Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 518719, 16 pages
doi:10.1155/2011/518719

Research Article

Analysis on a Stochastic Predator-Prey Model with Modified Leslie-Gower Response

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Received 30 October 2010; Accepted 30 March 2011

Academic Editor: Wing-Sum Cheung

This paper presents an investigation of asymptotic properties of a stochastic predator-prey model with modified Leslie-Gower response. We obtain the global existence of positive unique solution of the stochastic model. That is, the solution of the system is positive and not to explode to infinity in a finite time. And we show some asymptotic properties of the stochastic system. Moreover, the sufficient conditions for persistence in mean and extinction are obtained. Finally we work out some figures to illustrate our main results.

1. Introduction

The dynamic interaction between predators and their prey has been one of the dominant themes in mathematical biology due to its universal existence and importance. Much literature exists on the general problem of food chains in the classical Lotka-Volterra model. In [1, 2], Leslie introduced a predator-prey model where the capacity of the predators environment is proportional to the number of preys. Leslie stresses the fact that there are upper limits to the rates of increase of both prey and predator, which are not recognized in the Lotka-Volterra model. Broer et al. [3] studied the dynamical properties of a predator-prey model with nonmonotonic response function. Reference [4] considered two-species autonomous system which incorporated a modified Leslie-Gower functional response as well as that of the Holling II as follows:

$$\begin{aligned} \frac{dx}{dt} &= x\left[a - bx - \frac{cy}{x + k_1}\right], \\ \frac{dy}{dt} &= y\left[r - \frac{fy}{x + k_2}\right], \end{aligned} \tag{1.1}$$

where a, b, c, r, f, k_1, and k_2 are all positive constants and $x(t)$, $y(t)$ represent the population densities at time t.

Hsu and Huang [5] studied the global stability property of the following predator-prey system:

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right) - yp(x), \\ \frac{dy}{dt} &= y\left(s\left(1-\frac{hy}{x}\right)\right), \\ x_0 > 0,& \quad y_0 > 0, \quad r, s, k, h > 0. \end{aligned} \tag{1.2}$$

Recently, [6] discussed the following model with modified Leslie-Gower response:

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right) - \frac{n\lambda xy}{\lambda x + Ay}, \\ \frac{dy}{dt} &= y\left(s\left(1-\frac{hy}{\lambda x + b}\right)\right), \\ x_0 > 0,& \quad y_0 > 0, \end{aligned} \tag{1.3}$$

where $r, k, n, A, s, h, b,$ and λ are all positive constants and r, s are the growth rates of prey x and predator y, respectively. Here, we change the form of the predator-prey model above which reads

$$\begin{aligned} \frac{dx}{dt} &= x\left(a - bx - \frac{cy}{\lambda x + Ay}\right), \\ \frac{dy}{dt} &= y\left(f - \frac{gy}{\lambda x + h}\right), \\ x_0 > 0,& \quad y_0 > 0. \end{aligned} \tag{1.4}$$

As a matter of fact, population systems are often subject to environmental noise. Recently, more and more interest is focused on stochastic systems. Reference [7] investigated the predator-prey model with modified Leslie-Gower and Holling-type II schemes with stochastic perturbation:

$$\begin{aligned} dx &= x\left[a - bx - \frac{cy}{x+m}\right]dt + \sigma_1 x dB_1(t), \\ dy &= y\left[r - \frac{fy}{x+m}\right]dt - \sigma_2 y dB_2(t). \end{aligned} \tag{1.5}$$

By virtue of comparison theorem, [7] obtained some interesting results, including globally positive solutions, persistence in mean and extinction. Moreover, [8] continued to consider the stochastic ratio-dependent predator-prey system:

$$\begin{aligned} dx &= x\left[a - bx - \frac{cy}{x+my}\right]dt + \sigma_1 x dB_1(t), \\ dy &= y\left[-g + \frac{fx}{x+my}\right]dt - \sigma_2 y dB_2(t), \end{aligned} \tag{1.6}$$

where $B_i(t)$, $i = 1,2$, are independent standard Brownian motions. And [8] also obtained some nice conclusions on the stochastic model.

According to (1.4), taking into account the effect of randomly fluctuating environment, we will consider the corresponding autonomous stochastic system described by the Itô equation

$$\begin{aligned} dx(t) &= x(t)\left[a - bx(t) - \frac{cy(t)}{\lambda x(t) + Ay(t)}\right]dt + \sigma_1 x(t) dB_1(t), \\ dy(t) &= y(t)\left[f - \frac{gy(t)}{\lambda x(t) + h}\right]dt + \sigma_2 y(t) dB_2(t), \end{aligned} \tag{1.7}$$

where $B_i(t)$, $i = 1,2$, are independent standard Brownian motions and $a, b, c, f, g, \lambda, A, \sigma_1$, and σ_2 are all positive.

When white noise is taken into account in our model (1.7), we obtain the global existence of positive unique solution of the stochastic model, that is, the solution of the system is positive and not to explode to infinity in a finite time in Section 2. Section 3 shows some fundamental asymptotic properties of the stochastic system. Moreover, the sufficient conditions for persistence in mean and extinction are obtained in Section 3. The main contributions of this paper are therefore clear.

