p = \left\{x \in X \mid x \equiv a\left(\operatorname{mod}\ dp^N\right)\right\}, \end{gathered} \tag{1.6}$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \le a < dp^N$.

Let $\mathrm{UD}(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in \mathrm{UD}(\mathbb{Z}_p)$, the p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)\,d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x, \tag{1.7}$$

(see [3, 4, 15, 16]). By (1.5) and (1.7), the Witt's formula for the q-Bernoulli numbers with weight α is given by

$$\int_{\mathbb{Z}_p} [x]_{q^\alpha}^n d\mu_q(x) = \widetilde{\beta}_{n,q}^{(\alpha)}, \quad \text{where } n \in \mathbb{Z}_+. \tag{1.8}$$

The q-Bernoulli polynomials with weight α are also defined by

$$\widetilde{\beta}_{n,q}^{(\alpha)}(x) = \sum_{l=0}^{n} \binom{n}{l} [x]_{q^\alpha}^{n-l} q^{\alpha l x} \widetilde{\beta}_{l,q}^{(\alpha)}. \tag{1.9}$$

From (1.7), (1.8), and (1.9), we can derive the Witt's formula for $\widetilde{\beta}_{n,q}^{(\alpha)}(x)$ as follows:

$$\int_{\mathbb{Z}_p} [x+y]_{q^\alpha}^n d\mu_q(y) = \widetilde{\beta}_{n,q}^{(\alpha)}(x), \quad \text{where } n \in \mathbb{Z}_+. \tag{1.10}$$

For $n \in \mathbb{Z}_+$ and $d \in \mathbb{N}$, the distribution relation for the q-Bernoulli polynomials with weight α are known that

$$\widetilde{\beta}_{n,q}^{(\alpha)}(x) = \frac{[d]_{q^\alpha}^n}{[d]_q} \sum_{a=0}^{d-1} q^a \widetilde{\beta}_{n,q^d}^{(\alpha)}\left(\frac{x+a}{d}\right), \tag{1.11}$$

(see [3]). Recently, several authors have studied the p-adic q-Euler and Bernoulli measures on $\mathbb{Z}_p$ (see [8, 9, 11, 13, 14]). The purpose of this paper is to construct p-adic q-Bernoulli distribution with weight α (= p-adic q-Bernoulli unbounded measure with weight α) on $\mathbb{Z}_p$ and to study their integral representations. Finally, we construct the generalized q-Bernoulli numbers with weight α and investigate their properties related to p-adic q-L-functions.

2. p-Adic q-Bernoulli Distribution with Weight α

Let X be any compact-open subset of $\mathbb{Q}_p$, such as $\mathbb{Z}_p$ or $\mathbb{Z}_p^*$. A p-adic distribution μ on X is defined to be an additive map from the collection of compact open set in X to $\mathbb{Q}_p$:

$$\mu\left(\bigcup_{k=1}^{n} U_k\right) = \sum_{k=1}^{n} \mu(U_k) (\text{additivity}), \tag{2.1}$$

where $\{U_1, U_2, \ldots, U_n\}$ is any collection of disjoint compact opensets in X.

The set $\mathbb{Z}_p$ has a topological basis of compact open sets of the form $a + p^n\mathbb{Z}_p$.

Consequently, if U is any compact open subset of $\mathbb{Z}_p$, it can be written as a finite disjoint union of sets

$$U = \bigcup_{j=1}^{k} (a_j + p^n\mathbb{Z}_p), \tag{2.2}$$

where $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_k \in \mathbb{Z}$ with $0 \le a_i < p^n$ for $i = 1, 2, \ldots k$.

Indeed, the p-adic ball $a+p^n\mathbb{Z}_p$ can be represented as the union of smaller balls

$$a+p^n\mathbb{Z}_p=\bigcup_{b=0}^{p-1}\left(a+bp^n+p^{n+1}\mathbb{Z}_p\right). \tag{2.3}$$

Lemma 2.1. *Every map μ from the collection of compact-open sets in X to $\mathbb{Q}_p$ for which*

$$\mu\left(a+p^N\mathbb{Z}_p\right)=\bigcup_{b=0}^{p-1}\left(a+bp^N+dp^{N+1}\mathbb{Z}_p\right) \tag{2.4}$$

holds whenever $a+p^N\mathbb{Z}_p\subset X$, extends to a p-adic distribution (= p-adic unbounded measure) on X.

