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Research Article

Periodicity in a Class of Systems of Delay Difference Equations

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We study a system of delay difference equations modeling four-dimensional discrete-time delayed neural networks with no internal decay. Such a discrete-time system can be regarded as the discrete analog of a differential equation with piecewise constant argument. By using semicycle analysis method, it is shown that every bounded solution of this discrete-time system is eventually periodic. The obtained results are new, and they complement previously known results.

1. Introduction

Over the past decades, there has been increasing interest in the potential applications of the dynamics of artificial neural networks in signal and image processing. Among the most popular models in the literature of artificial neural networks is the following well-known Hopfield's model [1, 2]:

$$\frac{du_i(t)}{dt} = \mu_i u_i(t) + \sum_{j \neq i} T_{ij} f_j\left(u_j\left(t - \tau_{ij}\right)\right), \qquad 1 \leq i \leq n, \tag{1.1}$$

where $\mu_i \geq 0$ is the internal decay rate, $\tau_{ij} \geq 0$ is the delay incorporated by Marcus and Westervelt [3] to account for the finite switching speed of amplifiers (neurons), $f_j : \mathbf{R} \to \mathbf{R}$ (the set of all real numbers) is the signal functions, and T_{ij} represents the connection strengths between neurons and if the output from neuron j excites (resp., inhibits) neuron i, then $T_{ij} > 0$ (resp., < 0), $1 \leq i, j \leq n$.

Much work on Hopfield-type neural networks with constant time delays has been carried out. However, from the point of view of electronic implementation of neural networks and control, the time-varying delay case is more suitable for practical neural networks, and we can achieve this by using piecewise constant arguments. Because of the wide application of differential equations with piecewise constant argument in certain biomedical models (see, e.g., [4]), much progress has been made in the study of differential equations with piecewise constant arguments since the pioneering work of Cooke and Wiener [5] and Shah and Wiener [6]. For more details and references on this subject, interested readers may refer to a survey of Cooke and Wiener [7]. Motivated by these discussions, we propose

$$\frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = \mu_i u_i(t) + \sum_{j \neq i} T_{ij} f_j\left(u_j\left(\left[t-\tau_{ij}\right]\right)\right), \quad 1 \leq i \leq n, \tag{1.2}$$

as a new neural network model, where $[\cdot]$ denotes the greatest integer function and f_j $(1 \leq j \leq n)$ are the McCulloch-Pitts nonlinear function given by

$$f(u) = \begin{cases} 1 & \text{if } \mathrm{u} > \sigma, \\ -1 & \text{if } \mathrm{u} \leq \sigma. \end{cases} \tag{1.3}$$

Here, $\sigma \in R$ is referred as the threshold. The McCulloch-Pitts nonlinearity reflects the fact that the signal transmission is of digital nature: a neuron is either fully active or completely inactive.

Neural networks are complex and large-scale nonlinear dynamical systems, while the dynamics of the delayed neural networks is even richer and more complicated [8]. In order to obtain a deep and clear understanding of the dynamics of neural networks, the dynamics of delayed neural networks consisting of a few neurons has received increasing attention over the past years [9–14]. The studies on these neural networks of a few neurons are potentially useful because the complexities found in such models often provide promising information for the studies on more complicated neural networks with a large number of neurons [15]. So, we are inspired to consider a special case of (1.2),

$$\begin{aligned} x_1'(t) &= f(x_2([t-k])), \\ x_2'(t) &= f(x_3([t-k])), \\ x_3'(t) &= -f(x_4([t-k])), \\ x_4'(t) &= -f(x_1([t-k])), \end{aligned} \tag{1.4}$$

where f is the activation function with McCulloch-Pitts nonlinearity (1.3), k is a nonnegative integer and the negative (resp., positive) sign on the right indicates negative (resp., positive) feedback. (1.4) describes the dynamics of four interacting neurons with no internal decays.

To study (1.4), it is convenient to consider the following difference system:

$$\begin{aligned}
x_1(n+1) &= x_1(n) + f(x_2(n-k)),\\
x_2(n+1) &= x_2(n) + f(x_3(n-k)),\\
x_3(n+1) &= x_3(n) - f(x_4(n-k)),\\
x_4(n+1) &= x_4(n) - f(x_1(n-k)),
\end{aligned} \tag{1.5}$$

where k is a nonnegative integer and f satisfies (1.3). In fact, integrate (1.4) from n to $t \in [n, n+1)$ to get

$$\begin{aligned}
x_1(t) &= x_1(n) + \int_n^t f(x_2([s-k]))ds = x_1(n) + f(x_2(n-k))(t-n),\\
x_2(t) &= x_2(n) + \int_n^t f(x_3([s-k]))ds = x_2(n) + f(x_3(n-k))(t-n),\\
x_3(t) &= x_3(n) - \int_n^t f(x_4([s-k]))ds = x_3(n) - f(x_4(n-k))(t-n),\\
x_4(t) &= x_4(n) - \int_n^t f(x_1([s-k]))ds = x_4(n) - f(x_1(n-k))(t-n).
\end{aligned} \tag{1.6}$$

Letting $t \to n+1$, we get a special case of (1.5) with $k = 0$. Hence, system (1.5) can be regarded as a more general neural network model of four neurons with transmission delay.

