

Research Article

Time-Periodic Solution of the Weakly Dissipative Camassa-Holm Equation

Chunyu Shen^{1,2}

¹ Nonlinear Scientific Research Center, Jiangsu University, Xuefu Road 301, Zhenjiang, Jiangsu 212013, China

² Academic Affairs Office, Jiangsu University, Xuefu Road 301, Zhenjiang, Jiangsu 212013, China

Correspondence should be addressed to Chunyu Shen, shenchunyu82228@sina.com

Received 31 May 2011; Accepted 31 August 2011

Academic Editor: Shangbin Cui

Copyright © 2011 Chunyu Shen. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper is concerned with time-periodic solution of the weakly dissipative Camassa-Holm equation with a periodic boundary condition. The existence and uniqueness of a time periodic solution is presented.

1. Introduction

The Camassa-Holm equation

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad t > 0, \quad x \in \mathbb{R}, \quad (1.1)$$

modeling the unidirectional propagation of shallow water waves over a flat bottom, where $u(t, x)$ represents the fluid's free surface above a flat bottom (or equivalently, the fluid velocity at time $t \geq 0$ and in the spatial x direction).

Since the equation was derived physically by Camassa and Holm [1, 2], many researchers have paid extensive attention to it. The Camassa-Holm equation is also a model for the propagation of axially symmetric waves in hyperelastic rods [3, 4]. It has a bi-Hamiltonian structure [5, 6] and is completely integrable [1, 2, 7–11]. It is a reexpression of geodesic flow on the diffeomorphism group of the circle [12] and on the Virasoro group [13]. Its solitary waves are peaked [7], and they are orbitally stable and interact like solitons [14–16]. The peakons capture a characteristic of the traveling waves of greatest height-exact traveling solutions of the governing equations for water waves with a peak at their crest [17–19].

The Cauchy problem of the Camassa-Holm equation has been extensively studied. It has been shown that this equation is locally well posed [20–25] for initial data $u_0 \in H^s(\mathbb{R})$ with $s > 3/2$. Moreover, it has global strong solutions modeling permanent waves [20, 24–27] but also blow-up solutions modeling wave breaking [20–28]. On the other hand, it has global weak solutions with initial data $u_0 \in H^1$ [29–35]. Moreover, the initial-boundary value problem for the Camassa-Holm equation on the half-line and on a finite interval was discussed in [36, 37]. It is observed that if u is the solution of the Camassa-Holm equation with the initial data u_0 in $H^1(\mathbb{R})$, we have for all $t > 0$,

$$\|u(t, \cdot)\|_{L^\infty} \leq \frac{\sqrt{2}}{2} \|u(t, \cdot)\|_{H^1} \leq \frac{\sqrt{2}}{2} \|u_0(\cdot)\|_{H^1}. \quad (1.2)$$

It is worth pointing out that the advantage of the Camassa-Holm equation in comparison with the KdV equation lies in the fact that the Camassa-Holm equation has peaked solitons and models wave breaking [2, 20, 21].

In general, it is difficult to avoid energy dissipation mechanisms in a real world. Ott and Sudan [38] investigated how the KdV equation was modified by the presence of dissipation and the effect of such dissipation on the solitary solution of the KdV equation. Ghidaglia [39] investigated the long-time behavior of solutions to the weakly dissipative KdV equation as a finite-dimensional dynamical system.

The Camassa-Holm equation with dissipative term is

$$u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} + L(u) = f(t, x), \quad t > 0, x \in \mathbb{R}, \quad (1.3)$$

where $f(t, x)$ is the forcing term, $L(u)$ is a dissipative term, L can be a differential operator or a quasi-differential operator according to different physical situations.

With $f = 0$ and $L(u) = \gamma(1 - \partial_x^2)u$, (1.3) becomes weakly dissipative Camassa-Holm equation

$$u_t - u_{txx} + 3uu_x + \gamma(u - u_{xx}) = 2u_x u_{xx} + uu_{xxx}, \quad t > 0, x \in \mathbb{R}, \quad (1.4)$$

where $\gamma > 0$ is a constant.

