

Research Article

Role of Delay on Planktonic Ecosystem in the Presence of a Toxic Producing Phytoplankton

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Received 11 May 2011; Accepted 25 July 2011

Academic Editor: Gershon Wolansky

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A mathematical model is proposed to study the role of distributed delay on plankton ecosystem in the presence of a toxic producing phytoplankton. The model includes three state variables, namely, nutrient concentration, phytoplankton biomass, and zooplankton biomass. The release of toxic substance by phytoplankton species reduces the growth of zooplankton and this plays an important role in plankton dynamics. In this paper, we introduce a delay (time-lag) in the digestion of nutrient by phytoplankton. The stability analysis of all the feasible equilibria are studied and the existence of Hopf-bifurcation for the interior equilibrium of the system is explored. From the above analysis, we observe that the supply rate of nutrient and delay parameter play important role in changing the dynamical behaviour of the underlying system. Further, we have derived the explicit algorithm which determines the direction and the stability of Hopf-bifurcation solution. Finally, numerical simulation is carried out to support the theoretical result.

1. Introduction

The study of plankton system is an important area of research in marine ecology. Phytoplankton perform great service for earth. They provide food for marine life, oxygen for human being and also absorb half of the carbon dioxide which may be contributing to the global warming [1]. The dynamics of rapid (massive) increase or almost equal decrease of phytoplankton population is a common feature in marine plankton ecology and known as bloom. Blooms of red tide produce chemical toxin, a type of paralytic poison which can be harmful to zooplankton, fin fish, shellfish, fish, birds, marine mammals, and humans also. Some species, such as the dinoflagellate *Alexandrium tamarense* and the diatom *Pseudo-nitzschia australis* [2] produce potent toxins which are liberated into the water before they are eaten, and they may well affect zooplankton when they are in water. It is now

well established that a significant number of phytoplankton species produce toxin, such as *Pseudo-nitzschia*, *Gambierdiscus toxicus*, *Prorocentrum*, *Ostreopsis*, *Coolia monotis*, *Thecadinium*, *Amphidinium carterae*, *Dinophysis*, *Gymnodinium breve*, *Alexandrium*, *Gymnodinium catenatum*, *Pyrodinium bahamense*, *Pfiesteria piscicida*, *Chrysochromulina polylepis*, *Prymnesium patelliferum*, and *P. parvum* [3, 4]. Reduction of grazing pressure due to toxin is an important phenomena in plankton ecology [5, 6]. In aquatic system, toxin-producing phytoplankton may act as controlling factor in the phytoplankton-zooplankton interaction dynamics. Efforts have been made to study the role of toxin producing phytoplankton on the phytoplankton-zooplankton dynamics [7–9]. Toxicity may be the strong mediator of zooplankton feeding rate as shown by field studies [4, 10] and laboratory study [11].

Many researchers have been showing keen interest to investigate the direction and stability of Hopf-bifurcation of the system as refer their in [12, 13].

Our current study is motivated by [7, 8, 14], who have considered nutrient interaction with phytoplankton, phytoplankton interaction with zooplankton. The study of interacting species system with nutrient cycling which contributes to the growth of nutrient is carried out in [9, 14, 15]. A model can be more realistic if a delay effect (or time-lag) is being considered in the conversion from one species to another species [15–17]. The prey-predator systems with time delay are deeply considered by [18, 19] and many researchers have used distributed delay in their models [15, 16].

2. Mathematical Model Formulation

Let $N(t)$ denote the concentration of nutrient at time t , $x(t)$ denotes the biomass of toxic producing phytoplankton in the habitat that are partially harmful to zooplankton biomass, $y(t)$. We assume a constant supply rate of nutrient (i.e., N_0) in the system. The loss of nutrient due to leaching is assumed to be given by the term aN . We take a_1 as the growth rate of phytoplankton biomass, w as the rate of predation of phytoplankton by zooplankton, and w_1 denotes the corresponding conversion rate of zooplankton. The phytoplankton and zooplankton interaction is assumed to follow the Holling type-II functional response [20, 21] with D as half saturation constant. Again, the specific rate of nutrient uptake by per unit biomass of phytoplankton in unit time is considered to be bN and depletion of zooplankton biomass due to toxin-producing phytoplankton is given by the term c_1xy . We further assume that phytoplankton and zooplankton biomass deplete due to natural mortality at the rate of b_1 and a_2 , respectively, and k is the fraction of dead phytoplankton biomass that is being recycled back to the nutrient pool. With these assumption and notations, the resultant dynamics of the system under consideration is given by the following set of differential equations [14]:

$$\begin{aligned}\frac{dN}{dt} &= N_0 - aN - bNx + kb_1x, \\ \frac{dx}{dt} &= a_1Nx - b_1x - \frac{wxy}{D+x}, \\ \frac{dy}{dt} &= \frac{w_1xy}{D+x} - a_2y - c_1xy,\end{aligned}\tag{2.1}$$

with nonnegative initial conditions $N(0), x(0), y(0) > 0$. The above system of equations can be nondimensionalised using the relations $x_1 = N/D$, $x_2 = x/D$, $x_3 = y/D$, $\tau = at$, and

introducing the new parameters $\alpha = N_0/aD$, $\beta = bD/a$, $\gamma = b_1/a$, $\beta_1 = a_1D/a$, $\mu = w/a$, $\mu_1 = w_1/a$, $\xi = a_2/a$, and $\xi_1 = c_1D/a$. The non-dimensionalised equations of the above system (2.1) are as follows in which we have replaced τ by t and get

$$\begin{aligned}\frac{dx_1}{dt} &= \alpha - x_1 - \beta x_1 x_2 + k\gamma x_2, \\ \frac{dx_2}{dt} &= \beta_1 x_1 x_2 - \gamma x_2 - \frac{\mu x_2 x_3}{1 + x_2}, \\ \frac{dx_3}{dt} &= \frac{\mu_1 x_2 x_3}{1 + x_2} - \xi x_3 - \xi_1 x_2 x_3,\end{aligned}\tag{2.2}$$

with initial conditions: $x_1(0) > 0$, $x_2(0) > 0$, $x_3(0) > 0$, where, α , β , γ , β_1 , μ , μ_1 , ξ , ξ_1 , and $0 < k < 1$ are positive constants.