Our study is inspired by the work of Chen [16–18], who considered

$$\begin{aligned}
x(n+1) &= x(n) - f(x(n-k)),\\
x(n+1) &= x(n) + f(x(n-k)),\\
x(n+1) &= x(n) + f(y(n-k)),\\
y(n+1) &= y(n) - f(x(n-k)),
\end{aligned} \tag{1.7}$$

where $f : \mathbf{R} \to \mathbf{R}$ is given by (1.3).

The main objective of this paper is to study (1.5) with f satisfying (1.3). The discontinuity of f makes it difficult to apply directly the theory of discrete dynamical systems to system (1.5). However, by using semicycle analysis method, we obtain some interesting results for the periodicity of (1.5). It is shown that every bounded solution of system (1.5) is eventually periodic.

The rest of the paper is organized as follows. In Section 2, we introduce some necessary notations and lemmas that will be used later. In Section 3, we state and prove our main result.

2. Notations and Preliminaries

For the sake of simplicity, we introduce some notations. Let Z denote the set of all integers. For $a, b \in Z$, define $Z(a) = \{a, a+1, \dots\}$ and $Z(a,b) = \{a, a+1, \dots, b\}$ whenever $a \le b$.

By a solution of (1.5), we mean a sequence $\{(x_1(n), x_2(n), x_3(n), x_4(n))\}_{n=-k}^{+\infty}$ that satisfies (1.5). Clearly, for given $(\overline{x}_1(-k), \overline{x}_2(-k), \overline{x}_3(-k), \overline{x}_4(-k)), (\overline{x}_1(-k+1), \overline{x}_2(-k+1), \overline{x}_3(-k+1), \overline{x}_4(-k+1)), \dots, (\overline{x}_1(0), \overline{x}_2(0), \overline{x}_3(0), \overline{x}_4(0)) \in \mathbf{R}^4$, system (1.5) has a unique solution $\{(x_1(n), x_2(n), x_3(n), x_4(n))\}_{n=-k}^{+\infty}$ satisfying the initial condition:

$$(x_1(i), x_2(i), x_3(i), x_4(i)) = (\overline{x}_1(i), \overline{x}_2(i), \overline{x}_3(i), \overline{x}_4(i)) \quad \text{for } i \in Z(-k, 0). \tag{2.1}$$

If all terms of the sequence $\{v_n\}_{n=-k}^{+\infty}$ are integers, we say that it is an integer sequence. If $\{x_i(n)\}_{n=-k}^{\infty}$ for all $i \in Z(1,4)$ are integer sequences and also satisfy (1.5) and (1.3), then $\{(x_1(n), x_2(n), x_3(n), x_4(n))\}_{n=-k}^{+\infty}$ is said to be an integer sequence solution of (1.5) and (1.3).

Let $\{(x_1(n), x_2(n), x_3(n), x_4(n))\}_{n=-k}^{+\infty}$ be a solution of (1.5). Define

$$\tau(x) = \begin{cases} x - [x] & \text{for } x \neq 0, \\ 1 & \text{for } x = 0. \end{cases} \tag{2.2}$$

Clearly, $\tau(x) \ge 0$. From (1.5) and (1.3), we have $\tau(x_i(n) - \sigma) = \tau(x_i(0) - \sigma)$, $i \in Z(1,4)$. Denote $\widetilde{x}_i(n) = x_i(n) - \sigma - \tau(x_i(0) - \sigma) + 1$, $i \in Z(1,4)$. Then $\widetilde{x}_i(n) \in Z$ for all $n \in Z(-k)$, $i \in Z(1,4)$. Moreover, if $x_i(n) > \sigma$, then $\widetilde{x}_i(n) = x_i(n) - \sigma - \tau(x_i(0) - \sigma) + 1 \ge x_i(n) - \sigma > 0$, and if $x_i(n) \le \sigma$, then $\widetilde{x}_i(n) = x_i(n) - \sigma - \tau(x_i(0) - \sigma) + 1 \le -1 + 1 = 0$, $i \in Z(1,4)$. Hence, $\{(\widetilde{x_1}(n), \dots, \widetilde{x_4}(n))\}_{n=-k}^{+\infty}$ is an integer sequence solution of (1.5) and (1.3) with $\sigma = 0$.

