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Research Article

Modular Locally Constant Mappings in Vector Ultrametric Spaces

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We give some sufficient conditions for mappings defined on vector ultrametric spaces to be modular locally constant.

1. Introduction and Preliminaries

A metric space (X, d) in which the triangle inequality is replaced by

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}, \quad (x, y, z \in X), \tag{1.1}$$

is called an ultrametric space. Generalized ultrametric spaces were given in [1, 2] via partially ordered sets and some applications of them appeared in logic programming [3], computational logic [4], and quantitative domain theory [5].

In [6], the notion of a metric locally constant function on an ultrametric space was given in order to investigate certain groups of isometries and describe various Galois groups over local fields. Locally constant functions also appear in contexts such as higher ramification groups of finite extensions of $\mathbf{Q}_p$, and the Fontaine ring B^+_{dR}. Also, metric locally constant functions were studied in [7, 8]. On the other hand, vector ultrametric spaces are given in [9] as vectorial generalizations of ultrametrics. Hence, locally constant functions, in modular sense, can play the same role in vector ultrametric spaces as they do in usual ultrametric spaces.

In this paper, we introduce modular locally constant mappings in vector ultrametric spaces. Some sufficient conditions are given for mappings defined on vector ultrametric spaces to be modular locally constant.

We first present some basic notions.

Recall that a *modular* on a real linear space $\mathcal{A}$ is a real valued functional ρ on $\mathcal{A}$ satisfying the conditions:

(1) $\rho(x) = 0$ if and only if $x = 0$,

(2) $\rho(x) = \rho(-x)$,

(3) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$, for all $x, y \in \mathcal{A}$ and $\alpha, \beta \geq 0$, $\alpha + \beta = 1$.

Then, the linear subspace

$$\mathcal{A}_\rho = \{x \in \mathcal{A} : \rho(\alpha x) \longrightarrow 0 \text{ as } \alpha \longrightarrow 0\} \tag{1.2}$$

of $\mathcal{A}$ is called a *modular space*.

A sequence $(x_n)_{n=1}^{\infty}$ in $\mathcal{A}_\rho$ is called *ρ-convergent* (briefly, convergent) to $x \in \mathcal{A}_\rho$ if $\rho(x_n - x) \to 0$ as $n \to \infty$, and is called *Cauchy sequence* if $\rho(x_m - x_n) \to 0$ as $m, n \to \infty$. By a *ρ-closed* (briefly, closed) set in $\mathcal{A}_\rho$ we mean a set which contains the limit of each of its convergent sequences. Then, $\mathcal{A}_\rho$ is a *complete modular* space if every Cauchy sequence in $\mathcal{A}_\rho$ is convergent to a point of $\mathcal{A}_\rho$. We refer to [10, 11] for more details.

A *cone* $\mathcal{P}$ in a complete modular space $\mathcal{A}_\rho$ is a nonempty set such that

(i) $\mathcal{P}$ is ρ-closed, and $\mathcal{P} \neq \{0\}$;

(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in \mathcal{P} \Rightarrow ax + by \in \mathcal{P}$;

(iii) $\mathcal{P} \cap (-\mathcal{P}) = \{0\}$, where $-\mathcal{P} = \{-x : x \in \mathcal{P}\}$.

Let $\preceq$ be the partial order on $\mathcal{A}_\rho$ induced by the cone $\mathcal{P}$, that is, $x \preceq y$ whenever $y - x \in \mathcal{P}$. The cone $\mathcal{P}$ is called *normal* if

$$0 \preceq x \preceq y \Longrightarrow \rho(x) \leq \rho(y), \quad (x, y \in \mathcal{A}_\rho). \tag{1.3}$$

The cone $\mathcal{P}$ is said to be *unital* if there exists a vector $e \in \mathcal{P}$ with modular 1 such that

$$x \preceq \rho(x) e, \quad (x \in \mathcal{P}). \tag{1.4}$$

Example 1.1. Consider the real vector space $C[0,1]$ consisting of all real-valued continuous functions on $[0,1]$ equipped with the modular ρ defined by

$$\rho(x) = \max_{t \in [0,1]} |x(t)|^2, \quad (x \in C[0,1]). \tag{1.5}$$

It is not difficult to see that $C[0,1]$ is a complete modular space and

$$\mathcal{P} = \{x \in C[0,1] : x(t) \geq 0, \forall t \in [0,1]\} \tag{1.6}$$

is a normal cone in $C[0,1]$.