In this paper, our mathematical model is an extension of the system (2.2) which is studied earlier [14]. Now, we have introduced distributed delay in the digestion of nutrient by phytoplankton. The system (2.2) can be written as

$$\begin{aligned}\frac{dx_1}{dt} &= \alpha - x_1 - \beta x_1 x_2 + k\gamma x_2, \\ \frac{dx_2}{dt} &= \beta_1 x_2 \int_{-\infty}^t \alpha_1 \exp(-\alpha_1(t-s)) f(s) ds - \gamma x_2 - \frac{\mu x_2 x_3}{1 + x_2},\end{aligned}\tag{2.3}$$

where $f(s) = f(x_1) = x_1(s)$, α_1 is delay parameter,

$$\frac{dx_3}{dt} = \frac{\mu_1 x_2 x_3}{1 + x_2} - \xi x_3 - \xi_1 x_2 x_3,\tag{2.4}$$

putting

$$R(t) = \int_{-\infty}^t \alpha_1 \exp(-\alpha_1(t-s)) f(s) ds.\tag{2.5}$$

The above system of delay differential equation can be written as

$$\begin{aligned}\frac{dx_1}{dt} &= \alpha - x_1 - \beta x_1 x_2 + k\gamma x_2, \\ \frac{dx_2}{dt} &= \beta_1 x_2 R - \gamma x_2 - \frac{\mu x_2 x_3}{1 + x_2}, \\ \frac{dx_3}{dt} &= \frac{\mu_1 x_2 x_3}{1 + x_2} - \xi x_3 - \xi_1 x_2 x_3, \\ \frac{dR}{dt} &= \alpha_1(x_1 - R),\end{aligned}\tag{2.6}$$

with the additional initial condition: $R(0) > 0$.

3. Boundedness and Equilibria of the System

In this section, we will establish that the system (2.6) is bounded. We begin with the following lemma.

Lemma 3.1. *The system (2.6) is uniformly bounded in Ω_1 , where*

$$\Omega_1 = \left\{ (x_1, x_2, x_3, R) : 0 \leq x_1(t) + x_2(t) + x_3(t) + R(t) \leq \frac{\alpha}{\eta_1} \right\}, \quad (3.1)$$

$$\eta_1 = \min\{(1 - \alpha), \gamma(1 - k\beta_1/\beta), \xi, \alpha_1\}.$$

Proof. Let us consider a time dependent function:

$$W_1(t) = x_1(t) + \frac{\beta}{\beta_1} x_2(t) + \frac{\beta\mu}{\beta_1\mu_1} x_3(t) + R(t). \quad (3.2)$$

Clearly,

$$\frac{dW_1}{dt} = \frac{dx_1}{dt} + \frac{\beta}{\beta_1} \frac{dx_2}{dt} + \frac{\beta\mu}{\beta_1\mu_1} \frac{dx_3}{dt} + \frac{dR}{dt}. \quad (3.3)$$

Using (2.6) in the above expression we obtain

$$\begin{aligned} \frac{dW_1}{dt} &= [\alpha - x_1 - \beta x_1 x_2 + k\gamma x_2] + \frac{\beta}{\beta_1} \left[\beta_1 x_2 R - \gamma x_2 - \frac{\mu x_2 x_3}{1 + x_2} \right] \\ &\quad + \frac{\mu\beta}{\mu_1\beta_1} \left[\frac{\mu_1 x_2 x_3}{1 + x_2} - \xi x_3 - \xi_1 x_2 x_3 \right] + [\alpha_1(x_1 - R)], \\ &= \alpha - (1 - \alpha_1)x_1 - \gamma \left(1 - k \frac{\beta_1}{\beta} \right) \frac{\beta}{\beta_1} x_2 - \xi \frac{\mu\beta}{\mu_1\beta_1} x_3 - \xi_1 \frac{\mu\beta}{\mu_1\beta_1} x_2 x_3 - \alpha_1 R + (R - x_1)\beta x_2 \\ &\leq \alpha - \eta_1 W_1(t) + (R - x_1)\beta x_2, \end{aligned} \quad (3.4)$$

where η_1 is chosen as the minimum of $\{(1 - \alpha_1), \gamma(1 - k\beta_1/\beta), \xi, \alpha_1\}$. Thus,

$$\frac{dW_1}{dt} + \eta_1 W_1 \leq \alpha + (R - x_1)\beta x_2. \quad (3.5)$$

Now applying the theorem of differential inequalities [22], we obtain

$$0 < W_1(t) \leq W_1(0)e^{-\eta_1 t} + \frac{\alpha}{\eta_1} + \int (R - x_1)\beta x_2 e^{-\eta_1 t} dt, \quad (3.6)$$

as $t \rightarrow \infty$, $R \rightarrow x_1$, which gives

$$0 \leq W_1 \leq \frac{\alpha}{\eta_1}. \quad (3.7)$$

Hence the solution of the system (2.6) is bounded in Ω_1 . $\square$

We now consider the existence of possible equilibria of the system (2.6). The system of (2.6) has three feasible equilibria, namely,

- (i) boundary equilibrium: $E_1 \equiv (\alpha, 0, 0, \alpha)$,
- (ii) boundary equilibrium: $E_2 \equiv (\gamma/\beta_1, (\alpha\beta_1 - \gamma)/(\beta - k\beta_1)\gamma, 0, \gamma/\beta_1)$. The boundary equilibrium E_2 exists if either $\gamma/\alpha < \beta_1 < \beta/k$ or $\beta/k < \beta_1 < \gamma/\alpha$ is satisfied, that is, growth rate of phytoplankton biomass lies between the fraction of natural death rate of phytoplankton biomass to constant supply rate of nutrient and the fraction of nutrient uptake by phytoplankton biomass to fraction of dead phytoplankton biomass,
- (iii) positive interior equilibrium: $E_3 \equiv (x_1^*, x_2^*, x_3^*, R^*)$, where

$$\begin{aligned} x_1^* &= \frac{\alpha + k\gamma x_2^*}{1 + \beta x_2^*}, & x_3^* &= \frac{1}{\mu} [(\beta_1 x_1^* - \gamma)(1 + x_2^*)], & R^* &= \frac{\alpha + k\gamma x_2^*}{1 + \beta x_2^*}, \\ x_2^* &= \frac{-s_2 \pm \sqrt{s_2^2 - 4s_1 s_3}}{2s_1}, & s_1 &= \xi_1, & s_2 &= -(\mu_1 - (\xi + \xi_1)), & s_3 &= \xi. \end{aligned} \quad (3.8)$$

The positive interior equilibrium E_3 exists if $\mu_1 > \xi + \xi_1$, that is, the growth rate of zooplankton biomass is greater than the sum of natural death rate and death due to harmful phytoplankton, and also if $x_1^* > \gamma/\beta_1$ means that concentration of nutrient at equilibrium is greater than the fraction of natural death rate of phytoplankton biomass to the depletion rate of nutrient uptake by phytoplankton biomass.

4. Dynamic Behaviour and Hopf-Bifurcation

In the previous section we observed that the system of (2.6) have three equilibria, namely, $E_1(\alpha, 0, 0, \alpha)$, $E_2(\gamma/\beta, (\alpha\beta_1 - \gamma)/(\beta - k\beta_1)\gamma, 0, \gamma/\beta)$, and $E_3(x_1^*, x_2^*, x_3^*, R^*)$. We will now examine the dynamical behaviour of the system about all the feasible equilibria.