Throughout the paper, we use K to denote a positive constant whose exact value may be different in different appearances.

2. Positive and Global Solution

As $x(t)$, $y(t)$ of the SDE (1.7) are sizes of the species in the system at time t, it is obvious that the positive solutions are of interest. The coefficients of (1.7) are locally Lipschitz continuous and do not satisfy the linear growth condition, so the solution of (1.7) may explode at a finite time. The following Theorem shows that the solution will not explode at a finite time.

Theorem 2.1. *For a given initial value $X_0 = (x_0, y_0) \in R_+^2$, there is a unique positive solution $X(t) = (x(t), y(t))$ to (1.7) on $t \geq 0$, and the solution will remain in R_+^2 with probability one, namely, $X(t) \in R_+^2$ for all $t \geq 0$ almost surely.*

Proof. The proof is similar to [9, 10]. Since the coefficients of the equation are locally Lipschitz continuous, for a given initial value $X_0 = (x_0, y_0) \in R_+^2$, there is a unique local solution $X(t)$ on $t \in [0, \tau_e)$, where τ_e is the explosion time. To show that this solution is global, we need to show that $\tau_e = +\infty$ a.s. Let $k_0 > 0$ be sufficiently large for every component of $x(t)$ and $y(t)$ all lying within the interval $[1/k_0, k_0]$. For each integer $k \geq k_0$, define the stopping time

$$\tau_m = \inf\left\{t \in [0, \tau_e) : x(t) \notin \left(\frac{1}{k}, k\right) \text{ or } y(t) \notin \left(\frac{1}{k}, k\right)\right\}, \tag{2.1}$$

where throughout this paper we set $\inf \emptyset = \infty$. Obviously, τ_k is increasing as $k \to \infty$. Let $\tau_\infty = \lim_{k\to\infty} \tau_k$, whence $\tau_\infty \leq \tau_e$ a.s. If we can show that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ a.s. and $X(t) \in R_+^2$ a.s. for all $t \geq 0$. So we just prove that $\tau_\infty = \infty$ a.s. If not, there is $\epsilon \in (0, 1)$ and $T > 0$ such that

$$P\{\tau_\infty \leq T\} > \epsilon. \tag{2.2}$$

Hence, there is an integer $k_1 \geq k_0$ such that $P\{\tau_k \leq T\} \geq \epsilon$ for all $k \geq k_1$. Define a function $V : R_+^2 \to R_+$ by $V(x, y) = (x - 1 - \ln x) + (y - 1 - \ln y)$. The nonnegativity of this function can be seen from

$$u - 1 - \ln u \geq 0, \quad \text{on } u > 0. \tag{2.3}$$

If $X(t) = (x(t), y(t)) \in R_+^2$, by virtue of $u \leq 2[u - 1 - \ln u] + 2$ on $u > 0$, we obtain

$$\begin{aligned} LV(x, y) &= (x-1)\left(a - bx - \frac{cy}{\lambda x + Ay}\right) + (y-1)\left(f - \frac{gy}{\lambda x + h}\right) + \frac{\sigma_1^2 + \sigma_2^2}{2} \\ &= ax - bx^2 - \frac{cxy}{\lambda x + Ay} - a + bx + \frac{cy}{\lambda x + Ay} + fy - f - \frac{gy^2}{\lambda x + h} + \frac{gy}{\lambda x + h} + \frac{\sigma_1^2 + \sigma_2^2}{2} \\ &\leq K_1 V(x, y) + K_2, \end{aligned} \tag{2.4}$$

dropping t from $x(t)$ and $y(t)$. Making use of the Itô formula yields

$$EV(x(\tau_k \wedge T), y(\tau_k \wedge T)) \leq V(x_0, y_0) + K_2 T + K_1 \int_0^T EV(x(\tau_k \wedge T), y(\tau_k \wedge T)). \tag{2.5}$$

The Gronwall inequality yields

$$EV(x(\tau_k \wedge T), y(\tau_k \wedge T)) \leq [V(x_0, y_0) + K_2 T] e^{K_1 T}. \tag{2.6}$$

Set $\Omega_k = \tau_k \leq T$ for $k \geq k_1$; then $P(\Omega_k) \geq \epsilon$. Note that, for every $\omega \in \Omega$, there is $x(\tau_k, \omega)$ or $y(\tau_k, \omega)$ equal to either k or $1/k$, and hence $V(x(\tau_k, \omega))$ is no less than either

$$k - 1 - \ln k \tag{2.7}$$

or

$$\frac{1}{k} - 1 - \ln\left(\frac{1}{k}\right) = \frac{1}{k} - 1 + \ln k. \tag{2.8}$$

Therefore

$$V\left(x(\tau_k \wedge T), y(\tau_k \wedge T)\right) \geq \left([k - 1 - \ln k] \wedge \left[\frac{1}{k} - 1 + \ln k\right]\right). \tag{2.9}$$

