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Research Article

On the Neutrix Composition of the Delta and Inverse Hyperbolic Sine Functions

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Let F be a distribution in $\mathfrak{D}'$ and let f be a locally summable function. The composition $F(f(x))$ of F and f is said to exist and be equal to the distribution $h(x)$ if the limit of the sequence $\{F_n(f(x))\}$ is equal to $h(x)$, where $F_n(x) = F(x) * \delta_n(x)$ for $n = 1, 2, \dots$ and $\{\delta_n(x)\}$ is a certain regular sequence converging to the Dirac delta function. In the ordinary sense, the composition $\delta^{(s)}[(\sinh^{-1} x_+)^r]$ does not exists. In this study, it is proved that the neutrix composition $\delta^{(s)}[(\sinh^{-1} x_+)^r]$ exists and is given by $\delta^{(s)}[(\sinh^{-1} x_+)^r] = \sum_{k=0}^{sr+r-1} \sum_{i=0}^{k} \binom{k}{i} ((-1)^k r c_{s,k,i}/2^{k+1} k!) \delta^{(k)}(x)$, for $s = 0, 1, 2, \dots$ and $r = 1, 2, \dots$, where $c_{s,k,i} = (-1)^s s![(k-2i+1)^{rs-1} + (k-2i-1)^{rs+r-1}]/(2(rs+r-1)!)$. Further results are also proved.

1. Introduction

In the following, we let $\mathfrak{D}$ be the space of infinitely differentiable functions with compact support, let $\mathfrak{D}[a, b]$ be the space of infinitely differentiable functions with support contained in the interval $[a, b]$, and let $\mathfrak{D}'$ be the space of distributions defined on $\mathfrak{D}$.

Now, let $\rho(x)$ be a function in $\mathfrak{D}[-1, 1]$ having the following properties:

(i) $\rho(x) \geq 0$,

(ii) $\rho(x) = \rho(-x)$,

(iii) $\int_{-1}^{1} \rho(x)\, dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \dots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. Further, if F is an arbitrary distribution in $\mathfrak{D}'$ and $F_n(x) = F(x) * \delta_n(x) = \langle F(x-t), \varphi(t)\rangle$, then $\{F_n(x)\}$ is a regular sequence converging to $F(x)$.

Since the theory of distributions is a linear theory, thus we can extend some of the operations which are valid for ordinary functions to the space of distributions and such operations are called regular operations such as: addition, multiplication by scalars; see [1]. Other operations can be defined only for a particular class of distributions or for certain restricted subclasses of distributions; these are called irregular operations such as: multiplication of distributions, convolution products, and composition of distributions; see [2–4]. Thus, there have been several attempts recently to define distributions of the form $F(f(x))$ in $\mathfrak{D}'$, where F and f are distributions in $\mathfrak{D}'$; see for example [5–8]. In the following, we are going to consider an alternative approach. As a starting point, we look at the following definition which is a generalization of Gel'fand and Shilov's definition of the composition involving the delta function [9], and was given in [6].

Definition 1.1. Let F be a distribution in $\mathfrak{D}'$ and let f be a locally summable function. We say that the neutrix composition $F(f(x))$ exists and is equal to h on the open interval (a,b), with $-\infty < a < b < \infty$, if

$$N-\lim_{n\to\infty}\int_{-\infty}^{\infty} F_n(f(x))\varphi(x)dx = \langle h(x), \varphi(x)\rangle, \tag{1.1}$$

for all φ in $\mathfrak{D}[a,b]$, where $F_n(x) = F(x) * \delta_n(x)$ for $n = 1,2,\ldots$ and N is the neutrix, see [10], having domain N' the positive and range N'' the real numbers, with negligible functions which are finite linear sums of the functions

$$n^{\lambda}\ln^{r-1}n, \quad \ln^{r}n : \lambda > 0, \quad r = 1,2,\ldots \tag{1.2}$$

and all functions which converge to zero in the usual sense as n tends to infinity.

In particular, we say that the composition $F(f(x))$ exists and is equal to h on the open interval (a,b) if

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} F_n(f(x))\varphi(x)dx = \langle h(x), \varphi(x)\rangle, \tag{1.3}$$

for all φ in $\mathfrak{D}[a,b]$.