Now we define a map $\mu_{k,q}^{(\alpha)}$ on the balls in $\mathbb{Z}_p$ as follows:

$$\mu_{k,q}^{(\alpha)}(a+p^n\mathbb{Z}_p)=\frac{[p^n]_{q^\alpha}^k}{[p^n]_q}q^a f_{k,q^{p^n}}^{(\alpha)}\left(\frac{\{a\}_n}{p^n}\right), \tag{2.5}$$

where $\{a\}_n$ is the unique number in the set $\{0,1,2,\ldots,p^n-1\}$ such that $\{a\}_n\equiv a(\bmod\ p^n)$.

If $a\in\{0,1,2,\ldots,p^n-1\}$, then

$$\begin{aligned}\sum_{b=0}^{p-1}\mu_{k,q}^{(\alpha)}\left(a+bp^n+p^{n+1}\mathbb{Z}_p\right)&=\sum_{b=0}^{p-1}\frac{[p^{n+1}]_{q^\alpha}^k}{[p^{n+1}]_q}q^{a+bp^n}f_{k,q^{p^{n+1}}}^{(\alpha)}\left(\frac{a+bp^n}{p^{n+1}}\right)\\&=q^a\frac{[p^n]_{q^\alpha}^k}{[p^n]_q}\frac{[p]_{(q^{p^n})^\alpha}^k}{[p]_{q^{p^n}}}\sum_{b=0}^{p-1}q^{bp^n}f_{k,(q^{p^n})^p}^{(\alpha)}\left(\frac{(a/p^n)+b}{p}\right).\end{aligned} \tag{2.6}$$

From (2.6), we note that $\mu_{k,q}^{(\alpha)}$ is p-adic distribution on $\mathbb{Z}_p$ if and only if

$$\frac{[p]_{(q^{p^n})^\alpha}^k}{[p]_{q^{p^n}}}\sum_{b=0}^{p-1}q^{bp^n}f_{k,(q^{p^n})^p}^{(\alpha)}\left(\frac{(a/p^n)+b}{p}\right)=f_{k,q^{p^n}}^{(\alpha)}\left(\frac{a}{p^n}\right). \tag{2.7}$$

Theorem 2.2. *Let $\alpha\in\mathbb{N}$ and $k\in\mathbb{Z}_+$. Then we see that $\mu_{k,q}^{(\alpha)}(a+p^n\mathbb{Z}_p)$ is p-adic distribution on $\mathbb{Z}_p$ if and only if*

$$\frac{[p]_{(q^{p^n})^\alpha}^k}{[p]_{q^{p^n}}}\sum_{b=0}^{p-1}q^{bp^n}f_{k,(q^{p^n})^p}^{(\alpha)}\left(\frac{(a/p^n)+b}{p}\right)=f_{k,q^{p^n}}^{(\alpha)}\left(\frac{a}{p^n}\right). \tag{2.8}$$

One sets

$$f_{k,q^{p^n}}^{(\alpha)}(x)=\widetilde{\beta}_{k,q^{p^n}}^{(\alpha)}(x). \tag{2.9}$$

From (2.5) *and* (2.9), *one gets*

$$\mu_{k,q}^{(\alpha)}(a+p^n\mathbb{Z}_p)=\frac{[p^n]_{q^\alpha}^k}{[p^n]_q}q^a\widetilde{\beta}_{k,q^{p^n}}^{(\alpha)}\left(\frac{a}{p^n}\right). \tag{2.10}$$

By (1.11), (2.10), and Theorem 2.2, we obtain the following theorem.

Theorem 2.3. *Let* $\mu_{k,q}^{(\alpha)}$ *be given by*

$$\mu_{k,q}^{(\alpha)}\left(a+dp^N\mathbb{Z}_p\right)=\frac{[dp^N]_{q^\alpha}^k}{[dp^N]_q}q^a\widetilde{\beta}_{k,q^{dp^N}}^{(\alpha)}\left(\frac{a}{dp^N}\right). \tag{2.11}$$

Then $\mu_{k,q}^{(\alpha)}$ *extends to a* $\mathbb{Q}(q)$*-valued distribution on the compact open sets* $U\subset X$.