So

$$\begin{aligned} \left[V\left(x_0, y_0\right) + K_2 T\right] e^{K_1 T} &\geq E\left[1_{\Omega_k} V\left(x(\tau_k \wedge T), y(\tau_k \wedge T)\right)\right] \\ &\geq \epsilon\left([k - 1 - \ln k] \wedge \left[\frac{1}{k} - 1 + \ln k\right]\right), \end{aligned} \tag{2.10}$$

where 1_{Ω_k} is the indicator function of Ω_k. Letting $k \to \infty$ implies the contradiction

$$\infty > \left[V\left(x_0, y_0\right) + KT\right] = \infty. \tag{2.11}$$

So we have that $\tau_\infty = \infty$ a.s. The proof is complete. □

Theorem 2.1 shows that the solution of the SDE (1.7) will remain in the positive cone R_+^2 for any initial value $(x_0, y_0) \in R_+^2$. The conclusion is fundamental which will be used later.

3. Asymptotic Behavior

3.1. Limit Results

To begin our discussion, we impose the following assumption:

(H) $a - c/A - \sigma_1^2/2 > 0$, $f - \sigma_2^2/2 > 0$.

And we list the interesting lemma as follows.

Lemma 3.1 (see [7, 8]). *Consider one-dimensional stochastic differential equation*

$$dx = x[a - bx]dt + \sigma x dB(t), \tag{3.1}$$

where $a, b,$ *and* σ *are positive and* $B(t)$ *is standard Brownian motion. Under condition* $a > \sigma^2/2$, *for any initial value* $x_0 > 0$, *the solution* $x(t)$ *has the properties*

$$\lim_{t\to\infty}\frac{\ln x(t)}{t} = 0, \quad a.s,$$
$$\lim_{t\to\infty}\frac{1}{t}\int_0^t x(s)ds = \frac{a-\sigma^2/2}{b}, \quad a.s. \tag{3.2}$$

To demonstrate asymptotic properties of the stochastic system (1.7), we firstly discuss the long time behavior of $\ln x(t)/t$ and $\ln y(t)/t$.

On the one hand, by the comparison theorem of stochastic equations, it is obvious that

$$dx \le x[a - bx]dt + \sigma_1 x dB_1(t). \tag{3.3}$$

Denote by $X_2(t)$ the solution of the following stochastic equation:

$$dX_2 = X_2[a - bX_2]dt + \sigma_1 X_2 dB_1(t),$$
$$X_2(0) = x_0. \tag{3.4}$$

We have that

$$x(t) \le X_2(t), \quad t \in [0, +\infty), \text{ a.s.} \tag{3.5}$$

On the other hand, by the comparison theorem of stochastic equations, it is obvious that we denote by X_1 the solution of stochastic differential equation

$$dX_1 = X_1\left[a - \frac{c}{A} - bX_1\right]dt + \sigma_1 X_1 dB_1(t),$$
$$X_1(0) = x_0. \tag{3.6}$$

Consequently

$$x(t) \ge X_1(t), \quad t \in [0, +\infty), \text{ a.s.} \tag{3.7}$$

To sum up, we have that

$$X_1(t) \le x(t) \le X_2(t), \quad t \in [0, +\infty), \text{ a.s.} \tag{3.8}$$

So we have the explicit solutions of $X_1(t)$ and $X_2(t)$ as follows:

$$X_1(t) = \frac{e^{[(a-c/A-\sigma_1^2/2)t+\sigma_1 B_1(t)]}}{1/x_0 + b\int_0^t e^{[(a-c/A-\sigma_1^2/2)s+\sigma_1 B_1(s)]}ds}, \tag{3.9}$$

$$X_2(t) = \frac{e^{[(a-\sigma_1^2/2)t+\sigma_1 B_1(t)]}}{1/x_0 + b\int_0^t e^{[(a-\sigma_1^2/2)s+\sigma_1 B_1(s)]}ds}. \tag{3.10}$$

Theorem 3.2. *Under assumption (H), for any initial value $x_0 > 0$, the solutions $X_1(t)$ and $X_2(t)$ satisfy*

$$\begin{aligned} &\lim_{t\to\infty} \frac{\ln X_1(t)}{t} = 0, \quad a.s., \\ &\lim_{t\to\infty} \frac{\ln X_2(t)}{t} = 0, \quad a.s. \end{aligned} \tag{3.11}$$

Proof. By assumption (H) and Lemma 3.1, the assertion is straightforward. □

Theorem 3.3. *Under assumption (H), for any initial value $x_0 > 0$, the solution $x(t)$ satisfies*

$$\lim_{t\to\infty} \frac{\ln x(t)}{t} = 0, \quad a.s. \tag{3.12}$$

Proof. By virtue of (3.8) and Theorem 3.2, we can imply the desired assertion. □

Now let us continue to consider the asymptotic behavior of the species $y(t)$. By the comparison theorem of stochastic equations, we have that

$$dy(t) \le y(t)\left(f - \frac{gy(t)}{\lambda X_2(t) + h}\right)dt + \sigma_2 y(t)dB_2(t). \tag{3.13}$$