So, without loss of generality, in the paper we always assume that $\sigma = 0$, and a solution of (1.5) means an integer sequence solution.

A solution $\{(x_1(n), x_2(n), x_3(n), x_4(n))\}_{n=-k}^{+\infty}$ is said to be eventually periodic if there exist $n_0 \in \{-k, -k+1, -k+2, \dots\}$ and $\omega \in \{1, 2, 3, \dots\}$ such that

$$(x_1(n + n_0 + \omega), x_2(n + n_0 + \omega), \dots, x_4(n + n_0 + \omega)) = (x_1(n + n_0), x_2(n + n_0), \dots, x_4(n + n_0)), \tag{2.3}$$

for $n = 0, 1, 2, \dots$, and ω is called a period. The smallest such ω is called the minimal period of $\{(x_1(n), x_2(n), x_3(n), x_4(n))\}_{n=-k}^{+\infty}$.

Definition 2.1. For a sequence $\{u_n\}_{n=-k}^{+\infty}$, we say that $\{u_{n_1}, u_{n_1+1}, \dots, u_{n_1+l-1}\}$ with $l \ge 1$ is a positive semicycle with the length l if $u_n > 0$ for $n \in Z(n_1, n_1 + l - 1)$, and n_1 and l satisfy

$$\begin{aligned} &\text{either} \quad n_1 = -k \quad \text{or} \quad u_{n_1 - 1} \le 0, \\ &\text{either} \quad l = +\infty \quad \text{or} \quad u_{n_1 + l} \le 0. \end{aligned} \tag{2.4}$$

We call l the length of the positive semicycle. A negative semicycle is defined similarly (replace $>$ by $\le$ and vice versa).

To simplify the following arguments, for an integer sequence, we denote A_r^+ the positive semicycle $\{v_{n_1}, \dots, v_{n_1+2r}\}$ with $v_{n_1+r\pm m} = r - m + 1$ for $m \in Z(0, r)$ and A_r^- the negative semicycle $\{v_{n_1}, \dots, v_{n_1+2r}\}$ with $v_{n_1+r\pm m} = m - r$ for $m \in Z(0, r)$.

Definition 2.2. An integer sequence $\{v_n\}_{n=n_1}^{n_1+L-1}$ (may be infinite terms) is said to be an r-normal cycle if it possesses one of the representations as follows: $\{A_r^+, A_r^-, \ldots, A_r^+, A_r^-\}$, $\{A_r^+, A_r^-, \ldots, A_r^-, A_r^+\}$, $\{A_r^-, A_r^+, \ldots, A_r^-, A_r^+\}$, and $\{A_r^-, A_r^+, \ldots, A_r^+, A_r^-\}$. In addition, if $L = +\infty$, then it is said to be a total r-normal cycle.

Remark 2.3. For a r-normal cycle $\{v_n\}_{n=n_1}^{n_1+L-1}$, from Definition 2.2, it is easy to see that there must exist $i \in Z(0)$ such that $L = (2r+1)i$. If $L = +\infty$, that is, $\{v_n\}_{n=n_1}^{n_1+L-1}$ is a total r-normal cycle, then $i = +\infty$.

Throughout this section, we tacitly assume that $(S) = \{(x_1(n), x_2(n), x_3(n), x_4(n))\}_{n=-k}^{+\infty}$ is a periodic solution of system (1.5) satisfying the initial values:

$$(x_1(i), x_2(i), x_3(i), x_4(i)) \in Z^4 \quad \text{for } i \in Z(-k, 0). \tag{2.5}$$

In view of (1.5) and (1.3), it is easy to see that all terms $x_1(n)$, $x_2(n)$, $x_3(n)$, and $x_4(n)$ of (S) are integers, that is, (S) is an integer sequence solution of (1.5).

For a periodic solution (S) of (1.5), denote

$$P_{x_i}^+ = \sup\{x_i(n) : n \in Z(-k)\} - 1, \; P_{x_i}^- = -\inf\{x_i(n) : n \in Z(-k)\}, \tag{2.6}$$

and let $L_{x_i}^+$ and $L_{x_i}^-$ be the length of the longest positive semicycle of $\{x_i(n)\}_{n=-k}^{+\infty}$ and the length of the longest negative semicycle of $\{x_i(n)\}_{n=-k}^{+\infty}$, respectively, where $i = 1, 2, \ldots, 4$. Note that $L_{x_i}^+, L_{x_i}^- \in Z(1)$ for $i = 1, 2, \ldots, 4$.