From (2.11), *one notes that*

$$\begin{aligned}\int_X d\mu_{k,q}^{(\alpha)}(x)&=\lim_{N\to\infty}\sum_{x=0}^{dp^N-1}\mu_{k,q}^{(\alpha)}\left(x+dp^N\mathbb{Z}_p\right)\\&=\lim_{N\to\infty}\frac{[dp^N]_{q^\alpha}^k}{[dp^N]_q}\sum_{a=0}^{dp^N-1}q^a\widetilde{\beta}_{k,q^{dp^N}}^{(\alpha)}\left(\frac{a}{dp^N}\right).\end{aligned} \tag{2.12}$$

By (1.11) *and* (2.12), *one gets*

$$\int_X d\mu_{k,q}^{(\alpha)}(x)=\widetilde{\beta}_{k,q}^{(\alpha)}. \tag{2.13}$$

Therefore, we obtain the following theorem.

Theorem 2.4. *For* $\alpha\in\mathbb{N}$ *and* $k\in\mathbb{Z}_+$, *one has*

$$\int_X d\mu_{k,q}^{(\alpha)}(x)=\widetilde{\beta}_{k,q}^{(\alpha)}. \tag{2.14}$$

Let χ *be Dirichlet character with conductor* $d\in\mathbb{N}$. *Then one defines the generalized* q*-Bernoulli numbers attached to* χ *as follows:*

$$\begin{aligned}\widetilde{\beta}_{n,\chi,q}^{(\alpha)}&=\int_X\chi(x)[x]_{q^\alpha}^n d\mu_q(x)\\&=\frac{[d]_{q^\alpha}^n}{[d]_q}\sum_{a=0}^{d-1}q^a\chi(a)\widetilde{\beta}_{n,q^d}^{(\alpha)}\left(\frac{a}{d}\right).\end{aligned} \tag{2.15}$$

From (2.11) *and* (2.15), *one can derive the following equation;*

$$
\begin{aligned}
\int_X \chi(x)d\mu_{k,q}^{(\alpha)}(x) &= \lim_{N\to\infty}\sum_{x=0}^{dp^N-1}\chi(x)\mu_{k,q}^{(\alpha)}\left(x+dp^N\mathbb{Z}_p\right)\\
&= \lim_{N\to\infty}\frac{[dp^N]_{q^\alpha}^k}{[dp^N]_q}\sum_{x=0}^{dp^N-1}\chi(x)q^x\widetilde{\beta}_{k,q^{dp^N}}^{(\alpha)}\left(\frac{x}{dp^N}\right)\\
&= \frac{[d]_{q^\alpha}^k}{[d]_q}\sum_{a=0}^{d-1}q^a\chi(a)\left\{\lim_{N\to\infty}\frac{[p^N]_{q^{\alpha d}}^k}{[p^N]_{q^d}}\sum_{x=0}^{p^N-1}q^{dx}\widetilde{\beta}_{k,q^{dp^N}}\left(\frac{(a/d)+x}{p^N}\right)\right\}\\
&= \frac{[d]_{q^\alpha}^k}{[d]_q}\sum_{a=0}^{d-1}q^a\chi(a)\widetilde{\beta}_{k,q^d}^{(\alpha)}\left(\frac{a}{d}\right)=\widetilde{\beta}_{k,\chi,q}^{(\alpha)},\\
\int_{pX} \chi(x)d\mu_{k,q}^{(\alpha)}(x) &= \lim_{N\to\infty}\frac{[dp^{N+1}]_{q^\alpha}^k}{[dp^{N+1}]_q}\sum_{x=0}^{dp^N-1}\chi(px)q^{px}\widetilde{\beta}_{k,q^{dp^{N+1}}}^{(\alpha)}\left(\frac{px}{dp^{N+1}}\right)\\
&= \frac{[p]_{q^\alpha}^k}{[p]_q}\frac{[d]_{q^{p\alpha}}^k}{[d]_{q^p}}\sum_{a=0}^{d-1}\chi(pa)q^{pa}\lim_{N\to\infty}\frac{[p^N]_{q^{dp\alpha}}^k}{[p^N]_{q^{dp}}}\sum_{x=0}^{p^N-1}q^{pdx}\widetilde{\beta}_{k,q^{pdp^N}}^{(\alpha)}\left(\frac{p(xd+a)}{pdp^N}\right)\\
&= \frac{[p]_{q^\alpha}^k}{[p]_q}\frac{[d]_{q^{\alpha p}}^k}{[d]_{q^p}}\sum_{a=0}^{d-1}\chi(p)\chi(a)q^{pa}\widetilde{\beta}_{k,q^{pd}}^{(\alpha)}\left(\frac{a}{d}\right)=\chi(p)\frac{[p]_{q^\alpha}^k}{[p]_q}\widetilde{\beta}_{k,\chi,q^p}^{(\alpha)}.
\end{aligned}
\tag{2.16}
$$