The local well-posedness, global existence, and blow-up phenomena of the Cauchy problem of (1.4) on the line [40] and on the circle [41] were studied. A new global existence result and a new blow-up result for strong solutions to this equation with certain profiles are presented recently [42]. We found that the behaviors of (1.4) are similar to the Camassa-Holm equation in a finite interval of time, such as the local well-posedness and the blow-up phenomena, and that there are considerable differences between (1.4) and the Camassa-Holm equation in their long-time behaviors. The global solutions of (1.4) decay to zero as time goes to infinity provided the potential $y_0 = (1 - \partial_x^2)u_0$ is of one sign (see [40, 41]). This long-time behavior is an important feature that the Camassa-Holm equation does not possess. It is well known that the Camassa-Holm equation has peaked traveling wave solutions. But the fact that any global solution of (1.4) decays to zero means that there are no traveling wave solutions of (1.4).

Another difference between (1.4) and the Camassa-Holm equation is that (1.4) does not have the following conservation laws

$$I_1 = \int_S u \, dx, \quad I_2 = \int_S (u^2 + u_x^2) \, dx, \quad (1.5)$$

which play an important role in the study of the Camassa-Holm equation.

Equation (1.4) has the same blow-up rate as the Camassa-Holm equation does when the blow-up occurs [41]. This fact shows that the blow-up rate of the Camassa-Holm equation is not affected by the weakly dissipative term, but the occurrence of blow-up of (1.4) is affected by the dissipative parameter [40, 41].

In the paper, we would like to consider the following weakly dissipative Camassa-Holm equation

$$u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} + \gamma(u - u_{xx}) = f(t, x), \quad t > 0, \quad x \in \mathbb{R}, \quad (1.6)$$

$$u(t, x + L) = u(t, x), \quad t > 0, \quad x \in \mathbb{R}, \quad (1.7)$$

$$u(t + \omega, x) = u(t, x), \quad t > 0, \quad x \in \mathbb{R}, \quad (1.8)$$

where $\gamma(1 - \partial_x^2)u$ is the weakly dissipative term, $\gamma > 0$ is a constant, and the forcing term f is ω -periodic in time t and L -periodic in spatial x . Without loss of generality, we assume further $\int_{\Omega} f(t, x) \, dx = 0$, where $\Omega = [0, L]$. When system is periodically dependent on time t , we want to know whether there exists time-periodic solution with the same period for the system. In many nonlinear evolution equations, the study of time-periodic solution has attracted considerable interest (e.g., [43–45]). In this paper, we will prove that (1.6)–(1.8) have a solution by using the Galerkin method [46], and Leray-Schauder fixed point theorem [44].

Our paper is organized as follows. In Section 2, we give some notations and definition of some space used in this paper. In Section 3, we prove the existence of the approximate solution and give uniform a priori estimates needed where we prove the convergence of a sequence of the approximate solution. Section 4 is devoted to the study of the existence and uniqueness of time-periodic solution for (1.6)–(1.8).

2. Preliminaries

Before starting our work, it is appropriate to introduce some notations and inequalities that will be used in the paper.

Let X be a Banach space, we denote by $C^k(\omega; X)$ the set of ω -periodic X -valued measurable functions on \mathbb{R}^1 with continuous derivatives up to order k . The norm in the space $C^k(\omega; X)$ is $\|u\|_{C^k(\omega; X)} = \sup_{0 \leq t \leq \omega} \{\sum_{i=0}^k \|D_t^i u\|_X\}$.