The following several lemmas will be useful to the proof of the main result in this paper.

Lemma 2.4.

(a) *Either* $L_{x_i}^+ > L_{x_j}^+$ *or* $L_{x_i}^- > L_{x_j}^+$ *or* $\min\{L_{x_i}^+, L_{x_i}^-\} \geq L_{x_j}^+$ *for* $i, j = 1, 2, \ldots, 4$ *and* $i \neq j$.

(b) *Either* $L_{x_i}^+ > L_{x_j}^-$ *or* $L_{x_i}^- > L_{x_j}^-$ *or* $\min\{L_{x_i}^+, L_{x_i}^-\} \geq L_{x_j}^-$ *for* $i, j = 1, 2, \ldots, 4$ *and* $i \neq j$.

Proof. We only prove the conclusion: either $L_{x_1}^+ > L_{x_2}^+$ or $L_{x_1}^- > L_{x_2}^+$ or $\min\{L_{x_1}^+, L_{x_1}^-\} \geq L_{x_2}^+$. The other conclusions can be proved similarly. Now, we distinguish two cases to finish the proof.

Case 1. $L_{x_2}^+ = 2r$ for some $r > 0$. Without loss of generality, we assume that $x_2(0) \leq 0$, $x_2(1) > 0$, $x_2(2) > 0, \ldots, x_2(2r) > 0$, $x_2(2r+1) \leq 0, \ldots$. Then, from (1.5) and (1.3), we have

$$x_1(2r+k+1) = x_1(r+k+1) + \sum_{i=r+1}^{2r} f(x_2(i)) = x_1(r+k+1) + r, \tag{2.7}$$

$$x_1(r+k+1) = x_1(k+1) + \sum_{i=1}^{r} f(x_2(i)) = x_1(k+1) \; + r. \tag{2.8}$$

(i) If $x_1(r+k+1) \geq 1$, it follows from (2.7) that $x_1(2r+k+1) \geq r+1$. Since the distance between consecutive elements of $\{x_1(n)\}_{n=0}^{\infty}$ is 1, we have $L_{x_1}^+ \geq 2r+1 > L_{x_2}^+$.

(ii) If $x_1(r+k+1) \leq 0$, it follows from (2.8) that $x_1(k+1) \leq -r$. Thus, $L_{x_1}^- \geq 2r+1 > L_{x_2}^+$.

Case 2. $L^+_{x_2} = 2r+1$ for some $r > 0$. Without loss of generality, we assume that $x_2(0) \le 0, x_2(1) > 0,\ x_2(2) > 0, \ldots,\ x_2(2r+1) > 0$, and $x_2(2r+2) \le 0, \ldots$. Then, from (1.5) and (1.3), we have

$$x_1(k+2r+2) = x_1(r+k+1) + \sum_{i=1+r}^{2r+1} f(x_2(i)) = x_1(r+k+1) + (r+1), \tag{2.9}$$

$$x_1(r+k+1) = x_1(k+1) + \sum_{i=1}^{r} f(x_2(i)) = x_1(k+1) + r. \tag{2.10}$$

(i) If $x_1(r+k+1) > 0$, it follows from (2.9) that $x_1(k+2r+2) > r+1$. Thus, $L^+_{x_1} > 2r+1 = L^+_{x_2}$.

(ii) If $x_1(r+k+1) < 0$, it follows from (2.10) that $x_1(k+1) < -r$. Thus, $L^-_{x_1} > 2r+1 = L^+_{x_2}$.

(iii) If $x_1(r+k+1) = 0$, it follows from (2.9) and (2.10) that $x_1(k+2r+2) = r+1$ and $x_1(k+1) = -r$. Thus, $\min\{L^+_{x_1}, L^-_{x_1}\} \ge 2r+1 = L^+_{x_2}$.

Therefore, we have either $L^+_{x_1} > L^+_{x_2}$ or $L^-_{x_1} > L^+_{x_2}$ or $\min\{L^+_{x_1}, L^-_{x_1}\} \ge L^+_{x_2}$. This completes the proof. □

Lemma 2.5. *$L^+_{x_i} = L^-_{x_j}$ for $i, j = 1, \ldots, 4$. Moreover, $L^+_{x_1}$ is odd.*