The variational matrix for the system of (2.6) evaluated at E_1 is

$$V_1 = \begin{bmatrix} -1 & -(\alpha\beta - k\gamma) & 0 & 0 \\ 0 & (\alpha\beta_1 - \gamma) & 0 & 0 \\ 0 & 0 & -\xi & 0 \\ \alpha_1 & 0 & 0 & -\alpha_1 \end{bmatrix}. \quad (4.1)$$

The eigenvalues of the characteristic equation of V_1 are $\lambda_1 = -1$, $\lambda_2 = (\alpha\beta_1 - \gamma)$, $\lambda_3 = -\xi$, and $\lambda_4 = -\alpha_1$. It is seen from these eigenvalues that the equilibrium E_1 is locally

asymptotically stable if $\beta_1 < \gamma/\alpha$, which means that the growth rate of phytoplankton due to the availability of nutrient is less than the fraction of natural death rate of phytoplankton biomass to the constant input rate of nutrient.

The variational matrix for the system of (2.6) evaluated about E_2 is

$$V_2 = \begin{bmatrix} \frac{-\beta_1(\alpha\beta - k\gamma)}{\gamma(\beta - k\beta_1)} & \frac{-(\beta - k\beta_1)\gamma}{\beta_1} & 0 & 0 \\ 0 & 0 & \frac{-\mu(\alpha\beta_1 - \gamma)}{\gamma(\beta - k\beta_1) + (\alpha\beta_1 - \gamma)} & \frac{\beta_1(\alpha\beta_1 - \gamma)}{\gamma(\beta - k\beta_1)} \\ 0 & 0 & j_{33} & 0 \\ \alpha_1 & 0 & 0 & -\alpha_1 \end{bmatrix}, \quad (4.2)$$

where

$$j_{33} = \frac{-\left[\left(\sqrt{\xi}\gamma(\beta - k\beta_1) - \sqrt{\xi_1}(\alpha\beta_1 - \gamma)\right)^2 + \left(\left(\sqrt{\xi} + \sqrt{\xi_1}\right)^2 - \mu_1\right)\gamma(\alpha\beta_1 - \gamma)(\beta - k\beta_1)\right]}{(\gamma(\beta - k\beta_1) + (\alpha\beta_1 - \gamma))(\gamma(\beta - k\beta_1))}. \quad (4.3)$$

The eigenvalues λ_1 , λ_2 , and λ_3 of the above matrix are the roots of the following cubic equation:

$$\lambda^3 + \left[\frac{\beta_1(\alpha\beta - k\gamma)}{\gamma(\beta - k\beta_1)} + \alpha_1\right]\lambda^2 + \frac{\alpha_1\beta_1(\alpha\beta - k\gamma)}{\gamma(\beta - k\beta_1)}\lambda + \alpha_1(\alpha\beta_1 - \gamma) = 0, \quad (4.4)$$

and the fourth eigenvalue is given by $\lambda_4 = j_{33}$.

Clearly, λ_1 , λ_2 , and λ_3 have negative real parts if $(\alpha\beta - k\gamma)(\beta - k\beta_1) > 0$, $\beta_1 > \gamma/\alpha$, that is, growth rate of phytoplankton biomass is greater than the fraction of natural death rate of phytoplankton biomass to constant supply rate of nutrient and $\beta_1(\alpha\beta - k\gamma)(\beta_1(\alpha\beta - k\gamma) + \alpha_1\gamma(\beta - k\beta_1)) > (\beta - k\beta_1)^2(\alpha\beta_1 - \gamma)\gamma^2$ are satisfied, the eigenvalue λ_4 is negative when $\xi + \xi_1 + 2\sqrt{\xi}\sqrt{\xi_1} > \mu_1$. Thus, we can state the following theorem.

Theorem 4.1. *The second boundary equilibrium E_2 is linearly asymptotically stable if*

- (i) $(\alpha\beta - k\gamma)(\beta - k\beta_1) > 0$,
- (ii) $\beta_1 > \frac{\gamma}{\alpha}$,
- (iii) $\xi + \xi_1 + 2\sqrt{\xi}\sqrt{\xi_1} > \mu_1$,
- (iv) $\beta_1(\alpha\beta - k\gamma)(\beta_1(\alpha\beta - k\gamma) + \alpha_1\gamma(\beta - k\beta_1)) > (\beta - k\beta_1)^2(\alpha\beta_1 - \gamma)\gamma^2$,

are satisfied.

For the sake of convenience, the equilibrium points $(x_1^*, x_2^*, x_3^*, R^*)$ of the system is shifted to new points (n_1, n_2, n_3, n_4) through transformations $n_1 = x_1 - x_1^*$, $n_2 = x_2 - x_2^*$, $n_3 = x_3 - x_3^*$, $n_4 = R - R^*$.

In terms of the new variables, the dynamical equations (2.6) can be written as in matrix form as

$$\dot{X} = AX + B, \quad (4.6)$$

where $\dot{(\cdot)}$ cover X denotes the derivative with respect to time. Here AX is the linear part of the system and B represents the nonlinear part. Moreover,

$$X = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix}, \quad A = \begin{pmatrix} -(1 + \beta x_2^*) & -(\beta x_1^* - k\gamma) & 0 & 0 \\ 0 & \frac{\mu x_2^* x_3^*}{(1 + x_2^*)^2} & \frac{-\mu x_2^*}{1 + x_2^*} & \beta_1 x_2^* \\ 0 & \left(\frac{\mu_1}{(1 + x_2^*)^2} - \xi_1 \right) x_3^* & 0 & 0 \\ \alpha_1 & 0 & 0 & -\alpha_1 \end{pmatrix}, \quad (4.7)$$

$$B = \begin{pmatrix} -\beta n_1 n_2 \\ (\beta_1 n_2 n_4 + p_2 p_3 p_4^2 n_2^3 - p_2 p_3 n_2^2 + p_2 p_4 n_2 n_3 - p_2 p_4^2 n_2^2 n_3 + p_2 p_4^3 n_2^3 n_3 \\ + p_3 p_4 \mu n_2^2 - \mu p_4^2 p_3 n_2^3 - \mu p_4 n_2 n_3 + \mu p_4^2 n_2^2 n_3) \\ (p_3 p_5 x_2^* n_2^2 - p_3 p_4 p_5 x_2^* n_2^3 - p_5 x_2^* n_2 n_3 + p_4 p_5 x_2^* n_2^2 n_3 - p_4^2 p_5 x_2^* n_2^3 n_3 \\ - p_5 x_3^* n_2^2 + p_5 p_3 n_2^3 - \xi_1 n_2 n_3) \\ 0 \end{pmatrix}.$$