For $1 \leq p \leq \infty$, the space $L^p(\omega; X)$ is the set of ω -periodic X -valued measurable functions on \mathbb{R} such that

$$\|u\|_{L^p(\omega; X)} = \begin{cases} \left(\int_0^\omega \|u\|_X^p \, dt \right)^{1/p} < \infty, & 1 \leq p < \infty \\ \sup_{0 \leq t \leq \omega} \|u\|_X < \infty, & p = \infty. \end{cases} \quad (2.1)$$

The space $W^{k,p}(\omega; X)$ denote the set of functions which belong to $L^p(\omega; X)$ together with their derivatives up to order k , and we write $W^{k,2}(\omega; X) = H^k(\omega; X)$ in particular when X is a Hilbert space.

$L^p(\Omega)$ and $H^m(\Omega)$ are classical Sobolev spaces. For simplicity, we write $\|\cdot\|_{L^p(\Omega)}$ by $\|\cdot\|_p$ as $p \neq 2$ and $\|\cdot\|_{L^2(\Omega)}$ by $\|\cdot\|$.

The following inequalities (see [47]) will be used in the proofs later

$$\|u\|_{\infty} \leq k_1 \|u\|_{H^1}. \quad (2.2)$$

$$\|D^j u\|_p \leq k_2 \|u\|_{H^m}^{\theta} \|u\|^{1-\theta}, \quad (2.3)$$

where $D^j u = (\partial^j u)/(\partial x^j)$, $1/p = j + \theta(1/2 - m) + (1 - \theta)(1/2)$ as $0 \leq j < m$, $j/m \leq \theta \leq 1$.

$$\|u\| \leq k_3 \|u_x\|, \int_{\Omega} u(x) dx = 0. \quad (2.4)$$

3. A Priori Estimates

In this section, we first prove that (1.6)–(1.8) have a sequence of approximate solutions $\{u_n\}_{n=1}^{\infty}$, then give a prior, estimates about $\{u_n\}_{n=1}^{\infty}$.

We denote the unbounded linear operator $Au = -u_{xx}$ on $X = L^2 \cap \{u \mid u(x+L) = u(x), \int_{\Omega} u dx = 0\}$, then the set of all linearly independent eigenvectors $\{w_j\}_{j=0}^{\infty}$ of A , that is, $Aw_j = \lambda_j w_j$, with $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_j \rightarrow \infty$, is an orthonormal basis of $L^2(\Omega)$. For any n and a group of function $\{a_{jn}(t)\}_{j=1}^n$, where $a_{jn}(t) (j = 1, 2, \dots, n) \in C^1(\omega; \mathbb{R})$, the function $u_n = \sum_{j=1}^n a_{jn}(t) w_j \in C^1(\omega; H_n)$ is called an approximate solution to (1.6)–(1.8) if it satisfies the equation as follows:

$$(u_{nt} - u_{nxtt} + \gamma(u_n - u_{nxx}), w_j) = (Nu_n + f, w_j), \quad j = 1, \dots, n, \quad (3.1)$$

where $Nu_n = -3u_n u_{nx} + 2u_{nx} u_{nxx} + u_n u_{nxxx}$ and $H_n = \text{span}\{w_1, w_2, \dots, w_n\}$. By the classical theory of ordinary differential equations, for any fixed $v_n(t) = \sum_{j=1}^n b_{jn}(t) w_j \in C^1(\omega; H_n)$, the equation $(u_{nt} - u_{nxtt} + \gamma(u_n - u_{nxx}), w_j) = (Nv_n + f, w_j)$, $j = 1, \dots, n$ has a unique ω -periodic solution u_n and the mapping $F : v_n \rightarrow u_n$ is continuous and compact in $C^1(\omega; H_n)$. Hence by Leray-Schauder fixed point theorem, we want to prove the existence of an approximate solution only to show $\sup_{0 \leq t \leq \omega} \|u_n\|^2 \leq c$ for all possible solution of (3.1) replaced by $\lambda Nu_n (0 \leq \lambda \leq 1)$ instead of nonlinear term Nu_n , where c is a constant which only depends on $L, \varepsilon, \omega, \gamma$, and f .