Proof. The lemma follows immediately from Lemma 2.4 together with its proof. □

Lemma 2.6. *$P^+_{x_i} = P^-_{x_j}$ for $i, j = 1, \ldots, 4$.*

Proof. First, we claim that $P^+_{x_2} \le P^+_{x_1}$. Otherwise, $P^+_{x_2} > P^+_{x_1}$. Since the distance between consecutive elements of $\{x_2(n)\}_{n=0}^{\infty}$ is 1, it follows easily from the definitions of $P^+_{x_2}$ and $L^+_{x_2}$ that $L^+_{x_2} \ge 2P^+_{x_2} + 1$. By Lemma 2.5, there exists $r \ge 0$ such that $L^+_{x_2} = 2r+1 \ge 2P^+_{x_2} + 1$, which implies that $r \ge P^+_{x_2}$. Without loss of generality, we assume that $x_2(0) \le 0,\ x_2(1) > 0,\ x_2(2) > 0, \ldots,\ x_2(2r+1) > 0$, and $x_2(2r+2) \le 0, \ldots$. Note that $r \ge P^+_{x_2}$ and $P^+_{x_2} > P^+_{x_1}$. It follows from (2.9) and the definition of $P^+_{x_1}$ that

$$\begin{aligned} x_1(k+1+r) &= x_1(k+2r+2) - (r+1) \\ &\le x_1(k+2r+2) - P^+_{x_2} - 1 \\ &< x_1(k+2r+2) - P^+_{x_1} - 1 \\ &\le 0. \end{aligned} \tag{2.11}$$

Thus, $x_1(k+1+r) \le -1$, which, together with (2.10), implies that

$$x_1(k+1) = x_1(k+1+r) - r \le -(r+1). \tag{2.12}$$

It follows that $P^-_{x_1} \ge r+1$ and hence, $L^-_{x_1} \ge 2r+3 > L^+_{x_2}$, a contradiction to Lemma 2.5. This proves the claim. Repeating the same argument as that in the proof of the above claim, we have $P^+_{x_i} \le P^+_{x_j}$, $P^-_{x_i} \le P^-_{x_j}$, and $P^+_{x_i} \le P^-_{x_j}$ for $i, j = 1, \ldots, 4$ and $i \ne j$. Thus, we get $P^+_{x_i} = P^-_{x_j}$ for $i, j = 1, \ldots, 4$. This completes the proof. □

Lemma 2.7. $L^+_{x_1} = 2P^+_{x_1} + 1$.

Proof. It follows from the definitions of $P^+_{x_1}$ and $L^+_{x_1}$ that $L^+_{x_1} \geq 2P^+_{x_1} + 1$. We show $L^+_{x_1} = 2P^+_{x_1} + 1$ by way of contradiction. Assume that $L^+_{x_1} > 2P^+_{x_1} + 1$. By Lemma 2.5, there exists $r \geq 0$ such that $L^+_{x_1} = 2r + 1 \geq 2P^+_{x_1} + 1$, which implies that $r \geq P^+_{x_1}$. Without loss of generality, we assume that $x_1(0) \leq 0$, $x_1(1) > 0$, $x_1(2) > 0, \ldots,$ $x_1(2r+1) > 0$, and $x_1(2r+2) \leq 0, \ldots$. Now, we distinguish two cases to finish the proof.

Case 1. $x_4(r + k + 1) \geq 1$. For this case, from (1.5) and (1.3), we have

$$\begin{aligned} x_4(k+1) &= x_4(k+r+1) + \sum_{i=1}^{r} f(x_1(i)) \\ &\geq 1 + r \\ &> 1 + P^+_{x_1}, \end{aligned} \tag{2.13}$$

which implies that $P^+_{x_4} > P^+_{x_1}$, a contradiction to Lemma 2.6.

Case 2. $x_4(r + k + 1) \leq 0$. For this case, from (1.5) and (1.3), we have

$$\begin{aligned} x_4(k+2r+2) &= x_4(r+k+1) - \sum_{i=1+r}^{2r+1} f(x_1(i)) \\ &\leq -(1+r), \end{aligned} \tag{2.14}$$

which implies that $P^-_{x_4} \geq 1 + r > P^+_{x_1}$, a contradiction to Lemma 2.6. This completes the proof. □

Throughout the remaining part of this paper, we denote $P^+_{x_1} = p$.

Lemma 2.8.

(a) $x_1(n_1) = p+1$ *for some* $n_1 \in Z(-k)$. *Moreover,* $x_4(n_1+k) = 1$ *and* $x_4(n_1+k+p+1) = -p$.

(b) $x_4(n_2) = p+1$ *for some* $n_2 \in Z(-k)$. *Moreover,* $x_3(n_2+k) = 1$ *and* $x_3(n_2+k+p+1) = -p$.

(c) $x_2(n_3) = p+1$ *for some* $n_3 \in Z(-k)$. *Moreover,* $x_1(n_3+k) = 0$ *and* $x_1(n_3+k+p+1) = p+1$.