For $\beta(\neq 1)\in X^*$, *one has*

$$
\begin{aligned}
\int_{pX}\chi(x)d\mu_{k,q^{1/\beta}}^{(\alpha)}(\beta x) &= \chi\left(\frac{p}{\beta}\right)\frac{[p]_{q^{\alpha/\beta}}^k}{[p]_{q^{1/\beta}}}\widetilde{\beta}_{k,\chi,q^{p/\beta}}^{(\alpha)},\\
\int_X\chi(x)d\mu_{k,q^{1/\beta}}^{(\alpha)}(\beta x) &= \chi\left(\frac{1}{\beta}\right)\widetilde{\beta}_{k,\chi,q^{1/\beta}}^{(\alpha)}.
\end{aligned}
\tag{2.17}
$$

Therefore, we obtain the following theorem.

Theorem 2.5. *For* $\beta(\neq 1)\in X^*$, *one has*

$$
\begin{aligned}
\int_X\chi(x)d\mu_{k,q}^{(\alpha)}(x) &= \widetilde{\beta}_{k,\chi,q}^{(\alpha)},\\
\int_{pX}\chi(x)d\mu_{k,q}^{(\alpha)}(x) &= \chi(p)\frac{[p]_{q^\alpha}^k}{[p]_q}\widetilde{\beta}_{k,\chi,q^p}^{(\alpha)},
\end{aligned}
$$

$$\int_{pX} \chi(x) d\mu^{(\alpha)}_{k,q^{1/\beta}}(\beta x) = \chi\left(\frac{p}{\beta}\right) \frac{[p]^k_{q^{\alpha/\beta}}}{[p]_{q^{1/\beta}}} \widetilde{\beta}^{(\alpha)}_{k,\chi,q^{p/\beta}},$$
$$\int_{X} \chi(x) d\mu^{(\alpha)}_{k,q^{1/\beta}}(\beta x) = \chi\left(\frac{1}{\beta}\right) \widetilde{\beta}^{(\alpha)}_{k,\chi,q^{1/\beta}}. \tag{2.18}$$

Define

$$\mu^{(\alpha)}_{k,\beta,q}(U) = \mu^{(\alpha)}_{k,q}(U) - \beta^{-1} \frac{[\beta^{-1}]^k_{q^\alpha}}{[\beta^{-1}]_q} \mu^{(\alpha)}_{k,q^{1/\beta}}(\beta U). \tag{2.19}$$

By a simple calculation, one gets

$$\int_{X^*} \chi(x) d\mu^{(\alpha)}_{k,\beta,q}(x) = \int_X \chi(x) d\mu^{(\alpha)}_{k,q}(x) - \beta^{-1} \frac{[\beta^{-1}]^k_{q^\alpha}}{[\beta^{-1}]_q} \int_{pX} \chi(x) \mu^{(\alpha)}_{k,q^{1/\beta}}(x)$$
$$= \widetilde{\beta}^{(\alpha)}_{k,\chi,q} - \chi(p) \frac{[p]^k_{q^\alpha}}{[p]_q} \widetilde{\beta}^{(\alpha)}_{k,\chi,q^p},$$
$$\frac{[\beta^{-1}]^k_{q^\alpha}}{[\beta^{-1}]^k_q} \int_{X^*} \chi(x) d\mu^{(\alpha)}_{k,q^{1/\beta}}(\beta x) = \frac{[1/\beta]^k_{q^\alpha}}{[1/\beta]_q} \chi\left(\frac{1}{\beta}\right) \widetilde{\beta}^{(\alpha)}_{k,\chi,q^{1/\beta}}$$
$$- \chi\left(\frac{p}{\beta}\right) \frac{[p/\beta]^k_{q^\alpha}}{[p/\beta]_q} \widetilde{\beta}^{(\alpha)}_{k,\chi,q^{p/\beta}}. \tag{2.20}$$