Denote by $Y_2(t)$ the solution of the stochastic equation as follows:

$$\begin{aligned} dY_2 &= Y_2\left(f - \frac{gY_2}{\lambda X_2(t) + h}\right)dt + \sigma_2 Y_2 dB_2(t), \\ Y_2(0) &= y_0. \end{aligned} \tag{3.14}$$

We have that

$$y(t) \le Y_2(t), \quad t \in [0, +\infty), \ \text{a.s.} \tag{3.15}$$

On the other hand, applying the comparison theorem again, denote by Y_1 the solution of stochastic equation

$$dY_1 = Y_1\left(f - \frac{g}{h}Y_1\right)dt + \sigma_2 Y_1 dB_2(t),$$
$$Y_1(0) = y_0. \tag{3.16}$$

Consequently,

$$y(t) \geq Y_1(t), \quad t \in [0, +\infty), \text{ a.s.} \tag{3.17}$$

To sum up, we have that

$$Y_1(t) \leq y(t) \leq Y_2(t), \quad t \in [0, +\infty), \text{ a.s.} \tag{3.18}$$

Moreover, $Y_1(t)$ and $Y_2(t)$ have the explicit solutions, respectively,

$$Y_1(t) = \frac{e^{[(f-\sigma_2^2/2)t+\sigma_2 B_2(t)]}}{1/y_0 + (g/h)\int_0^t e^{[(f-\sigma_2^2/2)s+\sigma_2 B_2(s)]}ds}, \tag{3.19}$$

$$Y_2(t) = \frac{e^{[(f-\sigma_2^2/2)t+\sigma_2 B_2(t)]}}{1/y_0 + \int_0^t (g/(\lambda X_2(s)+h))e^{[(f-\sigma_2^2/2)s+\sigma_2 B_2(s)]}ds}. \tag{3.20}$$

Lemma 3.4. *Under assumption (H), for any initial value $y_0 > 0$, the solutions $Y_1(t)$ and $Y_2(t)$ satisfy*

$$\lim_{t\to\infty}\frac{\ln Y_1(t)}{t} = 0, \quad a.s.,$$
$$\limsup_{t\to\infty}\frac{\ln Y_2(t)}{t} \leq 0, \quad a.s. \tag{3.21}$$

Proof. The proof is motivated by [7]. Obviously, Lemma 3.1 and assumption (H) yield

$$\lim_{t\to\infty}\frac{\ln Y_1(t)}{t} = 0, \quad \text{a.s.} \tag{3.22}$$

On the other hand, it follows from (3.20) that

$$\frac{1}{Y_2(t)} = e^{[-(f-(\sigma_2^2/2))t-\sigma_2 B_2(t)]}\left[\frac{1}{y_0} + \int_0^t \frac{g}{\lambda X_2(s)+h}e^{[(f-\sigma_2^2/2)s+\sigma_2 B_2(s)]}ds\right]. \tag{3.23}$$

Choose T satisfying $e^{(a-(\sigma_1^2/2))t} \geq 2$ for $t \geq T$. Thus we have that $e^{(a-\sigma_1^2/2)t}/2 \leq e^{(a-\sigma_1^2/2)t} - 1$ for $t \geq T$. Then for $s \geq T$, from (3.10), we obtain

$$\begin{aligned}
X_2(t) &= \frac{e^{[(a-\sigma_1^2/2)t+\sigma_1 B_1(t)]}}{1/x_0 + b\int_0^t e^{[(a-\sigma_1^2/2)s+\sigma_1 B_1(s)]}ds} \\
&\leq \frac{e^{[(a-\sigma_1^2/2)t+\sigma_1 B_1(t)]}}{b\int_0^t e^{[(a-\sigma_1^2/2)s+\sigma_1 B_1(s)]}ds} \\
&\leq \frac{e^{[(a-\sigma_1^2/2)t+\sigma_1 B_1(t)]}}{be^{(\sigma_1 \min_{0\leq s\leq t} B_1(s))}\int_0^t e^{(a-\sigma_1^2/2)s}ds} \\
&= \frac{a-\sigma_1^2/2}{b}\frac{e^{[(a-\sigma_1^2/2)t+\sigma_1 B_1(t)]}}{e^{(\sigma_1 \min_{0\leq s\leq t} B_1(s))}\left[e^{(a-\sigma_1^2/2)t}-1\right]} \\
&\leq \frac{2\left(a-\sigma_1^2/2\right)}{b}\frac{e^{[(a-\sigma_1^2/2)t+\sigma_1 B_1(t)]}}{e^{(\sigma_1 \min_{0\leq s\leq t} B_1(s))}e^{(a-\sigma_1^2/2)t}} \\
&= \frac{2a-\sigma_1^2}{b}e^{\sigma_1(B_1(t)-\min_{0\leq s\leq t} B_1(s))}, \\
\int_T^t \frac{g}{\lambda X_2(s)+h}e^{[(f-\sigma_2^2/2)s+\sigma_2 B_2(s)]}ds &\geq \int_T^t \frac{g}{\lambda\left(\left(2a-\sigma_1^2\right)/b\right)e^{\sigma_1(B_1(s)-\min_{0\leq u\leq s} B_1(u))}+h} \\
&\quad\times e^{[(f-\sigma_2^2/2)s+\sigma_2 B_2(s)]}ds \\
&\geq \frac{bg}{\lambda\left(2a-\sigma_1^2\right)+bh}\int_T^t e^{-\sigma_1(B_1(s)-\min_{0\leq u\leq s} B_1(u))} \\
&\quad\times e^{[(f-\sigma_2^2/2)s+\sigma_2 B_2(s)]}ds \\
&\geq \frac{bg}{\lambda\left(2a-\sigma_1^2\right)+bh}e^{\sigma_1[\min_{0\leq s\leq t} B_1(s)-\max_{0\leq s\leq t} B_1(s)]+\sigma_2 \min_{0\leq s\leq t} B_2(s)} \\
&\quad\times \int_T^t e^{(f-\sigma_2^2/2)s}ds \\
&\geq \frac{2bg}{\left(\lambda\left(2a-\sigma_1^2\right)+bh\right)\left(2f-\sigma_2^2\right)} \\
&\quad\times \left(e^{(f-\sigma_2^2/2)t}-e^{(f-\sigma_2^2/2)T}\right) \\
&\quad\times e^{\sigma_1[\min_{0\leq s\leq t} B_1(s)-\max_{0\leq s\leq t} B_1(s)]+\sigma_2 \min_{0\leq s\leq t} B_2(s)}.
\end{aligned} \tag{3.24}$$