Lemma 3.1. *If $f \in C^1(\omega; H^{-1}(\Omega))$, then*

$$\sup_{0 \leq t \leq \omega} (\|u_n\|^2 + \|u_{nx}\|^2) \leq c_1, \quad (3.2)$$

where c_1 is a constant which only depends on $L, \omega, \varepsilon, \gamma, k_3$, and f , $M = \sup_{0 \leq t \leq \omega} \{\|f(t, x)\|_{H^{-1}(\Omega)}^2\}$ and $d_1 = \min\{2\gamma, 2\gamma - \varepsilon\} > 0$.

Proof. Multiplying (3.1) by $a_{jn}(t)$ and summing up over j from 1 to n , we obtain

$$(u_{nt} - u_{nxx} + \gamma(u_n - u_{nxx}), u_n) = (Nu_n + f, u_n). \quad (3.3)$$

Then, we can get

$$\frac{1}{2} \frac{d}{dt} (\|u_n\|^2 + \|u_{nx}\|^2) + \gamma (\|u_n\|^2 + \|u_{nx}\|^2) = (Nu_n + f, u_n). \quad (3.4)$$

Notice that $-3 \int_{\Omega} u_n^2 u_{nx} dx = 0$, $2 \int_{\Omega} u_n u_{nx} u_{nxx} dx + \int_{\Omega} u_n^2 u_{nxxx} dx = 0$.

From Young's inequality, we have $\int_{\Omega} f u_n dx \leq (\varepsilon/2) \|u_{nx}\|^2 + k_3^2 M/2\varepsilon$, where $\varepsilon > 0$ is a constant.

According to the above relations, we can derive from (3.4) that

$$\frac{d}{dt} (\|u_n\|^2 + \|u_{nx}\|^2) + d_1 (\|u_n\|^2 + \|u_{nx}\|^2) \leq \frac{k_3^2 M}{\varepsilon}, \quad (3.5)$$

where $d_1 = \min\{2\gamma, 2\gamma - \varepsilon\} > 0$.

Considering the time periodicity of u_n and integrating (3.5) over $[0, \omega]$, we get

$$d_1 \int_0^{\omega} (\|u_n\|^2 + \|u_{nx}\|^2) dt \leq \frac{\omega k_3^2 M}{\varepsilon}. \quad (3.6)$$

Hence, there exists $t^* \in [0, \omega)$ such that $\|u_n(t^*)\|^2 + \|u_{nx}(t^*)\|^2 \leq k_3^2 M/d_1 \varepsilon$.

From (3.5), we have $(d/dt)(\|u_n\|^2 + \|u_{nx}\|^2) \leq k_3^2 M/\varepsilon$.

Integrating the above inequality with respect to t from t^* to $t \in [t^*, t^* + \omega]$, we deduce that

$$\|u_n(t)\|^2 + \|u_{nx}(t)\|^2 \leq \|u_n(t^*)\|^2 + \|u_{nx}(t^*)\|^2 + \frac{\omega k_3^2 M}{\varepsilon} \leq \left(\frac{1}{d_1} + \omega\right) \frac{k_3^2 M}{\varepsilon}. \quad (3.7)$$

Hence, we infer

$$\sup_{0 \leq t \leq \omega} (\|u_n\|^2 + \|u_{nx}\|^2) \leq \left(\frac{1}{d_1} + \omega\right) \frac{k_3^2 M}{\varepsilon} \triangleq c_1, \quad (3.8)$$

which concludes our proof. \square

From Lemma 3.1 and Leray-Schauder fixed point theorem, (3.1) has solution $\{u_n\}_{n=1}^{\infty}$, which is also a sequence of approximate solutions of (1.6)–(1.8). In order to obtain the convergence of sequence $\{u_n\}_{n=1}^{\infty}$, we need to give a priori estimates for the high-order derivatives of $\{u_n\}_{n=1}^{\infty}$.