Note that taking the neutrix limit of a function $f(n)$ is equivalent to taking the usual limit of Hadamard's finite part of $f(n)$. The definition of the neutrix composition of distributions was originally given in [10] but was then simply called the composition of distributions.

The following three theorems were proved in [11], [8], and [12], respectively.

Theorem 1.2. *The neutrix composition $\delta^{(s)}(\operatorname{sgn} x|x|^{\lambda})$ exists and*

$$\delta^{(s)}\left(\operatorname{sgn} x|x|^{\lambda}\right) = 0, \tag{1.4}$$

for $s = 0, 1, 2, \ldots$ *and* $(s+1)\lambda = 1, 3, \ldots$, *and*

$$\delta^{(s)}\left(\operatorname{sgn} x|x|^{\lambda}\right) = \frac{(-1)^{(s+1)(\lambda+1)} s!}{\lambda[(s+1)\lambda - 1]!}\delta^{((s+1)\lambda-1)}(x), \tag{1.5}$$

for $s = 0, 1, 2, \ldots$, *and* $(s+1)\lambda = 2, 4, \ldots$.

Theorem 1.3. *The neutrix compositions* $\delta^{(2s-1)}(\operatorname{sgn} x|x|^{1/s})$ *and* $\delta^{(s-1)}(|x|^{1/s})$ *exist and*

$$\begin{aligned} \delta^{(2s-1)}\left(\operatorname{sgn} x|x|^{1/s}\right) &= \frac{1}{2}(2s)!\delta'(x), \\ \delta^{(s-1)}\left(|x|^{1/s}\right) &= (-1)^{s-1}\delta(x), \end{aligned} \tag{1.6}$$

for $s = 1, 2, \ldots$.

Theorem 1.4. *The neutrix composition* $\delta^{(s)}(\sinh^{-1} x_+^{1/r})$ *exists and*

$$\delta^{(s)}\left[\left(\sinh^{-1} x_+\right)^{1/r}\right] = \sum_{k=0}^{(s+1)/r-1}\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^k r c_{s,k,i}}{2^{k+1}k!}\delta^{(k)}(x), \tag{1.7}$$

for $s = 0, 1, 2, \ldots$ *and* $r = 1, 2, \ldots$, *where*

$$c_{r,s,k,i} = \frac{(-1)^s s!\left[(k-2i+1)^{rs+r-1} + (k-2i-1)^{rs+r-1}\right]}{2(rs+r-1)!}. \tag{1.8}$$

The next two theorems were proved in [13].

Theorem 1.5. *The neutrix composition* $\delta^{(s)}[\ln^r(1+|x|)]$ *exists and*

$$\delta^{(s)}\left[\ln^r(1+|x|)\right] = \sum_{k=0}^{sr+r-1}\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{s-i}\left[1+(-1)^k\right]s!(i+1)^{rs+r-1}}{2r(rs+r-1)!k!}\delta^{(k)}(x). \tag{1.9}$$

for $s = 0, 1, 2, \ldots$, *and* $r = 1, 2, \ldots$.

In particular, the composition $\delta[\ln(1+|x|)]$ *exists and*

$$\delta[\ln[1+|x|)] = \delta(x). \tag{1.10}$$

Theorem 1.6. *The neutrix composition* $\delta^{(s)}[\ln(1+|x^{1/r}|)]$ *exists and*

$$\delta^{(s)}\left[\ln\left(1+\left|x^{1/r}\right|\right)\right] = \sum_{k=0}^{m-1}\sum_{i=0}^{kr+r-1}\binom{kr+r-1}{i}\frac{(-1)^{r+s+i-1}\left[1+(-1)^k\right]r(i+1)^s}{2k!}\delta^{(k)}(x), \tag{1.11}$$

for $s = 0, 1, 2, \ldots$ *and* $r = 2, 3, \ldots$, *where* m *is the smallest non-negative integer greater than* $(s-r+1)r^{-1}$.