Example 1.2. The vector space $C^1[0,1]$ consisting of all continuously differentiable real-valued functions on $[0,1]$ equipped with the modular ρ defined by

$$\rho(x) = \max_{t\in[0,1]}|x(t)| + \max_{t\in[0,1]}\left|x'(t)\right|, \quad \left(x \in C^1[0,1]\right) \tag{1.7}$$

constitutes a complete modular space. The subset

$$\mathcal{P} = \left\{x \in C^1[0,1] : x(t) \geq 0, \forall t \in [0,1]\right\} \tag{1.8}$$

is a unital cone in $C^1[0,1]$ with unit 1. The cone $\mathcal{P}$ is not normal since, for example, $x(t) = t^n \preceq 1$, for $n \geq 1$ does not imply that $\rho(x) \leq \rho(1)$.

Throughout this note, we suppose that $\mathcal{P}$ is a cone in complete modular space $\mathcal{A}_\rho$, and $\preceq$ is the partial order induced by $\mathcal{P}$.

Definition 1.3. A *vector ultrametric* on a nonempty set $\mathcal{X}$ is a mapping $d : \mathcal{X} \times \mathcal{X} \to \mathcal{A}_\rho$ satisfying the conditions:

(CUM1) $d(x,y) \succeq 0$ for all $x, y \in \mathcal{X}$ and $d(x,y) = 0$ if and only if $x = y$;

(CUM2) $d(x,y) = d(y,x)$ for all $x, y \in \mathcal{X}$;

(CUM3) If $d(x,z) \preceq p$ and $d(y,z) \preceq p$, then $d(x,y) \preceq p$, for any $x, y, z \in \mathcal{X}$, and $p \in \mathcal{P}$.

Then the triple $(\mathcal{X}, d, \mathcal{P})$ is called a vector ultrametric space. If $\mathcal{P}$ is unital and normal, then $(\mathcal{X}, d, \mathcal{P})$ is called a *unital-normal* vector ultrametric space.

For unital-normal vector ultrametric space $(\mathcal{X}, d, \mathcal{P})$, since

$$d(x,y) \preceq \rho(d(x,y))e, \qquad d(y,z) \preceq \rho(d(y,z))e, \tag{1.9}$$

from (CUM3) we get

$$d(x,z) \preceq \max\{\rho(d(x,y)), \rho(d(y,z))\}e, \tag{1.10}$$

and therefore

$$\rho(d(x,z)) \leq \max\{\rho(d(x,y)), \rho(d(y,z))\}. \tag{1.11}$$

Let $(\mathcal{X}, d, \mathcal{P})$ be a unital-normal vector ultrametric space. If $x \in \mathcal{X}$ and $p \in \mathcal{P} \setminus \{0\}$, *the ball* $B(x,p)$ *centered at* x *with radius* p is defined as

$$B(x,p) := \{y \in \mathcal{X} : \rho(d(x,y)) \leq \rho(p)\}. \tag{1.12}$$

The unital-normal vector ultrametric space $(\mathcal{X}, d, \mathcal{P})$ is called *spherically complete* if every chain of balls (with respect to inclusion) has a nonempty intersection.

The following lemma may be easily obtained.

Lemma 1.4. *Let $(\mathcal{X}, d, \mathcal{P})$ be a unital-normal vector ultrametric space.*

(1) *If $a, b \in \mathcal{X}$, $0 \preceq p$ and $b \in B(a, p)$, then $B(a, p) = B(b, p)$.*

(2) *If $a, b \in \mathcal{X}$, $0 \prec p \preceq q$, then either $B(a, p) \bigcap B(b, q) = \emptyset$ or $B(a, p) \subseteq B(b, q)$.*

Definition 1.5. Let $(\mathcal{X}, d, \mathcal{P})$ be a unital-normal vector ultrametric space. A mapping $f : \mathcal{X} \to \mathcal{P} \setminus \{0\}$ is said to be *modular locally constant* provided that for any $x \in \mathcal{X}$ and any $y \in B(x, f(x))$ one has $\rho(f(x)) = \rho(f(y))$.