Lemma 3.2. *If $f \in C^1(\omega; H^{-1}(\Omega))$, then*

$$\sup_{0 \leq t \leq \omega} (\|u_{nx}\|^2 + \|u_{nxx}\|^2) \leq c_2, \quad (3.9)$$

where c_2 is a constant which only depends on L , ω , ε , γ , λ_n , k_1 , k_2 , k_3 , and f , $M = \sup_{0 \leq t \leq \omega} \{\|f(t, x)\|_{H^{-1}(\Omega)}^2\}$ and $d_2 = \min\{2\gamma - (13/2)\varepsilon\lambda_n, 2\gamma - (21/2)\varepsilon\} > 0$.

Proof. Multiplying (3.1) by $-\lambda_j a_{jn}(t)$ and summing up over j from 1 to n , we have

$$(u_{nt} - u_{nxtt} + \gamma(u_n - u_{nxx}), u_{nxx}) = (Nu_n + f, u_{nxx}). \quad (3.10)$$

The above equation yields

$$-\frac{1}{2} \frac{d}{dt} (\|u_{nx}\|^2 + \|u_{nxx}\|^2) - \gamma (\|u_{nx}\|^2 + \|u_{nxx}\|^2) = (Nu_n + f, u_{nxx}). \quad (3.11)$$

From Young's inequality, we have

$$\left| \int_{\Omega} f u_{nxx} dx \right| \leq \varepsilon \|u_{nxxx}\|^2 + \frac{k_3^2 M}{4\varepsilon}, \quad (3.12)$$

where $\varepsilon > 0$ is a constant.

From (2.2), (3.8), and Young's inequality, we can deduce that

$$\begin{aligned} \left| \int_{\Omega} u_n u_{nx} u_{nxx} dx \right| &\leq \|u_n\|_{\infty} \int_{\Omega} |u_{nx} u_{nxx}| dx \\ &\leq k_1 \|u_n\|_{H^1} \int_{\Omega} |u_{nx} u_{nxx}| dx \\ &\leq k_1 c_1^{1/2} \left(\frac{\varepsilon}{k_1 c_1^{1/2}} \|u_{nxx}\|^2 + \frac{k_1 c_1^{1/2}}{4\varepsilon} \|u_{nx}\|^2 \right) \\ &\leq \varepsilon \|u_{nxx}\|^2 + \frac{k_1^2 c_1^2}{4\varepsilon}. \end{aligned} \quad (3.13)$$

From (2.3), (3.8), Cauchy-Schwarz inequality, Young's inequality, and Lemma 3.1, we get

$$\begin{aligned} \left| \int_{\Omega} u_n u_{nxx} u_{nxxx} dx \right| &= \left| -\frac{1}{2} \int_{\Omega} u_{nx} u_{nxx}^2 dx \right| \leq \frac{1}{2} \|u_{nx}\| \|u_{nxx}\|_4^2 \\ &\leq \frac{1}{2} c_1^{1/2} k_2^2 \|u_n\|^{1/2} \|u_n\|_{H^3}^{3/2} \\ &\leq \frac{3}{4} \varepsilon \|u_n\|_{H^3}^2 + \frac{c_1^3 k_2^8}{64 \varepsilon^3} \\ &\leq \frac{3}{4} \varepsilon \|u_{nxx}\|^2 + \frac{3}{4} \varepsilon \|u_{nxxx}\|^2 + \frac{3}{4} \varepsilon c_1 + \frac{c_1^3 k_2^8}{64 \varepsilon^3}, \end{aligned} \quad (3.14)$$

$$\|u_{nxxx}\|^2 = \int_{\Omega} \left| \left(\sum_{j=1}^n a_{jn}(t) w_j \right)_{xxx} \right|^2 dx = \int_{\Omega} \left| \left(\sum_{j=1}^n \lambda_j a_{jn}(t) w_j \right)_x \right|^2 dx \leq \lambda_n \|u_{nx}\|^2. \quad (3.15)$$