In particular, the composition $\delta^{(s)}[\ln(1+|x^{1/r}|)]$ *exists and*

$$\delta^{(s)}\left[\ln\left(1+\left|x^{1/r}\right|\right)\right]=0, \tag{1.12}$$

for $s=0,1,2,\ldots,r-2$ *and* $r=2,3,\ldots$ *and*

$$\delta^{(r-1)}\left[\ln\left(1+\left|x^{1/r}\right|\right)\right]=(-1)^{r-1}r!\delta(x), \tag{1.13}$$

for $r=2,3,\ldots$.

2. Main Results

We now prove the following theorem.

Theorem 2.1. *The neutrix composition* $\delta^{(s)}[(\sinh^{-1}x_+)^r]$ *exists and*

$$\delta^{(s)}\left[\left(\sinh^{-1}x_+\right)^r\right]=\sum_{k=0}^{sr+r-1}\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^k r c_{s,k,i}}{2^{k+1}k!}\delta^{(k)}(x), \tag{2.1}$$

for $s=0,1,2,\ldots$ *and* $r=1,2,\ldots$, *where*

$$c_{r,s,k,i}=\frac{(-1)^s s!\left[(k-2i+1)^{rs+r-1}+(k-2i-1)^{rs+r-1}\right]}{2(rs+r-1)!}. \tag{2.2}$$

In particular, the neutrix composition $\delta(\sinh^{-1}x_+)$ *exists and*

$$\delta\left(\sinh^{-1}x_+\right)=\frac{1}{2}\delta(x). \tag{2.3}$$

Proof. To prove (2.1), we first of all evaluate

$$\int_{-1}^{1}\delta_n^{(s)}\left[\left(\sinh^{-1}x_+\right)^r\right]x^k\,dx. \tag{2.4}$$

We have

$$\begin{aligned}\int_{-1}^{1}\delta_n^{(s)}\left[\left(\sinh^{-1}x_+\right)^r\right]x^k dx&=n^{s+1}\int_{-1}^{1}\rho^{(s)}\left[\left(n\sinh^{-1}x_+\right)^r\right]x^k dx\\&=n^{s+1}\int_{0}^{1}\rho^{(s)}\left[n\left(\sinh^{-1}x\right)^r\right]x^k dx\\&\quad+n^{s+1}\int_{-1}^{0}\rho^{(s)}(0)x^k dx\\&=I_1+I_2.\end{aligned} \tag{2.5}$$

It is obvious that

$$N-\lim_{n\to\infty} I_2 = N-\lim_{n\to\infty}\int_{-1}^{0}\delta_n^{(s)}\left[\left(\sinh^{-1}x_+\right)^r\right]x^k\,dx = 0, \tag{2.6}$$

for $k = 0, 1, 2, \ldots$.

Making the substitution $t = n(\sinh^{-1}x)^r$, we have for large enough n

$$\begin{aligned} I_1 &= \frac{n^{s-r+1}}{r}\int_0^1 t^{1/(r-1)}\sinh^k\left(\frac{t}{n}\right)^{1/r}\cosh\left(\frac{t}{n}\right)^{1/r}\rho^{(s)}(t)dt \\ &\times\int_0^1 t^{1/(r-1)}\left\{\exp\left[(k-2i+1)\left(\frac{t}{n}\right)^{1/r}\right]+\exp\left[(k-2i-1)\left(\frac{t}{n}\right)^{1/r}\right]\right\}\rho^{(s)}(t)dt, \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} &n^{(s-1)/(r+1)}\int_0^1 t^{1/(r-1)}\left\{\exp\left[(k-2i+1)\left(\frac{t}{n}\right)^{1/r}\right]+\exp\left[(k-2i-1)\left(\frac{t}{n}\right)^{1/r}\right]\right\}\rho^{(s)}(t)dt \\ &\quad=\sum_{p=0}^{\infty}\int_0^1\frac{\left[(k-2i+1)^p+(k-2i-1)^p\right]t^{(p/r)+(1/r)-1}}{p!n^{(p/r)+(1/r)-s-1}}\rho^{(s)}(t)dt. \end{aligned} \tag{2.8}$$