(d) $x_3(n_4) = p+1$ *for some* $n_4 \in Z(-k)$. *Moreover,* $x_2(n_4+k) = 0$ *and* $x_2(n_4+k+p+1) = p+1$.

Proof. We only give the proof of the conclusion (c). The other conclusions can be proved similarly. By the definition of $P^+_{x_2}$, there must exist $n_3 \in Z(-k)$ such that $x_2(n_3) = P^+_{x_2}+1 = p+1$. Since the distance between consecutive elements of $\{x_2(n)\}_{n=0}^{\infty}$ is 1, it follows that $x_2(n_3 \pm j) > 0$ for all $j \in Z(1,p)$. We claim that $x_1(n_3+k) = 0$. Otherwise, $x_1(n_3+k) \leq -1$ or $x_1(n_3+\mathrm{k}) \geq 1$. If $x_1(n_3+k) \geq 1$, it follows from (1.5) and (1.3) that

$$x_1(n_3+k+p+1) = x_1(n_3+k) + \sum_{j=n_3}^{n_3+p} f(x_2(j)) = x_1(n_3+k) + (p+1). \tag{2.15}$$

Thus, $x_1(n_3+k+p+1) \geq p+2$, which implies that $P^+_{x_1} \geq p+2 > P^+_{x_2}$, a contradiction to Lemma 2.6. If $x_1(n_3+k) \leq -1$, it follows from (1.5) and (1.3) that

$$x_1(n_3+k) = x_1(n_3+k-p) + \sum_{j=n_3-p}^{n_3-1} f\left(x_2(j)\right) = x_1(n_3+k-p) + p. \tag{2.16}$$

Thus, $x_1(n_3+k-p) = x_1(n_3+k)-p \leq -1-p$, which implies that $P^-_{x_1} \geq p+1 > P^+_{x_2}$, a contradiction to Lemma 2.6. This proves the claim. By (1.5) and (1.3), we must have

$$x_1(n_3+k+p+1) = x_1(n_3+k) + \sum_{j=n_3}^{n_3+p} f\left(x_2(j)\right) = p+1. \tag{2.17}$$

This completes the proof. □

Arguing as in the proof of Lemma 2.8, we can get the following result.

Lemma 2.9.

(a) $x_1(n_1) = -p$ *for some* $n_1 \in Z(-k)$*. Moreover,* $x_4(n_1+k) = 0$ *and* $x_4(n_1+k+p+1) = p+1$.

(b) $x_4(n_2) = -p$ *for some* $n_2 \in Z(-k)$*. Moreover,* $x_3(n_2+k) = 0$ *and* $x_3(n_2+k+p+1) = p+1$.

(c) $x_2(n_3) = -p$ *for some* $n_3 \in Z(-k)$*. Moreover,* $x_1(n_3+k) = 1$ *and* $x_1(n_3+k+p+1) = -p$.

(d) $x_3(n_4) = -p$ *for some* $n_4 \in Z(-k)$*. Moreover,* $x_2(n_4+k) = 1$ *and* $x_2(n_4+k+p+1) = -p$.

Lemma 2.10. $p \leq 2k$.

Proof. Without loss of generality, we assume that $x_1(0) = 0$, $x_1(1) = 1, \ldots,$ $x_1(p) = p$, $x_1(p+1) = p+1$, $x_1(p+2) = p, \ldots,$ $x_1(2p) = 2$, $x_1(2p+1) = 1$, and $x_1(2p+2) = 0$. It follows from Lemma 2.8 that $x_4(p+k+1) = 1$. By (1.5) and (1.3), we have

$$x_4\left(p+k+1\right) = x_4(k+1) - \sum_{i=1}^{p} f\left(x_1(i)\right). \tag{2.18}$$

Thus, $x_4(k+1) = p+1$. It follows from Lemma 2.8 that $x_3(p+2k+2) = -p$, and hence, by Lemma 2.9, we have $x_2(p+3k+2) = 1$. From (1.5) and (1.3), we get

$$x_2\left(p+3k+2\right) = x_2(3k+2) + \sum_{i=2k+2}^{p+2k+1} f\left(x_3(i)\right) = x_2(3k+2) - p. \tag{2.19}$$

Thus, $x_2(3k+2) = p+1$. Therefore, by Lemma 2.8, we have $x_1(4k+2) = 0$. Since $x_1(2p+2) = 0$, it follows that $4k+2 \geq 2p+2$, and hence $2k \geq p$. This completes the proof. □

We hereafter use M_{x_i} to denote the length of the longest normal cycle of $\{x_i(n)\}_{n=-k}^{+\infty}$, where $i = 1, \ldots, 4$.