Taking (3.11)–(3.15) into account, we can infer that

$$\frac{d}{dt} \left(\|u_{nx}\|^2 + \|u_{nxx}\|^2 \right) + d_2 \left(\|u_{nx}\|^2 + \|u_{nxx}\|^2 \right) \leq \frac{k_3^2 M}{2\varepsilon} + \frac{3k_1^2 c_1^2}{2\varepsilon} + \frac{9\varepsilon c_1}{2} + \frac{3c_1^3 k_2^8}{32\varepsilon^3}, \quad (3.16)$$

where $d_2 = \min\{2\gamma - (13/2)\varepsilon\lambda_n, 2\gamma - (21/2)\varepsilon\} > 0$.

Integrating (3.16) about t from 0 to ω and considering the time periodicity of u_n , we get

$$d_2 \int_0^{\omega} \left(\|u_{nx}\|^2 + \|u_{nxx}\|^2 \right) dt \leq \left(\frac{k_3^2 M}{2\varepsilon} + \frac{3k_1^2 c_1^2}{2\varepsilon} + \frac{9\varepsilon c_1}{2} + \frac{3c_1^3 k_2^8}{32\varepsilon^3} \right) \omega. \quad (3.17)$$

Hence, there exists $t^* \in [0, \omega)$ such that

$$\|u_{nx}(t^*)\|^2 + \|u_{nxx}(t^*)\|^2 \leq \frac{1}{d_2} \left(\frac{k_3^2 M}{2\varepsilon} + \frac{3k_1^2 c_1^2}{2\varepsilon} + \frac{9\varepsilon c_1}{2} + \frac{3c_1^3 k_2^8}{32\varepsilon^3} \right). \quad (3.18)$$

From (3.16), we have

$$\|u_{nx}(t)\|^2 + \|u_{nxx}(t)\|^2 \leq \|u_{nx}(t^*)\|^2 + \|u_{nxx}(t^*)\|^2 + \left(\frac{k_3^2 M}{2\varepsilon} + \frac{3k_1^2 c_1^2}{2\varepsilon} + \frac{9\varepsilon c_1}{2} + \frac{3c_1^3 k_2^8}{32\varepsilon^3} \right) \omega. \quad (3.19)$$

Then we can obtain

$$\sup_{0 \leq t \leq \omega} \left(\|u_{nx}\|^2 + \|u_{nxx}\|^2 \right) \leq \left(\frac{1}{d_2} + \omega \right) \left(\frac{k_3^2 M}{2\varepsilon} + \frac{3k_1^2 c_1^2}{2\varepsilon} + \frac{9\varepsilon c_1}{2} + \frac{3c_1^3 k_2^8}{32\varepsilon^3} \right) \triangleq c_2, \quad (3.20)$$

which concludes our proof. \square

In the following, we continue to establish a priori estimates for high-order derivatives of the approximate solution $\{u_n\}_{n=1}^\infty$ by an inductive argument.

Lemma 3.3. For any $k \geq 0$, if $f \in C^1(\omega; H^{k-1}(\Omega))$, then

$$\sup_{0 \leq t \leq \omega} \left(\|D^{k+1}u_n\|^2 + \|D^{k+2}u_n\|^2 \right) \leq c, \quad (3.21)$$

where c is a constant which only depends on $L, \omega, \varepsilon, \gamma, \lambda_n, k, k_1, k_2, k_3, f$ and $d_3 = \{2\gamma - 16\varepsilon\lambda_n, 2\gamma - 14\varepsilon\} > 0$.

Proof. By Lemma 3.2, we know the conclusion of Lemma 3.3 holds for $k = 0$.

Assume that for $k \leq m - 1$ ($m \geq 2$) the conclusion of Lemma 3.3 holds, we want to prove that the same statement holds for $k = m$ also.