2. Main Theorem

Theorem 2.1. *Let $(\mathcal{X}, d, \mathcal{P})$ be a spherically complete unital-normal vector ultrametric space and $T : \mathcal{X} \to \mathcal{X}$ be a mapping such that for every $x, y \in \mathcal{X}$, $x \neq y$, either*

$$\rho(d(Tx, Ty)) < \max\{\rho(d(x, Tx)), \rho(d(y, Ty))\} \tag{2.1}$$

or

$$\rho(d(Tx, Ty)) \leq \rho(d(x, y)). \tag{2.2}$$

Then there exists a subset B of $\mathcal{X}$ such that $T : B \to B$ and the mapping

$$f(x) = d(x, Tx), \quad (x \in B) \tag{2.3}$$

is modular locally constant.

Proof. Let $\mathcal{E} = \{B_a\}_{a \in \mathcal{X}}$ where $B_a = B(a, d(a, Ta))$. Consider the partial order $\sqsubseteq$ on $\mathcal{E}$ defined by

$$B_a \sqsubseteq B_b \quad \text{iff } B_b \subseteq B_a, \tag{2.4}$$

where $a, b \in \mathcal{X}$. If $\mathcal{E}_1$ is any chain in $\mathcal{E}$, then the spherical completeness of $(\mathcal{X}, d, \mathcal{P})$ implies that the intersection Ω of elements of $\mathcal{E}_1$ is nonempty.

Suppose that (2.1) holds. Let $b \in \Omega$ and $B_a \in \mathcal{E}_1$. Obviously $b \in B_a$, so $\rho(d(a,b)) \leq \rho(d(a,Ta))$. For any $x \in B_b$, we have

$$\begin{aligned}
\rho(d(x,b)) &\leq \rho(d(b,Tb)) \\
&\leq \max\{\rho(d(b,a)), \rho(d(a,Ta)), \rho(d(Ta,Tb))\} \\
&< \max\{\rho(d(b,a)), \rho(d(a,Ta)), \max\{\rho(d(a,Ta)), \rho(d(b,Tb))\}\} \quad (\text{by } (2.1)) \\
&\leq \max\{\rho(d(b,a)), \rho(d(a,Ta)), \rho(d(b,Tb))\} \\
&\leq \max\{\rho(d(a,Ta)), \rho(d(b,Tb))\} \\
&= \rho(d(a,Ta)), \\
\rho(d(x,a)) &\leq \max\{\rho(d(x,b)), \rho(d(b,a))\} \leq \rho(d(a,Ta)).
\end{aligned} \tag{2.5}$$

So for every $B_a \in \mathcal{E}_1$, $B_b \subseteq B_a$; that is, B_b is an upper bound in $\mathcal{E}$ for the family $\mathcal{E}_1$. By Zorn's lemma, there exists a maximal element in $\mathcal{E}_1$, say B_z. If $b \in B_z$, $\rho(d(b,z)) \leq \rho(d(z,Tz))$, and we get

$$\begin{aligned}
\rho(d(b,Tb)) &\leq \max\{\rho(d(b,z)), \rho(d(z,Tz)), \rho(d(Tz,Tb))\} \\
&< \max\{\rho(d(b,z)), \rho(d(z,Tz)), \max\{\rho(d(z,Tz)), \rho(d(b,Tb))\}\} \quad (\text{by } (2.1)) \\
&\leq \max\{\rho(d(b,z)), \rho(d(z,Tz)), \rho(d(b,Tb))\} \\
&\leq \max\{\rho(d(z,Tz)), \rho(d(b,Tb))\} \\
&= \rho(d(z,Tz)).
\end{aligned} \tag{2.6}$$