The eigenvalues of the matrix A help to understand the stability of the system. The characteristic equation for the variational matrix A is given by

$$\lambda^4 + A_1 \lambda^3 + A_2 \lambda^2 + A_3 \lambda + A_4 = 0, \quad (4.8)$$

where

$$\begin{aligned} A_1 &= p_1 - p_2 p_3, & A_2 &= g_2 \alpha_1 + p_2 p_5 x_3^* - p_2 (p_1 p_3 + \xi_1 x_3^*), \\ A_3 &= \alpha_1 \beta_1 p_6 x_2^* + p_1 p_2 p_5 x_3^* - p_2 (p_1 \xi_1 x_3^* + g_2 p_3 \alpha_1), & A_4 &= p_2 (p_5 - \xi_1) \alpha_1 g_2 x_3^*, \\ p_1 &= 1 + \beta x_2^* + \alpha_1, & p_2 &= \frac{\mu x_2^*}{1 + x_2^*}, & p_3 &= \frac{x_3^*}{1 + x_2^*}, & p_4 &= \frac{1}{1 + x_2^*} g_2 = 1 + \beta x_2^*, \\ p_5 &= \frac{\mu_1}{(1 + x_2^*)^2}, & p_6 &= \frac{\alpha \beta - k\gamma}{1 + \beta x_2^*}. \end{aligned} \quad (4.9)$$

Using the Routh-Hurwitz criteria [21, 23], we derive that the equilibrium E_3 is locally asymptotically stable, if $A_1 > 0$, $A_3 > 0$, $A_1A_2 > A_3$, and $A_1A_2A_3 - A_3^2 - A_1^2A_4 > 0$. Here the conditions $A_1 > 0$, $A_3 > 0$, $A_1A_2 > A_3$, and $A_1A_2A_3 - A_3^2 - A_1^2A_4 > 0$ requires

$$\begin{aligned} \mu_1 p_1 &> \xi_1 \mu x_2^* x_3^*, \quad r_1 + r_3 > r_2 + r_4, \quad p_1 p_4 \alpha_1 + L_1 > p_2^2 p_3 p_5 x_3^* + L_2, \\ r_1 L_1 + r_2 L_3 + r_3 L_4 + r_4 L_2 &> r_1 L_3 + r_2 L_1 + r_3 L_3 + r_4 L_4, \end{aligned} \quad (4.10)$$

where $r_1 = p_1 p_2 p_5 x_3^*$, $r_2 = p_1 p_2 \xi_1 x_3^*$, $r_3 = \alpha_1 \beta_1 p_6 x_2^*$, $r_4 = p_2 p_3 g_2 \alpha_1$, $L_1 = p_1 p_2^2 p_3 + p_2^2 p_3 \xi_1 x_3^*$, $L_2 = p_1^2 p_2 p_3 + \alpha_1 \beta_1 p_6 x_2^*$, $L_3 = L_2 + p_2^2 p_3 p_5 x_3^*$, $L_4 = L_1 + p_1 g_2 \alpha_1$.

Thus, we can state the following theorem.

Theorem 4.2. *The interior equilibrium E_3 if it exists is linearly asymptotically stable when*

$$\begin{aligned} \text{(i)} \quad &\mu_1 p_1 > \xi_1 \mu x_2^* x_3^*, \\ \text{(ii)} \quad &r_1 + r_3 > r_2 + r_4, \\ \text{(iii)} \quad &p_1 g_2 \alpha_1 + L_1 > p_2^2 p_3 p_5 x_3^* + L_2, \\ \text{(iv)} \quad &r_1 L_1 + r_2 L_3 + r_3 L_4 + r_4 L_2 > r_1 L_3 + r_2 L_1 + r_3 L_3 + r_4 L_4, \end{aligned} \quad (4.11)$$

are satisfied.

Now, we will study the Hopf-bifurcation [24, 25] of the system given by (2.6), taking α (i.e., constant input rate of nutrient) as the bifurcation parameter. The necessary and sufficient conditions for the existence of the Hopf-bifurcation for $\alpha = \alpha^*$, if it exists, are (i) $A_i(\alpha^*) > 0$, $i = 1, 2, 3, 4$ (ii) $A_1(\alpha^*)A_2(\alpha^*) > A_3(\alpha^*)$ (iii) $A_1(\alpha^*)A_2(\alpha^*)A_3(\alpha^*) - A_3^2(\alpha^*) - A_1^2(\alpha^*)A_4(\alpha^*) = 0$, and (iv) the eigenvalues of the characteristic equation (4.8) should be of the form $\lambda_i = u_i + iv_i$, where $du_i/d\alpha \neq 0$, $i = 1, 2, 3, 4$. The condition $A_1A_2A_3 - A_3^2 - A_1^2A_4 = 0$ becomes

$$-G_1 \alpha^3 + G_2 \alpha^2 + G_3 \alpha + G_4 = 0, \quad (4.12)$$

where $G_1 = h_1 + f_1$, $G_2 = h_2 - q_5^2 - f_2$, $G_3 = h_3 - 2q_5 q_6 - f_3$, $G_4 = h_4 - q_6^2 - f_4$, $f_1 = q_2^2 q_7$, $f_2 = q_2^2 q_8 - 2q_1 q_2 q_7$, $f_3 = q_1^2 q_7 - 2q_1 q_2 q_8$, $f_4 = q_1^2 q_8$, $h_1 = l_2 q_5$, $h_2 = l_1 q_5 - l_2 q_6$, $h_3 = l_1 q_6 + l_3 q_5$, $h_4 = l_3 q_6$, $q_1 = p_1 - k_2 g_6$, $q_2 = k_2 g_5$, $q_3 = (p_2 k_6 - p_1 k_2) g_5$, $q_4 = (p_2 k_6 - p_1 k_2) g_6 + g_2 \alpha_1$, $q_5 = (p_2 k_6 p_1 - g_2 \alpha_1 k_2) g_5 + \beta \beta_1 \alpha_1 g_3 x_2^*$, $q_6 = (p_2 k_6 p_1 - g_2 \alpha_1 k_2) g_6 + (\beta g_4 - k \gamma) \beta_1 \alpha_1 x_2^*$, $q_7 = p_2 g_2 \alpha_1 k_6 g_5$, $q_8 = p_2 k_6 g_2 \alpha_1 g_6$, $g_1 = k \gamma x_2^*$, $g_3 = 1/g_2$, $g_4 = g_1/g_2$, $g_5 = (1 + x_2^*) g_3 \beta_1 / \mu_1$, $g_6 = (1 + x_2^*) (g_4 \beta_1 - \gamma) / \mu$, $k_2 = \mu x_2^* / (1 + x_2^*)^2$, $k_6 = \mu_1 / (1 + x_2^*)^2 - \xi_1$. Therefore, one pair of eigenvalues of the characteristic equation (4.8) at $\alpha = \alpha^*$ are of the form $\lambda_{1,2} = \pm iv$, where v is positive real number.