Thus,

$$\begin{aligned}
\frac{1}{Y_2(t)} &\geq e^{[-(f-\sigma_2^2/2)(t-T)-\sigma_2(B_2(t)-B_2(T))]} \\
&\quad\times\left[\frac{1}{y(T)}+\frac{2bg\left(e^{(f-\sigma_2^2/2)t}-e^{(f-\sigma_2^2/2)T}\right)}{\left(\lambda\left(2a-\sigma_1^2\right)+bh\right)\left(2f-\sigma_2^2\right)}e^{\sigma_1[\min_{0\leq s\leq t} B_1(s)-\max_{0\leq s\leq t} B_1(s)]+\sigma_2 \min_{0\leq s\leq t} B_2(s)}\right]
\end{aligned}$$

$$\geq \frac{2bge^{(f-\sigma_2^2/2)T\sigma_2 B_2(T)}\left(1-e^{-(f-\sigma_2^2/2)(t-T)}\right)}{\left(\lambda\left(2a-\sigma_1^2\right)+bh\right)\left(2f-\sigma_2^2\right)}$$
$$\times e^{\sigma_1[\min_{0\leq s\leq t}B_1(s)-\max_{0\leq s\leq t}B_1(s)]+\sigma_2[\min_{0\leq s\leq t}B_2(s)-\max_{0\leq s\leq t}B_2(s)]}$$
$$:= K(t)e^{\sigma_1[\min_{0\leq s\leq t}B_1(s)-\max_{0\leq s\leq t}B_1(s)]+\sigma_2[\min_{0\leq s\leq t}B_2(s)-\max_{0\leq s\leq t}B_2(s)]}, \tag{3.25}$$

where $K(t) = 2bge^{(f-\sigma_2^2/2)T\sigma_2 B_2(T)}(1-e^{-(f-\sigma_2^2/2)(t-T)})/(\lambda(2a-\sigma_1^2)+bh)(2f-\sigma_2^2)$. So we derive

$$-\ln Y_2(t) \geq \ln K(t) + \sigma_1\left[\min_{0\leq s\leq t} B_1(s) - \max_{0\leq s\leq t} B_1(s)\right] + \sigma_2\left[\min_{0\leq s\leq t} B_2(s) - \max_{0\leq s\leq t} B_2(s)\right]. \tag{3.26}$$

Dividing t on both sides yields

$$\begin{aligned}\frac{\ln Y_2(t)}{t} \leq -\frac{\ln K(t)}{t} &- \sigma_1\left[\frac{\min_{0\leq s\leq t}B_1(s)}{t} - \frac{\max_{0\leq s\leq t}B_1(s)}{t}\right]\\ &- \sigma_2\left[\frac{\min_{0\leq s\leq t}B_2(s)}{t} - \frac{\max_{0\leq s\leq t}B_2(s)}{t}\right].\end{aligned} \tag{3.27}$$

The distributions of $\max_{0\leq s\leq t}B_1(s)$ and $\max_{0\leq s\leq t}B_2(s)$ are that same as $|B_1(t)|$ and $|B_2(t)|$, respectively, and $\min_{0\leq s\leq t}B_1(s)$ and $\min_{0\leq s\leq t}B_2(s)$ have the same distributions as $-\max_{0\leq s\leq t}B_1(s)$ and $-\max_{0\leq s\leq t}B_1(s)$, respectively.