Lemma 2.11. $M_{x_i} = +\infty$, $i = 1, \ldots, 4$.

Proof. We show that $M_{x_i} = +\infty$ for $i \in Z(1,4)$ by way of contradiction. Without loss of generality, we assume that $M_{x_1} < +\infty$. We will distinguish several cases to finish the proof.

Case 1. $M_{x_i} = +\infty$, $i = 2,3,4$. Without loss of generality, we assume that $\{x_2(i)\}_{n=1}^{+\infty}$ is a total p-normal cycle and $x_2(1) = 1$. Then, $x_2(2i(2p+1)+p+1) = p+1$ for $i \in Z(0)$, and $x_2(2i(2p+1)+3p+2) = -p$ for $i \in Z(0)$. By Lemmas 2.8 and 2.9, we have

$$x_1\left(2i(2p+1)+(p+1)+(k+p+1)\right) = p+1, \qquad x_1\left(2i(2p+1)+(3p+2)+(k+p+1)\right) = -p, \tag{2.20}$$

for $i \in Z(0)$. Therefore, $\{x_1(n)\}_{n=p+k+2}^{+\infty}$ is a total p-normal cycle, which is a contradiction to $M_{x_1} < +\infty$.

Case 2. $M_{x_i} < +\infty$, $i = 2,3,4$. Without loss of generality, we assume that $M_{x_2} \geq \max\{M_{x_1}, M_{x_3}, M_{x_4}\}$. Then, by the definition of M_{x_2}, there exists $L \geq 1$ such that $M_{x_2} = (2p+1)L$ is the length of the longest normal cycle of $\{x_2(n)\}_{n=-k}^{+\infty}$. Without loss of generality, we assume that $\{x_2(n)\}_{n=1}^{M_{x_2}}$ is a p-normal cycle and $x_{M_{x_2}-p} = p+1, x_{M_{x_2}-p+1} = p, \ldots, x_{M_{x_2}-1} = 2, x_{M_{x_2}} = 1$. To finish the proof of Case 2, we will distinguish two subcases.

Subcase 1. $x_2(1) = 1$. Then there exists $q \geq 0$ such that $L = 2q+1$. Thus,

$$\begin{aligned} x_2\left(2i(2p+1)+p+1\right) &= p+1 \quad \text{for } i \in Z(0,q), \\ x_2\left(2i(2p+1)+3p+2\right) &= -p \quad \text{for } i \in Z(0,q-1). \end{aligned} \tag{2.21}$$

By Lemmas 2.8 and 2.9, we have $x_1(2i(2p+1)+(p+1)+(k+p+1)) = p+1$ for $i \in Z(0,q)$ and $x_1(2i(2p+1)+(3p+2)+(k+p+1)) = -p$ for $i \in Z(0,q-1)$. It follows from $x_2(p+1) = p+1$ that $x_1(p+k+1) = 0$. This, combined with

$$x_1(p+k+1) = x_1(k+1) + \sum_{i=1}^{p} f(x_2(i)) = x_1(k+1) + p, \tag{2.22}$$

gives $x_1(k+1) = -p$. By Lemma 2.10, we have $k-p+1 \geq k-2k+1 > -k$. Therefore, the total number of $p+1$ and $-p$ contained in $\{x_1(i)\}_{i=k+1-p}^{(2q+1)(2p+1)+k+1+p}$ is $2q+2$. Since

$$(2q+1)(2p+1)+k+1+p-(k-p+1)+1 = (2q+1)(2p+1)+2p+1, \tag{2.23}$$

it follows that $\{x_1(i)\}_{i=k+1-p}^{(2q+1)(2p+1)+k+1+p}$ is a normal cycle with the length of $M_{x_2}+2p+1$, and hence $M_{x_1} > M_{x_2}$, which leads to a contradiction.

Subcase 2. $x_2(1) = 0$. Then, using a similar argument to that in the proof of Subcase 1, we can show that this is also a contradiction.

If $M_{x_2} < +\infty$, $M_{x_3} = +\infty$, $M_{x_4} = +\infty$ or $M_{x_2} < +\infty$, $M_{x_3} < +\infty$, $M_{x_4} = +\infty$, then we have the same contradiction by similar arguments. This completes the proof. □

3. Main Result and Proof

Theorem 3.1. *For $k \in Z(0)$, every bounded solution of* (1.5) *is eventually periodic with the minimal period* $4r + 2$ *for some* $r \in \Omega(k)$, *where* $\Omega(k) = \{r \in Z(0) : (2k + 1)/(2r + 1) \in Z(0)\}$.