Now, we will verify the Hopf-bifurcation condition (iv), putting $\lambda = u + iv$ in (4.8), we get

$$(u + iv)^4 + A_1(u + iv)^3 + A_2(u + iv)^2 + A_3(u + iv) + A_4 = 0. \quad (4.13)$$

On separating the real and imaginary parts and eliminating v between real and imaginary parts, we have

$$u^4 + v^4 - 6u^2v^2 + A_1u^3 - 3A_1uv^2 + A_2(u^2 - v^2) + A_3u + A_4 = 0, \quad (4.14)$$

$$-4uv^3 + 4u^3v - A_1v^3 + 3A_1u^2v + 2A_2uv + A_3v = 0. \quad (4.15)$$

Substituting the value of v^2 from (4.15) in (4.14), we get

$$\begin{aligned} & u^4(4u + A_1)^2 + \left(4u^3 + 3A_1u^2 + 2A_2u + A_3\right)^2 + \left(4u^3 + 3A_1u^2 + 2A_2u + A_3\right) \\ & \times \left(-6u^2 - 3A_1u - A_2\right)(4u + A_1) + \left(A_1u^3 + A_2u^2 + A_3u + A_4\right)(4u + A_1)^2 = 0, \end{aligned} \quad (4.16)$$

differentiating with respect to α and putting $\alpha = \alpha^*$, we have

$$\left[\frac{du}{d\alpha}\right]_{\alpha=\alpha^*} = \frac{(-d/d\alpha)(A_1A_2A_3 - A_3^2 - A_1^2A_4)}{4A_3(A_2 - A_3) - 2A_1(A_2^2 - A_1A_3) + 8A_1A_2} \neq 0. \quad (4.17)$$

Hence we can state the following theorem.

Theorem 4.3. *The system (2.6) has a Hopf-bifurcation at $\alpha^* > 0$ such that*

$$A_1(\alpha^*)A_2(\alpha^*)A_3(\alpha^*) - A_3^2(\alpha^*)^2 - A_1^2(\alpha^*)A_4(\alpha^*) = 0, \quad \left[\frac{du}{d\alpha}\right]_{\alpha=\alpha^*} \neq 0. \quad (4.18)$$

At the Hopf-bifurcation point, the equilibrium state loses its stability and bifurcates to a periodic orbit. We obtain the value of α at the Hopf-bifurcation point denoted as α^* and solve the equation

$$A_1A_2A_3 - A_3^2 - A_1^2A_4 = 0. \quad (4.19)$$

At the Hopf-bifurcation point, where the real parts of complex conjugate eigenvalues are zero, the roots of (4.8) are

$$\lambda_{1,2} = \pm vi, \quad \lambda_3 = -m_1, \quad \lambda_4 = -m_2, \quad (4.20)$$

where $v = \sqrt{A_3}/\sqrt{A_1}$, $m_1 = (-A_1^2 - \sqrt{A_1}\sqrt{A_1^3 - 4A_1A_2 + 4A_3})/2A_1$, and $m_2 = (-A_1^2 + \sqrt{A_1}\sqrt{A_1^3 - 4A_1A_2 + 4A_3})/2A_1$.

Next, we seek a transformation matrix P which reduces the matrix A to the form

$$P^{-1}AP = \begin{pmatrix} 0 & -v & 0 & 0 \\ v & 0 & 0 & 0 \\ 0 & 0 & -m_1 & 0 \\ 0 & 0 & 0 & -m_2 \end{pmatrix}, \quad (4.21)$$

where the nonsingular matrix P is given as

$$P = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ a_{31} & 0 & a_{31} & a_{31} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}, \quad (4.22)$$

where, $a_{11} = v^2c - ed\alpha_1 - \alpha_1v$, $a_{12} = \alpha_1vc + v^3 + edw$, $a_{13} = (\alpha_1 - m_1)(cm_1 + m_1^2 - ed)$, $a_{14} = (\alpha_1 - m_2)(cm_2 + m_2^2 - ed)$, $a_{22} = \alpha_1v\beta_1x_2^*$, $a_{23} = \alpha_1m_1\beta_1x_2^*$, $a_{24} = \alpha_1m_2\beta_1x_2^*$, $a_{31} = \alpha_1e\beta_1x_2^*$, $a_{41} = (\omega^2 + ed)\alpha_1$, $a_{42} = vc\alpha_1$, $a_{43} = (ed - cm_1 - m_1^2)\alpha_1$, $a_{44} = (ed - cm_2 - m_2^2)\alpha_1$, $c = \mu x_2^*x_3^*/(1 + x_2^*)^2$, $d = -\mu x_2^*/(1 + x_2^*)$, $e = (\mu_1/(1 + x_2^*)^2 - \xi_1)x_3^*$.

To achieve the normal form of (4.6), we make another change of variable, that is, $X = PY$, where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$.

Through some algebraic manipulations, (4.6) takes the form

$$\dot{Y} = \Omega Y + F, \quad (4.23)$$

where $\Omega = P^{-1}AP$ and

$$F = P^{-1}f = \begin{pmatrix} F^1(y_1, y_2, y_3, y_4) \\ F^2(y_1, y_2, y_3, y_4) \\ F^3(y_1, y_2, y_3, y_4) \\ F^4(y_1, y_2, y_3, y_4) \end{pmatrix}, \quad f \text{ is given by } f = \begin{pmatrix} f^1(y_1, y_2, y_3, y_4) \\ f^2(y_1, y_2, y_3, y_4) \\ f^3(y_1, y_2, y_3, y_4) \\ f^4(y_1, y_2, y_3, y_4) \end{pmatrix}, \quad (4.24)$$