From the representation of $K(t)$, we can simplify it as follows:

$$K(t) = A_1 e^{A_2 B_2(T)}\left(1 - A_3 e^{A_4 t}\right). \tag{3.28}$$

By assumption (H), constants A_i $(i = 1, 2, 3, 4)$ satisfy $A_1 > 0$, $A_2 > 0$, $A_3 > 0$, and $A_4 < 0$. Then,

$$\ln K(t) = \ln A_1 + A_2 B_2(T) + \ln\left(1 - A_3 e^{A_4 t}\right). \tag{3.29}$$

It follows from $(\ln B_2(t)/t) \to 0,\ t \to \infty$, that

$$\frac{\ln K(t)}{t} \longrightarrow 0, \quad t \longrightarrow \infty. \tag{3.30}$$

Hence, letting $t \to \infty$ and by the strong law of large numbers, we have that

$$\begin{aligned}&\frac{\min_{0\leq s\leq t}B_1(s)}{t} \longrightarrow 0, \quad \frac{\max_{0\leq s\leq t}B_1(s)}{t} \longrightarrow 0, \quad t \longrightarrow \infty,\\ &\frac{\min_{0\leq s\leq t}B_2(s)}{t} \longrightarrow 0, \quad \frac{\max_{0\leq s\leq t}B_2(s)}{t} \longrightarrow 0, \quad t \longrightarrow \infty.\end{aligned} \tag{3.31}$$

Then,

$$\limsup_{t\to\infty}\frac{\ln Y_2(t)}{t}\le 0,\quad \text{a.s.,} \tag{3.32}$$

as desired. □

Theorem 3.5. *Under assumption (H), for any initial value $y_0 > 0$, the solution $y(t)$ of* (1.7) *has the property*

$$\lim_{t\to\infty}\frac{\ln y(t)}{t}=0,\quad a.s. \tag{3.33}$$

Proof. It follows from (3.18) and Lemma 3.4 that

$$0\le\liminf_{t\to\infty}\frac{\ln Y_1(t)}{t}\le\liminf_{t\to\infty}\frac{\ln y(t)}{t}\le\limsup_{t\to\infty}\frac{\ln y(t)}{t}\le\limsup_{t\to\infty}\frac{\ln Y_2(t)}{t}\le 0,\quad \text{a.s.} \tag{3.34}$$

Consequently,

$$\lim_{t\to\infty}\frac{\ln y(t)}{t}=0,\quad \text{a.s.} \tag{3.35}$$

The proof is complete. □

3.2. Persistent in Mean and Extinction

As we know, the property of persistence is more desirable since it represents the long-term survival to a population dynamics. Now we present the definition of persistence in mean proposed in [7, 11].

Definition 3.6. System (1.7) is said to be persistent in mean if

$$\liminf_{t\to\infty}\frac{\int_0^t x(t)ds}{t}>0,\quad \liminf_{t\to\infty}\frac{\int_0^t y(t)ds}{t}>0,\quad \text{a.s.} \tag{3.36}$$

Theorem 3.7. *Assume that condition (H) holds. Then system* (1.7) *is persistent in mean.*

Proof. Define the function $V=\ln x$; by the Itô formula, we get

$$\ln x(t)-\ln x_0=\left(a-\frac{\sigma_1^2}{2}\right)t-b\int_0^t x(s)ds-\int_0^t\frac{cy(s)}{\lambda x(s)+Ay(s)}ds+\sigma_1 B_1(t). \tag{3.37}$$

That is,

$$\begin{aligned} b\int_0^t x(s)ds &= -\ln x(t) + \ln x_0 + \left(a - \frac{\sigma_1^2}{2}\right)t - \int_0^t \frac{cy(s)}{\lambda x(s) + Ay(s)}ds + \sigma_1 B_1(t) \\ &\geq -\ln x(t) + \ln x_0 + \left(a - \frac{\sigma_1^2}{2}\right)t - \frac{ct}{A} + \sigma_1 B_1(t). \end{aligned} \tag{3.38}$$

Dividing t on both sides and using the strong law of large numbers, it follows from Theorem 3.3 that

$$\liminf_{t\to\infty} \frac{\int_0^t x(t)ds}{t} \geq \frac{a - c/A - \sigma_1^2/2}{b} > 0, \quad \text{a.s.} \tag{3.39}$$

Moreover, define the function $V = \ln y$; using the Itô formula again, we have that

$$\ln y(t) - \ln y_0 = \left(f - \frac{\sigma_2^2}{2}\right)t - \int_0^t \frac{gy(s)}{\lambda x(s) + h}ds + \sigma_2 B_2(t). \tag{3.40}$$

Thus,

$$\frac{g}{h}\int_0^t y(s)ds \geq \int_0^t \frac{gy(s)}{\lambda x(s) + h}ds = -\ln y(t) + \ln y_0 + \left(f - \frac{\sigma_2^2}{2}\right)t + \sigma_2 B_2(t). \tag{3.41}$$

Dividing both sides by t and letting $t \to \infty$ and also by the strong law of large numbers and Theorem 3.5, we have that

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t y(s)ds \geq \frac{h(f - \sigma_2^2/2)}{g} > 0, \quad \text{a.s.} \tag{3.42}$$

So the system is persistent in mean and we complete the proof. □

Under condition (H), we show that the system is persistent in mean. To a large extent, (H) is the condition that stands for small environmental noises. That is, small stochastic perturbation does not change the persistence of the system. Here, we will consider that large noises may make the system extinct.