Then

$$\rho(d(b,Tb)) \leq \rho(d(z,Tz)). \tag{2.7}$$

Since $b \in B_b \cap B_z$, we have $B_b \subseteq B_z$ by Lemma 1.4. But $Tb \in B_b$, so $T : B_z \to B_z$. Now we show that $\rho(f(b)) = \rho(f(z))$ for every $b \in B_z$. It is clear that $\rho(d(b,Tb)) \leq \rho(d(z,Tz))$, for all $b \in B_z$. Suppose $\rho(d(b,Tb)) < \rho(d(z,Tz))$ for some $b \in B_z$. We have $\rho(d(b,z)) \leq \rho(d(z,Tz))$, and

$$\begin{aligned}
\rho(d(z,Tz)) &\leq \max\{\rho(d(z,b)), \rho(d(b,Tb)), \rho(d(Tb,Tz))\} \\
&< \max\{\rho(d(b,z)), \rho(d(b,Tb)), \max\{\rho(d(b,Tb)), \rho(d(z,Tz))\}\} \quad (\text{by } (2.1)) \\
&\leq \max\{\rho(d(b,z)), \rho(d(b,Tb)), \rho(d(z,Tz))\} \\
&\leq \max\{\rho(d(b,Tb)), \rho(d(z,Tz))\} \\
&= \rho(d(z,Tz)).
\end{aligned} \tag{2.8}$$

which is a contradiction. Thus f is modular locally constant on B_z.

Suppose that (2.2) holds. As above, let $b \in \Omega$ and $B_a \in \mathcal{E}_1$. Obviously $b \in B_a$, so $\rho(d(a,b)) \leq \rho(d(a,Ta))$. For any $x \in B_b$, we have

$$\begin{aligned} \rho(d(x,b)) &\leq \rho(d(b,Tb)) \\ &\leq \max\{\rho(d(b,a)),\rho(d(a,Ta)),\rho(d(Ta,Tb))\} \\ &\leq \max\{\rho(d(b,a)),\rho(d(a,Ta))\} \quad (\text{by } (2.2)) \\ &= \rho(d(a,Ta)). \end{aligned} \tag{2.9}$$

Thus

$$\begin{aligned} \rho(d(x,a)) &\leq \max\{\rho(d(x,b)),\rho(d(b,a))\} \leq \rho(d(a,Ta)), \\ \rho(d(x,a)) &\leq \max\{\rho(d(x,b)),\rho(d(b,a))\} \leq \rho(d(a,Ta)). \end{aligned} \tag{2.10}$$

So, for every $B_a \in \mathcal{E}_1$, $B_b \subseteq B_a$; that is, B_b is an upper bound for the family $\mathcal{E}_1$. Again, by Zorn's lemma there exists a maximal element in $\mathcal{E}_1$, say B_z. For any $b \in B_z$, we have

$$\begin{aligned} \rho(d(b,Tb)) &\leq \max\{\rho(d(b,z)),\rho(d(z,Tz)),\rho(d(Tz,Tb))\} \\ &\leq \max\{\rho(d(b,z)),\rho(d(z,Tz)),\rho(d(z,b))\} \quad (\text{by } (2.2)) \\ &= \rho(d(z,Tz)). \end{aligned} \tag{2.11}$$

This implies that $b \in B_b \cap B_z$, and Lemma 1.4 gives $B_b \subseteq B_z$. Since $Tb \in B_b$, so $T : B_z \to B_z$.

If $z = Tz$, then $f(x) = 0$ on B_z and this yields the result. If $z \neq Tz$, we show that $\rho(f(b)) = \rho(f(z))$ for every $b \in B_z$. Since $\rho(d(b,Tb)) \leq \rho(d(z,Tz))$ for any $b \in B_z$, let us suppose that for some $b \in B_z$, $\rho(d(b,Tb)) < \rho(d(z,Tz))$. So $\rho(d(b,z)) \leq \rho(d(z,Tz))$ and

$$\begin{aligned} \rho(d(z,Tz)) &\leq \max\{\rho(d(z,b)),\rho(d(b,Tb)),\rho(d(Tb,Tz))\} \\ &\leq \max\{\rho(d(b,z)),\rho(d(b,Tb)),\rho(d(z,b))\} \quad (\text{by } (2.2)) \\ &= \rho(d(b,z)), \end{aligned} \tag{2.12}$$

thus $\rho(d(b,z)) = \rho(d(z,Tz))$. But $\rho(d(b,z)) = \rho(d(z,Tz)) > \rho(d(b,Tb))$ implies that $z \in B_z$, but $z \notin B_b$ and hence $B_b \subsetneq B_z$which contradicts the maximality of B_z. This completes the proof. □

In the following, we assume that $(\mathcal{X}, d, \mathcal{D})$ is a spherically complete unital-normal vector ultrametric space.