It follows that

$$\begin{aligned} &N-\lim_{n\to\infty} n^{s-1/r+1}\int_0^1 t^{1/(r-1)}\left\{\exp\left[(k-2i+1)\left(\frac{t}{n}\right)^{1/r}\right]+\exp\left[(k-2i-1)\left(\frac{t}{n}\right)^{1/r}\right]\right\}\rho^{(s)}(t)\,dt \\ &\quad=\frac{(-1)^s s!\left[(k-2i+1)^{rs+r-1}+(k-2i-1)^{rs+r-1}\right]}{2(rs+r-1)!} \\ &\quad= c_{r,s,k,i}, \end{aligned} \tag{2.9}$$

and by applying the neutrix limit we obtain

$$N-\lim_{n\to\infty} I_1 = N-\lim_{n\to\infty}\int_0^1\delta_n^{(s)}\left[\left(\sinh^{-1}x_+\right)^r\right]x^k\,dx = \frac{1}{2^{k+1}r}\sum_{i=0}^{k}\binom{k}{i}(-1)^i c_{r,s,\mathrm{k},i} \tag{2.10}$$

for $k = 0, 1, 2, \ldots$.

When $k = sr + r$, we have

$$\begin{aligned}
|I_1| &= \int_0^1 \left|\delta_n^{(s)}\left[\left(\sinh^{-1}x_+\right)^r\right]x^{sr+r}\right|dx \\
&= n^{s+1}\int_0^1 \left|\rho_n^{(s)}\left[n\left(\sinh^{-1}x\right)^r\right]x^{sr+r}\right|dx \\
&\leq \frac{n^{(s-1)/(r+1)}}{2^{sr+r}r}\exp(sr+r+1)\int_0^1 \left|\left[1-\exp\left[-2\left(\frac{t}{n}\right)^{1/r}\right]^{sr+r}\rho^{(s)}(t)\right]\right|dt \\
&= \frac{n^{(s-1)/(r+1)}}{2^{sr+r}r}\exp(sr+r+1)\int_0^1 \left[2\left(\frac{t}{n}\right)^{1/r}+O\left(n^{-2/r}\right)\right]^{sr+r}\left|\rho^{(s)}(t)\right|dt \\
&\leq n^{-1/r}\exp(sr+r+1)\int_0^1 \left[1+O\left(n^{-2/r}\right)\right]\left|\rho^{(s)}(t)\right|dt \\
&= O\left(n^{-1/r}\right).
\end{aligned} \tag{2.11}$$

Thus, if φ is an arbitrary continuous function, then

$$\lim_{n\to\infty}\int_0^1 \delta_n^{(s)}\left[\left(\sinh^{-1}x_+\right)^r\right]x^{rs+r}\varphi(x)dx = 0. \tag{2.12}$$

We also have

$$\int_{-1}^0 \delta_n^{(s)}\left[\left(\sinh^{-1}x_+\right)^r\right]\varphi(x)dx = n^{s+1}\int_{-1}^0 \rho^{(s)}(0)\varphi(x)dx, \tag{2.13}$$

and it follows that

$$N-\lim_{n\to\infty}\int_{-1}^0 \delta_n^{(s)}\left[\left(\sinh^{-1}x_+\right)^r\right]\varphi(x)dx = 0. \tag{2.14}$$

If now φ is an arbitrary function in $\mathfrak{D}[-1,1]$, then by Taylor's Theorem, we have

$$\varphi(x) = \sum_{k=0}^{sr+r-1}\frac{\varphi^{(k)}(0)}{k!}x^k + \frac{x^{rs+r}}{(rs+r)!}\varphi^{(rs+r)}(\xi x), \tag{2.15}$$