Theorem 3.8. *Assume that condition $a - \sigma_1^2/2 < 0$, $f - \sigma_2^2/2 < 0$ holds. Then system* (1.7) *will become extinct exponentially with probability one.*

Proof. Define the function $V = \ln x$; by the Itô formula, we get

$$\ln x(t) - \ln x_0 = \left(a - \frac{\sigma_1^2}{2}\right)t - b\int_0^t x(s)ds - \int_0^t \frac{cy(s)}{\lambda x(s) + Ay(s)}ds + \sigma_1 B_1(t). \tag{3.43}$$

Then,

$$\ln x(t) - \ln x_0 \leq \left(a - \frac{\sigma_1^2}{2}\right)t + \sigma_1 B_1(t). \tag{3.44}$$

By the strong law of large numbers of martingales, we have that

$$\lim_{t\to\infty} \frac{B_1(t)}{t} = 0, \quad \text{a.s.} \tag{3.45}$$

Therefore,

$$\limsup_{t\to\infty} \frac{\ln x(t)}{t} \leq a - \frac{\sigma_1^2}{2} < 0, \quad \text{a.s.} \tag{3.46}$$

On the other hand, by the Itô formula again, we derive

$$\begin{aligned} \ln y(t) &= \ln y_0 + \left(f - \frac{\sigma_2^2}{2}\right)t - \int_0^t \frac{g y(s)}{\lambda x(s) + h} ds + \sigma_2 B_2(t) \\ &\leq \ln y_0 + \left(f - \frac{\sigma_2^2}{2}\right)t + \sigma_2 B_2(t). \end{aligned} \tag{3.47}$$

Applying the strong law of large numbers of martingales, we obtain

$$\limsup_{t\to\infty} \frac{\ln y(t)}{t} \leq f - \frac{\sigma_2^2}{2} < 0, \quad \text{a.s.} \tag{3.48}$$

The proof is complete. □

We continue to discuss the asymptotic behaviors of the stochastic system (1.7).

Theorem 3.9. *Assume that condition* $a - c/A - \sigma_1^2/2 > 0$, $f - \sigma_2^2/2 < 0$ *holds. Then the prey* $x(t)$ *of system* (1.7) *is persistent in mean; however, the predator* $y(t)$ *will become extinct exponentially with probability one.*

Proof. Define the function $V = \ln x$; by the Itô formula, we get

$$\ln x(t) - \ln x_0 = \left(a - \frac{\sigma_1^2}{2}\right)t - b\int_0^t x(s)ds - \int_0^t \frac{c y(s)}{\lambda x(s) + A y(s)} ds + \sigma_1 B_1(t). \tag{3.49}$$

Thus,

$$b\int_0^t x(s)ds \geq -\ln x(t) + \ln x_0 + \left(a - \frac{\sigma_1^2}{2} - \frac{c}{A}\right)t + \sigma_1 B_1(t). \tag{3.50}$$

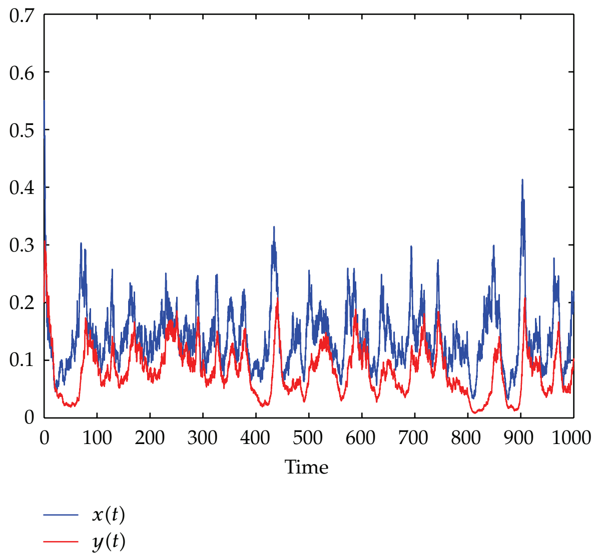

Figure 1

Under condition $a - c/A - \sigma_1^2/2 > 0$, it follows from the proof of Theorem 3.3 that

$$\lim_{t \to \infty} \frac{\ln x(t)}{t} = 0, \quad \text{a.s.} \tag{3.51}$$

So

$$\liminf_{t \to \infty} \frac{\int_0^t x(t) ds}{t} \geq \frac{a - c/A - \sigma_1^2/2}{b} > 0, \quad \text{a.s.} \tag{3.52}$$

That is, the prey $x(t)$ is persistent in mean. However, under condition $f - \sigma_2^2/2 < 0$, from the proof of Theorem 3.8, we have that

$$\limsup_{t \to \infty} \frac{\ln y(t)}{t} \leq f - \frac{\sigma_2^2}{2} < 0, \quad \text{a.s.} \tag{3.53}$$

That is, the predator $y(t)$ will become extinct exponentially with probability one. □

4. Numerical Simulations

In this section, some simulation figures are introduced to support the main results in our paper.