where, $f^1(y_1, y_2, y_3, y_4) = -\beta(a_{11}y_1 + a_{12}y_2 + a_{13}y_3 + a_{14}y_4)(a_{22}y_2 + a_{23}y_3 + a_{24}y_4)$, $f^2(y_1, y_2, y_3, y_4) = -\mu p_4(a_{22}y_2 + a_{23}y_3 + a_{24}y_4)(a_{31}y_1 + a_{31}y_3 + a_{31}y_4) + \mu p_4^2(a_{22}y_2 + a_{23}y_3 + a_{24}y_4)^2(a_{31}y_1 + a_{31}y_3 + a_{31}y_4) - p_2p_4^2(a_{22}y_2 + a_{23}y_3 + a_{24}y_4)^2(a_{31}y_1 + a_{31}y_3 + a_{31}y_4) + p_2p_4^3(a_{22}y_2 + a_{23}y_3 + a_{24}y_4)^3(a_{31}y_1 + a_{31}y_3 + a_{31}y_4) + (a_{22}y_2 + a_{23}y_3 + a_{24}y_4)(a_{41}y_1 + a_{42}y_2 + a_{43}y_3 + a_{44}y_4)\beta_1 + p_2p_4(a_{22}y_2 + a_{23}y_3 + a_{24}y_4)x_3^* + \mu p_4^2(a_{22}y_2 + a_{23}y_3 + a_{24}y_4)^2x_3^* - p_2p_4^2(a_{22}y_2 + a_{23}y_3 + a_{24}y_4)^2x_3^* - \mu p_4^3(a_{22}y_2 + a_{23}y_3 + a_{24}y_4)^3x_3^* + p_2p_4^3(a_{22}y_2 + a_{23}y_3 + a_{24}y_4)^3x_3^*$, $f^3(y_1, y_2, y_3, y_4) = -p_2p_4(a_{22}y_2 + a_{23}y_3 + a_{24}y_4)(a_{31}y_1 + a_{31}y_3 + a_{31}y_4) + p_2p_4^2(a_{22}y_2 + a_{23}y_3 + a_{24}y_4)^2(a_{31}y_1 + a_{31}y_3 + a_{31}y_4) - (a_{22}y_2 + a_{23}y_3 + a_{24}y_4)(a_{31}y_1 + a_{31}y_3 + a_{31}y_4)\xi_1 - p_2p_4^3(a_{22}y_2 + a_{23}y_3 + a_{24}y_4)^3(a_{31}y_1 + a_{31}y_3 + a_{31}y_4)x_3^* - p_2p_4(a_{22}y_2 + a_{23}y_3 + a_{24}y_4)^2x_3^* + p_2p_4^2(a_{22}y_2 + a_{23}y_3 + a_{24}y_4)^2x_3^* + p_2p_4^2(a_{22}y_2 + a_{23}y_3 + a_{24}y_4)^2x_3^* + p_2p_4^2(a_{22}y_2 + a_{23}y_3 + a_{24}y_4)^2x_3^* - p_2p_4^3(a_{22}y_2 + a_{23}y_3 + a_{24}y_4)^3x_3^*$, $f^4(y_1, y_2, y_3, y_4) = 0$.

Equation (4.23) is the normal form of (4.6) from which the stability coefficient can be computed. In (4.6), on the right hand side the first term is linear and the second one is nonlinear in y 's. For evaluating the direction of bifurcating solution, we can evaluate the following quantities at $\alpha = \alpha^*$ and origin

$$g_{11} = \frac{1}{4} \left(\frac{\partial^2 F^1}{\partial y_1^2} + \frac{\partial^2 F^1}{\partial y_2^2} + i \left(\frac{\partial^2 F^2}{\partial y_1^2} + \frac{\partial^2 F^2}{\partial y_2^2} \right) \right), \quad (4.25)$$

$$g_{02} = \frac{1}{4} \left(\frac{\partial^2 F^1}{\partial y_1^2} - \frac{\partial^2 F^1}{\partial y_2^2} - 2 \frac{\partial^2 F^2}{\partial y_1 \partial y_2} + i \left(\frac{\partial^2 F^2}{\partial y_1^2} - \frac{\partial^2 F^2}{\partial y_2^2} + 2 \frac{\partial^2 F^1}{\partial y_1 \partial y_2} \right) \right), \quad (4.26)$$

$$g_{20} = \frac{1}{4} \left(\frac{\partial^2 F^1}{\partial y_1^2} - \frac{\partial^2 F^1}{\partial y_2^2} + 2 \frac{\partial^2 F^2}{\partial y_1 \partial y_2} + i \left(\frac{\partial^2 F^2}{\partial y_1^2} - \frac{\partial^2 F^2}{\partial y_2^2} - 2 \frac{\partial^2 F^1}{\partial y_1 \partial y_2} \right) \right), \quad (4.27)$$

$$G_{21} = \frac{1}{8} \left(\frac{\partial^3 F^1}{\partial y_1^3} + \frac{\partial^3 F^2}{\partial y_2^3} + \frac{\partial^3 F^2}{\partial y_1^2 \partial y_2} + \frac{\partial^3 F^1}{\partial y_1 \partial y_2^2} + i \left(\frac{\partial^3 F^2}{\partial y_1^3} + \frac{\partial^3 F^1}{\partial y_2^3} + \frac{\partial^3 F^1}{\partial y_1^2 \partial y_2} + \frac{\partial^3 F^2}{\partial y_1 \partial y_2^2} \right) \right),$$

$$h_{11}^1 = \frac{1}{4} \left(\frac{\partial^2 F^3}{\partial y_1^2} + \frac{\partial^2 F^3}{\partial y_2^2} \right), \quad h_{11}^2 = \frac{1}{4} \left(\frac{\partial^2 F^4}{\partial y_1^2} + \frac{\partial^2 F^4}{\partial y_2^2} \right),$$

$$h_{20}^1 = \frac{1}{4} \left(\frac{\partial^2 F^3}{\partial y_1^2} - \frac{\partial^2 F^3}{\partial y_2^2} - 2i \frac{\partial^2 F^3}{\partial y_1 \partial y_2} \right), \quad h_{20}^2 = \frac{1}{4} \left(\frac{\partial^2 F^4}{\partial y_1^2} - \frac{\partial^2 F^4}{\partial y_2^2} - 2i \frac{\partial^2 F^4}{\partial y_1 \partial y_2} \right),$$

$$w_{11}^1 = \frac{h_{11}^1}{m_1}, \quad w_{11}^2 = \frac{h_{11}^2}{m_2}, \quad w_{20}^1 = \frac{h_{20}^1}{m_1 + 2i\omega}, \quad w_{20}^2 = \frac{h_{20}^2}{m_2 + 2i\omega},$$

$$G_{110}^1 = \frac{1}{2} \left(\frac{\partial^2 F^1}{\partial y_1 \partial y_3} + \frac{\partial^2 F^2}{\partial y_2 \partial y_3} + i \left(\frac{\partial^2 F^2}{\partial y_1 \partial y_3} - \frac{\partial^2 F^1}{\partial y_2 \partial y_3} \right) \right),$$

$$G_{110}^2 = \frac{1}{2} \left(\frac{\partial^2 F^1}{\partial y_1 \partial y_4} + \frac{\partial^2 F^2}{\partial y_2 \partial y_4} + i \left(\frac{\partial^2 F^2}{\partial y_1 \partial y_4} - \frac{\partial^2 F^1}{\partial y_2 \partial y_4} \right) \right),$$

$$G_{101}^1 = \frac{1}{2} \left(\frac{\partial^2 F^1}{\partial y_1 \partial y_3} - \frac{\partial^2 F^2}{\partial y_2 \partial y_3} + i \left(\frac{\partial^2 F^2}{\partial y_1 \partial y_3} + \frac{\partial^2 F^1}{\partial y_2 \partial y_3} \right) \right),$$

$$G_{101}^2 = \frac{1}{2} \left(\frac{\partial^2 F^1}{\partial y_1 \partial y_4} - \frac{\partial^2 F^2}{\partial y_2 \partial y_4} + i \left(\frac{\partial^2 F^2}{\partial y_1 \partial y_4} + \frac{\partial^2 F^1}{\partial y_2 \partial y_4} \right) \right),$$