where $0 < \xi < 1$, and so

$$
\begin{aligned}
&\mathrm{N}\text{-}\lim_{n\to\infty}\left\langle \delta_n^{(s)}\left[\left(\sinh^{-1}x_+\right)^{1/r}\right], \varphi(x)\right\rangle \\
&= \mathrm{N}\text{-}\lim_{n\to\infty}\sum_{k=0}^{sr+r-1}\frac{\varphi^{(k)}(0)}{k!}\int_0^1 \delta_n^{(s)}\left[\left(\sinh^{-1}x_+\right)^{r}\right]x^k\,dx \\
&\quad + \mathrm{N}\text{-}\lim_{n\to\infty}\sum_{k=0}^{sr+r-1}\frac{\varphi^{(k)}(0)}{k!}\int_{-1}^0 \delta_n^{(s)}\left[\left(\sinh^{-1}x_+\right)^{r}\right]x^k\,dx \\
&\quad + \lim_{n\to\infty}\frac{1}{(sr+r)!}\int_0^1 \delta_n^{(s)}\left[\left(\sinh^{-1}x_+\right)^{r}\right]x^{sr+r}\varphi^{(sr+r)}(\xi x)\,dx \\
&\quad + \lim_{n\to\infty}\frac{1}{(sr+r)!}\int_{-1}^0 \delta_n^{(s)}\left[\left(\sinh^{-1}x_+\right)^{r}\right]x^{sr+r}\varphi^{(sr+r)}(\xi x)\,dx \\
&= \sum_{k=0}^{sr+r-1}\sum_{i=0}^{k}\binom{k}{i}\frac{r c_{r,s,k,i}\varphi^{(k)}(0)}{2^{k+1}k!} + 0 \\
&= \sum_{k=0}^{sr+r-1}\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^k r c_{r,s,k,i}}{2^{k+1}k!}\left\langle \delta^{(k)}(x), \varphi(x)\right\rangle,
\end{aligned}
\tag{2.16}
$$

on using (2.3) to (2.14). This proves (2.1) on the interval $(-1, 1)$.

It is clear that $\delta^{(s)}[(\sinh^{-1}x_+)^r] = 0$ for $x > 0$ and so (2.1) holds for $x > -1$.

Now, suppose that φ is an arbitrary function in $\mathfrak{D}[a, b]$, where $a < b < 0$. Then,

$$
\int_a^b \delta_n^{(s)}\left[\left(\sinh^{-1}x_+\right)^{r}\right]\varphi(x)\,dx = n^{s+1}\int_a^b \rho^{(s)}(0)\varphi(x)dx \tag{2.17}
$$

and so

$$
\mathrm{N}\text{-}\lim_{n\to\infty}\int_a^b \delta_n^{(s)}\left[\left(\sinh^{-1}x_+\right)^{r}\right]\varphi(x)dx = 0. \tag{2.18}
$$

It follows that $\delta^{(s)}[(\sinh^{-1}x_+)^r] = 0$ on the interval (a, b). Since a and b are arbitrary, we see that (2.1) holds on the real line. This completes the proof of the theorem. □

Corollary 2.2. *The neutrix composition $\delta^{(s)}[(\sinh^{-1}|x|)^r]$ exists and*

$$
\delta^{(s)}\left[\left(\sinh^{-1}|x|\right)^{r}\right] = \sum_{k=0}^{sr+r-1}\sum_{i=0}^{k}\binom{k}{i}\frac{\left[(-1)^k+1\right]c_{r,s,k,i}}{2^{k+1}k!}\delta^{(k)}(x), \tag{2.19}
$$

for $s = 0, 1, 2, \ldots$ and $r = 1, 2, \ldots$.

In particular, the composition $\delta(\sinh^{-1}|x|)$ *exists and*

$$\delta\left(\sinh^{-1}|x|\right) = \frac{1}{2}\delta(x). \tag{2.20}$$

Proof. To prove (2.19), we note that

$$\begin{aligned}\int_{-1}^{1} \delta_n^{(s)}\left[\left(\sinh^{-1}|x|\right)^r\right] x^k\,dx &= n^{s+1}\int_{-1}^{1} \rho^{(s)}\left[\left(n\sinh^{-1}|x|\right)^r\right] x^k\,dx \\ &= n^{s+1}\left[1+(-1)^k\right]\int_{0}^{1} \rho^{(s)}\left[n\left(\sinh^{-1}x\right)^r\right] x^k\,dx,\end{aligned} \tag{2.21}$$

and (2.19) now follows as above.