Proof. Let $\{(x_1(n), x_2(n), x_3(n), x_4(n))\}_{n=-k}^{+\infty}$ be a bounded solution of (1.5). Then there exists a positive integer $M_0 > 0$ such that $|x_1(n)| + \cdots + |x_4(n)| < M_0$ for all $n \in Z(-k)$. Denote $x_j^n(i) = x_j(n - i)$ for $i \in Z(0, k)$ and $j \in Z(1, \ldots, 4)$, and

$$X_n = \left(x_1^n(0), x_1^n(1), \ldots, x_1^n(k), x_2^n(0), x_2^n(1), \ldots, x_2^n(k), \ldots, x_4^n(0), \ldots, x_4^n(k)\right). \tag{3.1}$$

Define $P_i : Z^{4k+4} \to Z$ by setting $P_i(v_1, v_2, \ldots, v_{4k+4}) = v_i$ for $v_i \in Z$ and $i \in Z(1, 4k + 4)$, and define $F = (F_1, F_2, \ldots, F_{4k+4})$ as

$$F_i(X) = \begin{cases} P_1X + f(P_{2k+2}X) & \text{for } i = 1, \\ P_{i-1}X & \text{for } 2 \leq i \leq \mathrm{k} + 1, \\ P_{k+2}X + f(P_{3k+3}X) & \text{for } i = \mathrm{k} + 2, \\ P_{i-1}X & \text{for } \mathrm{k} + 3 \leq i \leq 2\mathrm{k} + 2, \\ P_{2k+3}X - f(P_{4k+4}X) & \text{for } i = 2\mathrm{k} + 3, \\ P_{i-1}X & \text{for } 2\mathrm{k} + 4 \leq i \leq 3\mathrm{k} + 3, \\ P_{3k+4}X - f(P_{k+1}X) & \text{for } i = 3\mathrm{k} + 4, \\ P_{i-1}X & \text{for } 3\mathrm{k} + 5 \leq i \leq 4\mathrm{k} + 4. \end{cases} \tag{3.2}$$

Then, $X_{n+1} = F(X_n)$. Since $|x_1(n)| + \cdots + |x_4(n)| < M_0$ for all $n \in Z(-k)$ and $x_j(n) \in Z$ for all $n \in Z(-k)$ and $j \in Z(1, \ldots, 4)$, it follows that $\{X_n\}_{n=-k}^{+\infty}$ is a finite set. Therefore, there exist $n_1 > n_2 \geq 1$ such that $X_{n_2} = X_{n_1} = X_{n_2+n_1-n_2}$. This, combined with $X_{n+1} = F(X_n)$, gives $X_n = X_{n+n_1-n_2}$ for $n \geq n_2 + 1$. Hence, $\{X_n\}_{n=-k}^{+\infty}$ is eventually periodic with the period $(n_1 - n_2)$. It follows that $\{P_jX_n\}_{n=-k}^{+\infty}$ are also eventually periodic, $j = 1,\ k + 2,\ 2k + 3,\ 3k + 4$. Therefore, $\{(x_1(n), \ldots, x_4(n))\}_{n=-k}^{+\infty}$ is eventually periodic with the minimal period $T \in Z(1)$. By Lemma 2.11, there exists some $r \in Z(0)$ such that $T = 4r + 2$. Without loss of generality, we assume that $\{x_1(n)\}_{n=1}^{+\infty}$ is a total r-normal cycle, and $x_1(1) = 1$. Then, $x_1(r + 1) = r + 1$. By Lemma 2.8, we can get that $x_4(r+k+1) = 1$ and $x_4(2r+2+k) = -r$. It follows from Lemma 2.9 that $x_3(2r+2k+2) = 0$ and $x_3(3r+2k+3) = r+1$, and so, again by Lemma 2.8, $x_2(3r+3k+3) = 0$ and $x_2(4r+3k+4) = r+1$. By Lemma 2.8, we have $x_1(4r+4k+4) = 0$ and $x_1(5r+4k+5) = r+1$. From (1.5) and (1.3), we can get $x_1(4r + 4k + 5) = x_1(4r + 4k + 4) + f(x_2(4r + 3k + 4))$. Thus, $x_1(4r+4k+5) = 0+1 = 1$. Therefore, $\{x_1(i)\}_{i=1}^{4r+4k+4}$ consists of $A_r^+, A_r^-, \ldots, A_r^+, A_r^-$, consecutively, in total of N semicycles, in which N is a positive even integer. Consequently, there exists an integer $M \geq 0$ such that $4r + 4k + 4 = (2r + 1) \cdot (2M + 2)$, that is, $(2k + 1)/(2r + 1) = M$, which implies that $r \in \Omega(k)$. This completes the proof. □

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