(4.28)

$$g_{21} = G_{21} + (2G_{110}^1 w_{11}^1 + G_{101}^1 w_{20}^1 + 2G_{110}^2 w_{11}^2 + G_{101}^2 w_{20}^2). \quad (4.29)$$

Thus, we can determine g_{11} , g_{20} , g_{02} , g_{21} from (4.25), (4.26), (4.27), and (4.29), respectively. Thus, we can compute the following values:

$$C_1(0) = \frac{i}{2\omega} \left[g_{20} g_{11} - |g_{11}|^2 - \frac{1}{3} |g_{02}|^2 \right] + \frac{g_{21}}{2}, \quad (4.30)$$

$$\mu_2 = \frac{-\operatorname{Re} C_1(0)}{\operatorname{Re} \lambda'(\alpha^*)}, \quad \tau_2 = -\frac{(\operatorname{Im} C_1(0) + \mu_2 \operatorname{Im} \lambda'(\alpha^*))}{v \alpha^*}, \quad \beta_2 = 2 \operatorname{Re} C_1(0),$$

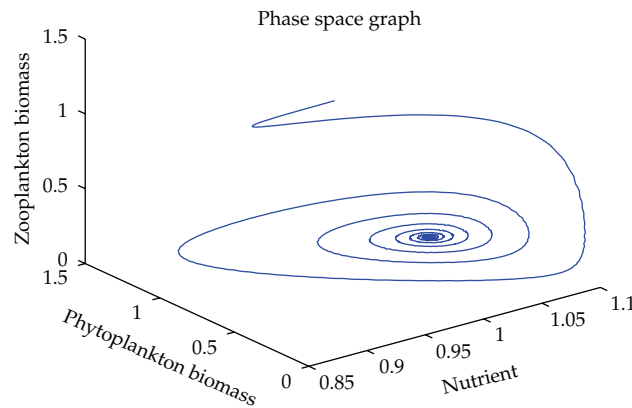


Figure 1: Phase-space diagram for nutrient, phytoplankton, and zooplankton at $\alpha = 1.1$.

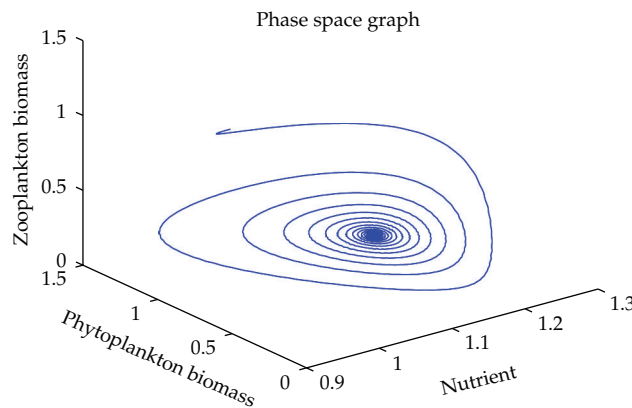


Figure 2: Phase-space diagram for nutrient, phytoplankton, and zooplankton at $\alpha = 1.1775$.

which determine the qualities of bifurcation periodic solution in the center manifold at the critical value α^* .

Theorem 4.4. *The parameter μ_2 determines the direction of the Hopf-bifurcation if $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf-bifurcation is supercritical (subcritical) and the bifurcation periodic solutions exist for $\alpha > \alpha^*$ ($\alpha < \alpha^*$); β_2 determines the stability of bifurcating periodic solution; the bifurcation periodic solutions are orbitally asymptotically stable (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$); τ_2 determines the period of the bifurcating periodic solution; the period increases (decreases) if $\tau_2 > 0$ ($\tau_2 < 0$).*

5. Numerical Example

In this section, we have performed numerical simulation for both systems (2.2) as well as (2.6) using MATLAB. We are taking parametric values $\beta = 0.3$, $\gamma = 0.2$, $k = 0.3$, $\beta_1 = 0.25$, $\mu = 0.2$, $\mu_1 = 0.19$, $\xi = 0.05$, $\xi_1 = 0.01$ for both systems. The behavior of the system (2.2) with different values of α . From Figures 1, 2, 3, 4, it is observed that as α increases, the stable system starts oscillating. Again, with these set of values and $\alpha_1 = 0.12$, we get a positive root

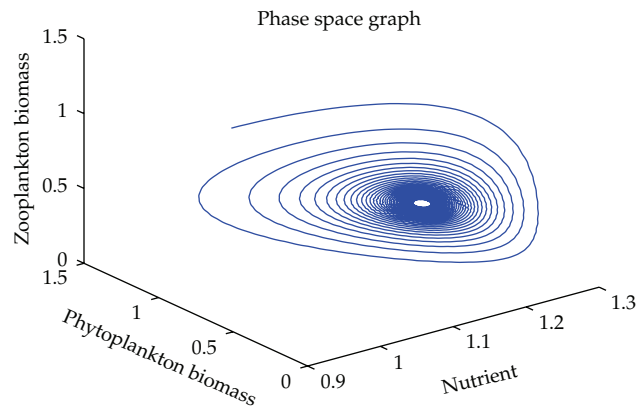


Figure 3: Stable phase-space diagram for the system with nutrient, phytoplankton, and zooplankton at $\alpha = 1.25$.

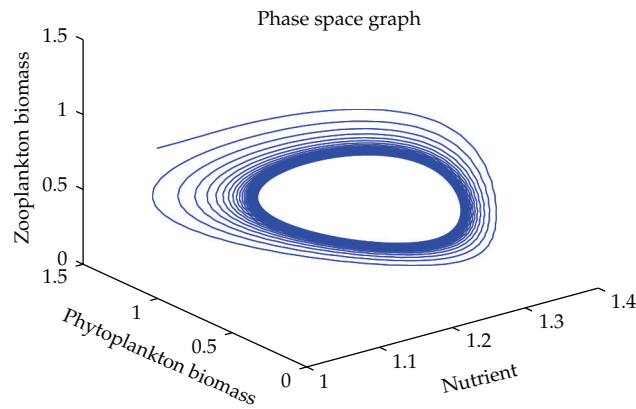


Figure 4: Phase-space diagram for nutrient, phytoplankton, and zooplankton at $\alpha = 1.3$.