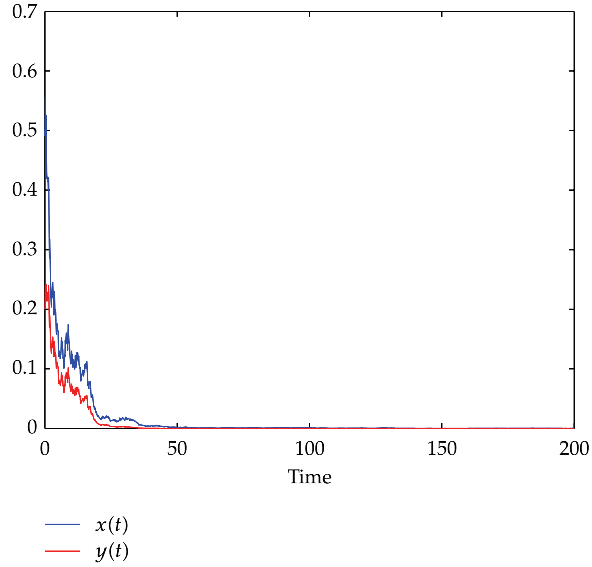

Figure 2

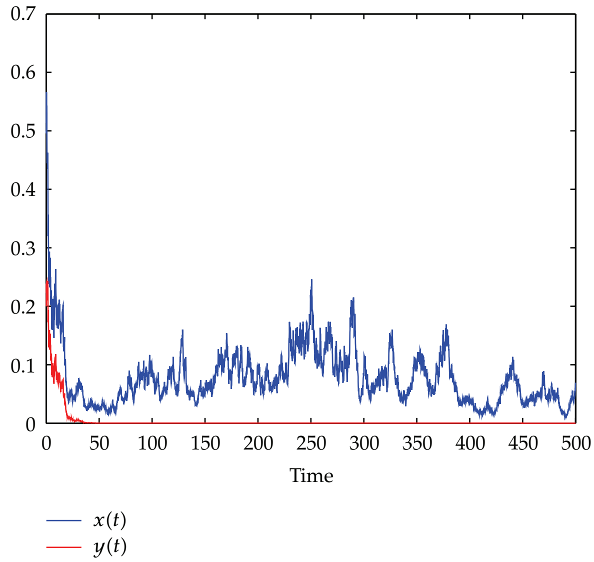

Figure 3

For model (1.7), we consider the discretization equations

$$
\begin{aligned}
x_{k+1} &= x_k + x_k \left[a - b x_k - \frac{c y_k}{\lambda x_k + A y_k} \right] \Delta t + \sigma_1 x_k \sqrt{\Delta t} \xi_k + \frac{\sigma_1^2}{2} x_k \left(\xi_k^2 - 1 \right) \Delta t, \\
y_{k+1} &= y_k + y_k \left[f - \frac{g y_k}{\lambda x_k + h} \right] \Delta t + \sigma_2 y_k \sqrt{\Delta t} \eta_k + \frac{\sigma_2^2}{2} y_k \left(\eta_k^2 - 1 \right) \Delta t,
\end{aligned} \tag{4.1}
$$

where ξ_k and η_k are Gaussian random variables that follow $N(0,1)$.

In Figure 1, we choose $a = 0.4$, $f = 0.3$, $c = 0.1$, $A = 0.5$, $\sigma_1(t)^2/2 = \sigma_2^2(t)/2 = 0.01$, and $(x_0, y_0) = (0.5, 0.2)$. By virtue of Theorem 3.7, the system will be persistent in mean. What we mentioned above can be seen from Figure 1. The difference between the conditions of Figure 1 and Figure 2 is that the values of σ_1 and σ_2 are different. In Figure 1, we choose $\sigma_1^2/2 = \sigma_2^2/2 = 0.01$. In Figure 2, we choose $\sigma_1^2/2 = \sigma_2^2/2 = 1$. In view of Theorem 3.8, both species x and y will go to extinction. Figure 2 confirms this.

In Figure 3, we choose $a = 0.4$, $f = 0.3$, $c = 0.1$, $A = 0.5$, $\sigma_1(t)^2/2 = 0.01$, $\sigma_2^2/2 = 1$, and $(x_0, y_0) = (0.5, 0.2)$. Then the prey $x(t)$ is persistent in mean; however, the predator $y(t)$ will become extinct. Figure 3 confirms the assertion (Theorem 3.9).

By comparing Figures 1 and 2, with Figure 3, we can observe that small environmental noise can retain the stochastic system permanent; however, sufficiently large environmental noise makes the stochastic system extinct.

Remark 4.1. White noise is taken into account in our model in this paper. It tells us that, when the intensities of environmental noises are not too big, some nice properties such as nonexplosion and permanence are desired. However, Theorem 3.8 reveals that a large white noise will force the population to become extinct while the population may be persistent under a relatively small white noise. To some extent, Theorem 3.9 shows that, though the predator $y(t)$ has plenty of food $x(t)$, they may be extinct because of large environmental noise.

Acknowledgment

This research is supported by China Postdoctoral Science Foundation (no. 20100481000).

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