Equation (2.20) follows on noting that in the particular case $s = 0$, the usual limit holds in (2.10). This completes the proof of the corollary. □

Theorem 2.3. *The neutrix composition* $\delta^{(2s-1)}[\sinh^{-1}(\operatorname{sgn} x \cdot x^2)]$ *exists and*

$$\delta^{(2s-1)}\left[\sinh^{-1}\left(\operatorname{sgn} x \cdot x^2\right)\right] = \sum_{k=0}^{2s-1}\sum_{i=0}^{k+1}\binom{k}{i}\frac{(-1)^k b_{s,k,i}}{2^{k+1}(2k+1)!}\delta^{(k)}(x), \tag{2.22}$$

for $s = 1, 2, \ldots$, *where*

$$b_{s,k,i} = (k-2i+1)^{2s-1} + (k-2i-1)^{2s-1}. \tag{2.23}$$

Proof. To prove (2.22), we now have to evaluate

$$\int_{-1}^{1} \delta_n^{(2s-1)}\left[\sinh^{-1}\left(sgn\, x \cdot x^2\right)\right] x^k\,dx. \tag{2.24}$$

We have

$$\begin{aligned}\int_{-1}^{1} \delta_n^{(2s-1)}\left[\sinh^{-1}\left(\operatorname{sgn} x \cdot x^2\right)\right] x^k\,dx &= n^{2s}\int_{-1}^{1} \rho^{(2s-1)}\left[n\sinh^{-1}\left(\operatorname{sgn} x \cdot x^2\right)\right] x^k\,dx \\ &= \begin{cases} 2n^{2s}\displaystyle\int_{0}^{1} \rho^{(2s-1)}\left[n\left(\sinh^{-1}x^2\right)\right] x^k\,dx, & k \text{ odd}, \\ 0, & k \text{ even}. \end{cases}\end{aligned} \tag{2.25}$$

Making the substitution $t = n(\sinh^{-1} x^2)$, we have for large enough n

$$\begin{aligned}
&\int_{-1}^{1} \delta_n^{(2s-1)} \left[\sinh^{-1}\left(sgn\, x \cdot x^2\right)\right] x^k dx \\
&\quad = 2n^{2s} \int_0^1 \rho^{(2s-1)} \left[n\left(\sinh^{-1} x^2\right)\right] x^{2k+1} dx \\
&\quad = \frac{n^{2s-1}}{2^{k+1}} \sum_{i=0}^{k} k_{\;i}\, (-1)^i \int_0^1 \left\{\exp\left[\frac{(k-2i+1)t}{n}\right] + \exp\left[\frac{(k-2i-1)t}{n}\right]\right\} \rho^{(2s-1)}(t)dt,
\end{aligned} \tag{2.26}$$

where

$$\begin{aligned}
&n^{2s-1} \int_0^1 \left\{\exp\left[\frac{(k-2i+1)t}{n}\right] + \exp\left[\frac{(k-2i-1)t}{n}\right]\right\} \rho^{(s)}(t)dt \\
&\quad = \sum_{p=0}^{\infty} \int_0^1 \frac{\left[(k-2i+1)^p + (k-2i-1)^p\right] t^p}{p!\, n^{p-2s+1}} \rho^{(2s-1)}(t)dt.
\end{aligned} \tag{2.27}$$

It follows that

$$\begin{aligned}
&N - \lim_{n\to\infty} n^{2s-1} \int_0^1 \left\{\exp\left[\frac{(k-2i+1)t}{n}\right] + \exp\left[\frac{(k-2i-1)t}{n}\right]\right\} \rho^{(s)}(t)\; dt \\
&\quad = N - \lim_{n\to\infty} \sum_{p=0}^{\infty} \int_0^1 \frac{\left[(k-2\mathrm{i}+1)^p + (k-2i-1)^p\right] t^p}{p!\, n^{p-2s+1}} \rho^{(2s-1)}(t)\; dt \\
&\quad = \frac{-(k-2i+1)^{2s-1} + (k-2i-1)^{2s-1}}{2} \\
&\quad = \frac{b_{s,k,i}}{2},
\end{aligned} \tag{2.28}$$

and so by using the neutrix limit, we have

$$N - \lim_{n\to\infty} \int_{-1}^{1} \delta_n^{(2s-1)} \left[\sinh^{-1}\left(\operatorname{sgn} x \cdot x^2\right)\right] x^{2k+1} dx = \sum_{i=0}^{k} \binom{k}{i} \frac{(-1)^{i+1} b_{s,k,i}}{2^{k+1}}, \tag{2.29}$$

for $k = 0, 1, 2, \ldots$.