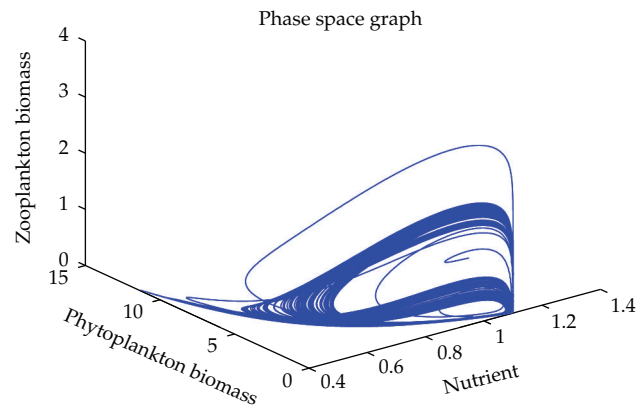


Figure 5: Phase-space diagram for nutrient, phytoplankton, and zooplankton at $\alpha = 1.1$, $\alpha_1 = 0.01$.

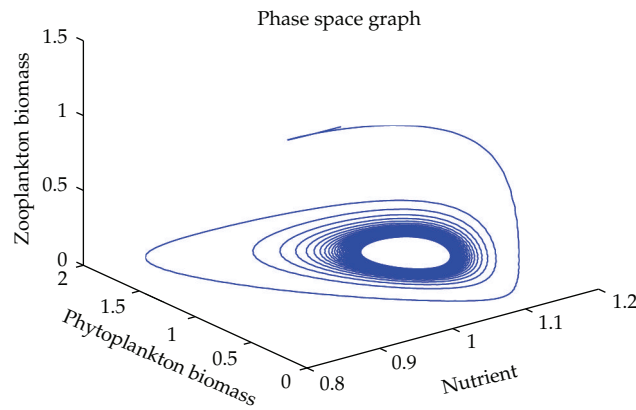


Figure 6: Phase-space diagram for nutrient, phytoplankton, and zooplankton at $\alpha = 1.1$, $\alpha_1 = 0.07$.

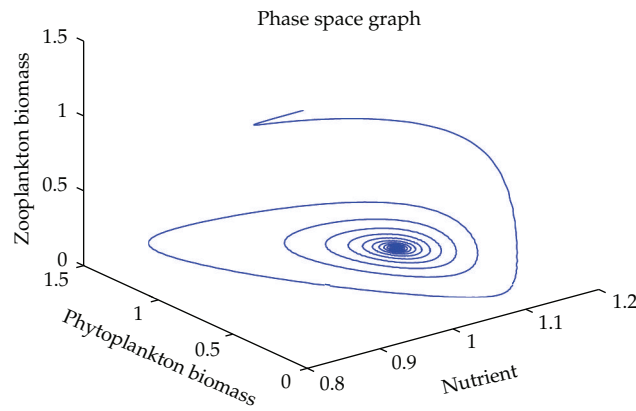


Figure 7: Stable phase-space diagram for the system with nutrient, phytoplankton, and zooplankton at $\alpha = 1.1$, $\alpha_1 = 0.12$.

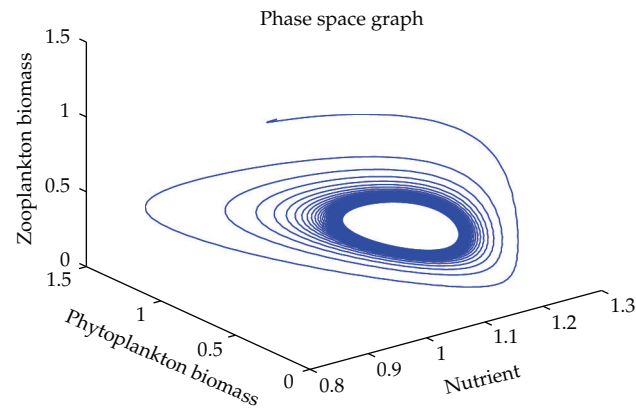


Figure 8: Phase-space diagram for nutrient, phytoplankton, and zooplankton at $\alpha = 1.1775$, $\alpha_1 = 0.12$.

of (4.12) is $\alpha = \alpha^* = 1.1680$. Therefore, one pair of eigenvalues of the characteristic equation (4.8) at $\alpha = \alpha^* = 1.1680$ are of the form $\lambda_{1,2} = \pm iv$, where v is positive real number. Here the positive interior equilibrium is $E_3(1.06505, 0.3967, 0.462745)$. It follows from Section 4, Theorem 4.3, that $\alpha^* = 1.1680$, the positive equilibrium E_3 is stable when $\alpha < \alpha^*$ (see Figure 7) and the system (2.6) undergoes a Hopf-bifurcation at $\alpha > \alpha^*$ (see Figure 8). Now keeping $\alpha = 1.1$, further reduction of α_1 at 0.01 and 0.07 are shown on Figures 5-6. Moreover, we can determine the stability and direction of periodic bifurcation from the positive equilibrium at the critical point α^* . For instance, when $\alpha = \alpha^* = 1.1680$, $g_{11} = -0.0000105695 - 0.0000200596i$, $g_{20} = 0.0000362402 - 5.6110710^{-6}i$, $g_{02} = -0.0000151012 + 0.0000457302i$, $g_{21} = -6.90210^{-8} - 4.0204810^{-8}i$, $C_1(0) = -4.0826910^{-8} + 1.6292810^{-9}i$, $u'(\alpha^*) = 0.00115975$. It follows from (4.30) that $\mu_2 > 0$ and $\beta_2 < 0$. Therefore, the bifurcation takes place when α crosses α^* to the right ($\alpha > \alpha^*$), and the corresponding periodic orbits are orbitally asymptotically stable, as depicted in Figure 8.

6. Conclusion

In this paper, we have studied the role of delay on plankton ecosystem in the presence of a toxic producing phytoplankton. In this system, it has been assumed that the toxic phytoplankton reduces the growth of zooplankton and studied the effect of delay in the digestion of nutrient by phytoplankton biomass. The local stability conditions of all the feasible equilibria of the system are established. The interior equilibrium are locally asymptotically stable under certain conditions. We have shown numerically with the set of parameters that at $\alpha_1 = 0.01$, the system exhibits the chaotic behavior (see Figure 5). Figure 6 exhibits the oscillatory behavior of the system and further it is observed that the increase in α_1 makes the system stable (see Figure 7). For the comparison of the system (2.6), without delay in the system, a numerical simulation of the system (2.2) is shown in Figures 1, 2, 3, 4. By applying the normal form theory and the center manifold theorem, we define the explicit formulae that determine the stability and direction of the bifurcating periodic solutions. For numerical experiment, it is observed that when the input rate of nutrient, α , exceeds the critical value $\alpha^*(1.1680)$, the system (2.6) leads to oscillatory behaviour shown in Figure 8. Thus, the quantitative level of abundance of system populations depends crucially on the input rate of nutrient. Further from Theorem 4.4, we can determine the direction and stability of Hopf-bifurcation. For these chosen set of parametric values, the Hopf-bifurcation is supercritical and stable. Hence, we conclude that the supply rate (α) of nutrient and delay parameter (α_1) play an important role in changing the dynamical behaviour of the system.

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