Corollary 2.2. *Let $T : \mathcal{X} \to \mathcal{X}$ be a mapping such that for all $x, y \in \mathcal{X}$, $x \neq y$,*

$$\rho(d(Tx,Ty)) < \max\{\rho(d(y,Tx)),\rho(d(x,Ty))\}. \tag{2.13}$$

Then there exists a subset B of $\mathcal{X}$ such that $T : B \to B$ and the mapping f defined in (2.3) *is modular locally constant.*

Proof. Since

$$\begin{aligned}\rho(d(y,Tx)) &\leq \max\{\rho(d(y,x)),\rho(d(x,Tx))\},\\ \rho(d(x,Ty)) &\leq \max\{\rho(d(x,y)),\rho(d(y,Ty))\},\end{aligned} \tag{2.14}$$

for all $x, y \in \mathcal{X}, \quad x \neq y$, we get

$$\rho(d(x,y)) \leq \max\{\rho(d(x,Tx)),\rho(d(Tx,Ty)),\rho(d(Ty,y))\} \tag{2.15}$$

for all $x, y \in \mathcal{X}, \quad x \neq y$. Now, if

$$\max\{\rho(d(x,Tx)),\rho(d(y,Ty))\} < \rho(d(Tx,Ty)), \tag{2.16}$$

then

$$\begin{aligned}\rho(d(Tx,Ty)) &< \max\{\rho(d(y,Tx)),\rho(d(x,Ty))\} && (\text{by } (2.13))\\ &\leq \max\{\rho(d(x,y)),\rho(d(x,Tx)),\rho(d(y,Ty))\} && (\text{by } (2.14))\\ &\leq \max\{\rho(d(x,Tx)),\rho(d(Tx,Ty)),\rho(d(Ty,y))\}\\ &= \rho(d(Tx,Ty)), && (\text{by } (2.16))\end{aligned} \tag{2.17}$$

which is a contradiction. Thus $\rho(d(Tx,Ty)) \leq \max\{\rho(d(x,Tx)),\rho(d(y,Ty))\}$, and so

$$\rho(d(x,y)) \leq \max\{\rho(d(x,Tx)),\rho(d(y,Ty))\}. \tag{2.18}$$

Therefore

$$\begin{aligned}\rho(d(Tx,Ty)) &< \max\{\rho(d(y,Tx)),\rho(d(x,Ty))\}, && (\text{by } (2.13))\\ &\leq \max\{\rho(d(x,y)),\rho(d(x,Tx)),\rho(d(y,Ty))\} && (\text{by } (2.14))\\ &\leq \max\{\rho(d(x,Tx)),\rho(d(y,Ty))\}. && (\text{by } (2.18))\end{aligned} \tag{2.19}$$

Now, Theorem 2.1 completes the proof. □

Corollary 2.3. *Let $T : \mathcal{X} \to \mathcal{X}$ be a mapping such that for all $x, y \in \mathcal{X}, \quad x \neq y$,*

$$\rho(d(Tx,Ty)) < \rho(d(x,y)). \tag{2.20}$$

Then there exists a subset B of $\mathcal{X}$ such that $T : B \to B$ and the mapping f defined in (2.3) *is modular locally constant.*

Proof. We have

$$\begin{aligned}\rho(d(x,y)) &\le \max\{\rho(d(x,Tx)),\rho(d(Tx,Ty)),\rho(d(Ty,y))\}\\ &< \max\{\rho(d(x,y)),\rho(d(x,Tx)),\rho(d(Ty,y))\} \quad \text{(by (2.20))}\\ &\le \max\{\rho(d(x,Tx)),\rho(d(y,Ty))\},\end{aligned} \tag{2.21}$$

for all $x, y \in \mathcal{X}$, $\quad x \neq y$. Again, Theorem 2.1, completes the proof. □

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