When $k = 2s$, we have

$$\begin{aligned}\int_{-1}^{1}\left|\delta_n^{(2s-1)}\left[\sinh^{-1}\left(\operatorname{sgn} x\cdot x^2\right)\right]x^{4s+1}\right|dx &= n^{2s}\int_{-1}^{1}\rho^{(2s-1)}\left[n\left(\sinh^{-1}x^2\right)\right]x^{4s+1}\,dx\\ &\leq \frac{n^{2s-1}}{2^{s-1}}\exp(s+1)\int_{-1}^{1}\left|\left[1-\exp\left(-\frac{2t}{n}\right)\right]^{2s}\rho^{(2s-1)}(t)\right|\,dt\\ &= \frac{n^{2s-1}}{2^{s-1}}\exp(s+1)\int_{-1}^{1}\left|\left[\frac{2t}{n}+O\left(n^{-2}\right)\right]^{2s}\rho^{(2s-1)}(t)\right|dt\\ &\leq 2^{2s+1}n^{-1}\exp(s+1)\int_{-1}^{1}\left[1+O\left(n^{-2/r}\right)\right]\left|\rho^{(2s-1)}(t)\right|\,dt\\ &= O\left(n^{-1}\right).\end{aligned} \tag{2.30}$$

Thus, if φ is an arbitrary continuous function, then

$$\lim_{n\to\infty}\int_{-1}^{1}\delta_n^{(2s-1)}\left[\sinh^{-1}\left(\operatorname{sgn} x\cdot x^2\right)\right]x^{4s+1}\varphi(x)dx = 0. \tag{2.31}$$

If now φ is an arbitrary function in $\mathfrak{D}[-1,1]$, then by Taylor's Theorem, we have

$$\varphi(x) = \sum_{k=0}^{4s}\frac{\varphi^{(k)}(0)}{k!}x^k + \frac{x^{4s+1}}{(4s+1)!}\varphi^{(4s+1)}(\xi x), \tag{2.32}$$

where $0 < \xi < 1$, and so

$$\begin{aligned}&N-\lim_{n\to\infty}\left\langle\delta_n^{(2s-1)}\left[\sinh^{-1}\left(\operatorname{sgn} x\cdot x^2\right)\right],\varphi(x)\right\rangle\\ &\quad= N-\lim_{n\to\infty}\sum_{k=0}^{2s-1}\frac{\varphi^{(2k+1)}(0)}{(2k+1)!}\int_{-1}^{1}\delta_n^{(2s-1)}\left[\sinh^{-1}\left(\operatorname{sgn} x\cdot x^2\right)\right]x^{2k+1}dx\\ &\qquad+\lim_{n\to\infty}\frac{1}{(4s+1)!}\int_{-1}^{1}\delta_n^{(4s+1)}\left[\sinh^{-1}\left(\operatorname{sgn} x\cdot x^2\right)\right]x^{4s+1}\varphi^{(4s+1)}(\xi x)dx\\ &\quad= \sum_{k=0}^{2s-1}\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{i+1}b_{s,k,i}\varphi^{(k)}(0)}{2^{k+1}(2k+1)!}+0\\ &\quad= \sum_{k=0}^{2s-1}\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{i+k+1}b_{s,k,i}}{2^{k+1}(2k+1)!}\left\langle\delta^{(k)}(x),\varphi(x)\right\rangle,\end{aligned} \tag{2.33}$$

on using (2.25) to (2.31), proving (2.22) on the interval $(-1,1)$. However, it is clear that $\delta_n^{(2s-1)}[\sinh^{-1}(\operatorname{sgn}x\cdot x^2)] = 0$ for $|x| > 0$ and so (2.22) holds on the real line, completing the proof of the theorem. □

Corollary 2.4. *The composition* $\delta'[\sinh^{-1}\operatorname{sgn} x \cdot x^2)]$ *exists and*

$$\delta'\left[\sinh^{-1}\left(\operatorname{sgn} x \cdot x^2\right)\right] = \frac{\delta'(x)}{4.3!} - 2\delta(x). \tag{2.34}$$

Proof. To prove (2.34) note that in the particular case $s = 1$, the usual limits hold and then (2.34) is a particular case of (2.22). This completes the proof of the corollary. ☐

For further related results on the neutrix operation of distributions, see [12–22] and [2, 3, 23].

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