By (2.19) *and* (2.20), *one gets*

$$\int_{X^*} \chi(x) d\mu^{(\alpha)}_{k,\beta,q}(\beta x) = \int_X \chi(x) d\mu^{(\alpha)}_{k,q}(x) - \beta^{-1} \frac{[\beta^{-1}]^k_{q^\alpha}}{[\beta^{-1}]_q} \int_{pX} \chi(x) \mu^{(\alpha)}_{k,q^{1/\beta}}(\beta x)$$
$$= \widetilde{\beta}^{(\alpha)}_{k,\chi,q} - \chi(p) \frac{[p]^k_{q^\alpha}}{[p]_q} \widetilde{\beta}^{(\alpha)}_{k,\chi,q^p} - \frac{1}{\beta} \frac{[1/\beta]^k_{q^\alpha}}{[1/\beta]_q} \chi\left(\frac{1}{\beta}\right) \widetilde{\beta}^{(\alpha)}_{k,\chi,q^{1/\beta}} \tag{2.21}$$
$$+ \chi\left(\frac{p}{\beta}\right) \frac{[p/\beta]^k_{q^\alpha}}{[p/\beta]_q} \widetilde{\beta}^{(\alpha)}_{k,\chi,q^{p/\beta}}.$$

Now one defines the operator $\chi^y = \chi^{y,k,\alpha:q}$ *on* $f(q)$ *by*

$$\chi^y f(q) = \chi^{y,k,\alpha:q} f(q) = \frac{[y]^k_{q^\alpha}}{[y]_q} \chi(y) f(q^y). \tag{2.22}$$

Thus, by (2.22), *one gets*

$$\begin{aligned}\chi^{x,k,\alpha:q}\circ\chi^{y,k,\alpha:q}f(q)&=\chi^{x,k,\alpha:q}\frac{[y]_{q^\alpha}^k}{[y]_q}\chi(y)f(q^y)\\&=\frac{[y]_{q^\alpha}^k}{[y]_q}\chi(y)\chi(x)\frac{[y]_{q^{\alpha y}}^k}{[y]_{q^y}}\chi(y)f(q^{xy})\\&=\frac{[xy]_{q^\alpha}^k}{[xy]_q}\chi(xy)f(q^{xy})\\&=\chi^{xy,k,\alpha:q}f(q)\\&=\chi^{xy}f(q).\end{aligned}\tag{2.23}$$

Let us define $\chi^x\chi^y=\chi^{x,k,\alpha:q}\circ\chi^{y,k,\alpha:q}$. *Then one has*

$$\chi^x\chi^y=\chi^{xy}.\tag{2.24}$$

From the definition of χ^x, *one can easily derive the following equation;*

$$(1-\chi^p)\left(1-\frac{1}{\beta}x^{1/\beta}\right)=1-\frac{1}{\beta}x^{1/\beta}-\chi^p+\frac{1}{\beta}x^{p/\beta}.\tag{2.25}$$

Let $f(q)=\widetilde{\beta}_{k,\chi,q}^{(\alpha)}$. *Then one gets*

$$\begin{aligned}(1-\chi^p)\left(1-\frac{1}{\beta}x^{1/\beta}\right)\widetilde{\beta}_{k,\chi,q}^{(\alpha)}&=\widetilde{\beta}_{k,\chi,q}^{(\alpha)}-\frac{1}{\beta}\frac{[1/\beta]_{q^\alpha}^k}{[1/\beta]_q}\chi\left(\frac{1}{\beta}\right)\widetilde{\beta}_{k,\chi,q}^{(\alpha)}-\frac{[p]_{q^\alpha}^k}{[p]_q}\chi(p)\widetilde{\beta}_{k,\chi,q^p}^{(\alpha)}\\&\quad+\frac{1}{\beta}\frac{[p/\beta]_{q^\alpha}^k}{[p/\beta]_q}\chi\left(\frac{p}{\beta}\right)\widetilde{\beta}_{k,\chi,q^{p/\beta}}^{(\alpha)}.\end{aligned}\tag{2.26}$$

By (2.21) *and* (2.26), *one obtains the following equation:*

$$\int_{X^*}\chi(x)d\mu_{k,\beta,q}^{(\alpha)}(\beta x)=(1-\chi^p)\left(1-\frac{1}{\beta}x^{1/\beta}\right)\widetilde{\beta}_{k,\chi,q}^{(\alpha)},\tag{2.27}$$

where $\beta(\neq 1)\in X^*$.

References

[1] L. Carlitz, "Expansions of q-Bernoulli numbers," *Duke Mathematical Journal*, vol. 25, pp. 355–364, 1958.

[2] L. Carlitz, "q-Bernoulli numbers and polynomials," *Duke Mathematical Journal*, vol. 15, pp. 987–1000, 1948.
[3] T. Kim, "On the weighted q-Bernoulli numbers and polynomials," *Advanced Studies in Contemporary Mathematics*, vol. 21, pp. 207–215, 2011.
[4] T. Kim, "On a q-analogue of the p-adic log gamma functions and related integrals," *Journal of Number Theory*, vol. 76, no. 2, pp. 320–329, 1999.
[5] Y.-H. Kim, "On the p-adic interpolation functions of the generalized twisted (h,q)-Euler numbers," *International Journal of Mathematical Analysis*, vol. 3, no. 17–20, pp. 897–904, 2009.
[6] A. Bayad, "Modular properties of elliptic Bernoulli and Euler functions," *Advanced Studies in Contemporary Mathematics*, vol. 20, no. 3, pp. 389–401, 2010.
[7] A. Bayad, "Arithmetical properties of elliptic Bernoulli and Euler numbers," *International Journal of Algebra*, vol. 4, no. 5-8, pp. 353–372, 2010.
[8] H. Ozden, Y. Simsek, S.-H. Rim, and I. N. Cangul, "A note on p-adic q-Euler measure," *Advanced Studies in Contemporary Mathematics*, vol. 14, no. 2, pp. 233–239, 2007.
[9] H. Ozden, Y. Simsek, and I. N. Cangul, "Euler polynomials associated with p-adic q-Euler measure," *General Mathematics*, vol. 15, no. 2, pp. 24–37, 2007.
[10] Y. Simsek, "Special functions related to Dedekind-type DC-sums and their applications," *Russian Journal of Mathematical Physics*, vol. 17, no. 4, pp. 495–508, 2010.
[11] S.-H. Rim and T. Kim, "A note on p-adic Euler measure on $\mathbb{Z}_p$," *Russian Journal of Mathematical Physics*, vol. 13, no. 3, pp. 358–361, 2006.
[12] S.-H. Rim, J.-H. Jin, E.-J. Moon, and S.-J. Lee, "Some identities on the q-Genocchi polynomials of higher-order and q-Stirling numbers by the fermionic p-adic integral on $\mathbb{Z}_p$," *International Journal of Mathematics and Mathematical Sciences*, vol. 2010, Article ID 860280, 14 pages, 2010.
[13] N. Koblitz, "On Carlitz's q-Bernoulli numbers," *Journal of Number Theory*, vol. 14, no. 3, pp. 332–339, 1982.
[14] T. Kim, "On explicit formulas of p-adic q-L-functions," *Kyushu Journal of Mathematics*, vol. 48, no. 1, pp. 73–86, 1994.
[15] T. Kim, "q-Bernoulli numbers and polynomials associated with Gaussian binomial coefficients," *Russian Journal of Mathematical Physics*, vol. 15, no. 1, pp. 51–57, 2008.
[16] T. Kim, "q-Volkenborn integration," *Russian Journal of Mathematical Physics*, vol. 9, no. 3, pp. 288–299, 2002.