Multiplying (3.1) by $(-1)^{m+1} \lambda_j^{m+1} a_{jn}(t)$ and summing up over j from 1 to n , we have

$$\begin{aligned} (-1)^{m+1} \frac{1}{2} \frac{d}{dt} \left(\|D^{m+1}u_n\|^2 + \|D^{m+2}u_n\|^2 \right) + (-1)^{m+1} \gamma \left(\|D^{m+1}u_n\|^2 + \|D^{m+2}u_n\|^2 \right) \\ = (Nu_n + f, D^{2(m+1)}u_n). \end{aligned} \quad (3.22)$$

Follow the same methods discussed in Lemma 3.2, we have

$$\left| \int_{\Omega} f D^{2(m+1)}u_n dx \right| = \left| \int_{\Omega} D^{m-1} f D^{m+3}u_n dx \right| \leq \varepsilon \|D^{m+3}u_n\|^2 + \frac{1}{4\varepsilon} \|D^{m-1}f\|^2. \quad (3.23)$$

From the conclusion of Lemma 3.3 for $k \leq m - 1$, (2.2), (2.4) and Young's inequality, we can deduce that

$$\begin{aligned} \left| \int_{\Omega} u_n u_{nx} D^{2(m+1)}u_n dx \right| &= \left| \int_{\Omega} \left(\sum_{i=0}^{m+1} C_{m+1}^i D^i u_n D^{m+1-i} u_{nx} \right) D^{m+1}u_n dx \right| \\ &\leq \int_{\Omega} |u_n D^{m+2}u_n D^{m+1}u_n| dx \\ &\quad + \int_{\Omega} \left| \left(\sum_{i=1}^{m+1} C_{m+1}^i D^i u_n D^{m+1-i} u_{nx} \right) D^{m+1}u_n \right| dx \\ &\leq \varepsilon \|D^{m+2}u_n\|^2 + c(\varepsilon, k_1, k_3) \|D^{m+1}u_n\|^2 + c(m, k_1) \\ &\leq \varepsilon \|D^{m+2}u_n\|^2 + c(\varepsilon, k_1, k_3, m). \end{aligned} \quad (3.24)$$

Similarly, we can also deduce that

$$\begin{aligned}
\left| \int_{\Omega} u_n u_{nxxx} D^{2(m+1)} u_n dx \right| &= \left| \int_{\Omega} \left(\sum_{i=0}^m C_m^i D^i u_n D^{m-i} u_{nxxx} \right) D^{m+2} u_n dx \right| \\
&\leq \int_{\Omega} \left| u_n D^{m+3} u_n D^{m+2} u_n \right| dx + \int_{\Omega} \left| C_m^1 D u_n (D^{m+2} u_n)^2 \right| dx \\
&\quad + \int_{\Omega} \left| C_m^2 D^2 u_n D^{m+1} u_n D^{m+2} u_n \right| dx \\
&\quad + \int_{\Omega} \left| \left(\sum_{i=3}^m C_m^i D^i u_n D^{m+3-i} u_n \right) D^{m+2} u_n \right| dx \\
&\leq \|u_n\|_{\infty} \int_{\Omega} \left| D^{m+3} u_n D^{m+2} u_n \right| dx + m \|D u_n\|_{\infty} \int_{\Omega} \left| D^{m+2} u_n \right|^2 dx \\
&\quad + C_m^2 \|D^2 u_n\|_{\infty} \int_{\Omega} \left| D^{m+1} u_n D^{m+2} u_n \right| dx \\
&\quad + \sum_{i=3}^m C_m^i \|D^i u_n\|_{\infty} \|D^{m+3-i} u_n\|_{\infty} \int_{\Omega} \left| D^{m+2} u_n \right| dx \\
&\leq c(k_1, k_3, m) \left(\int_{\Omega} \left| D^{m+3} u_n D^{m+2} u_n \right| dx + \int_{\Omega} \left| D^{m+2} u_n \right|^2 dx \right. \\
&\quad \left. + \int_{\Omega} \left| D^{m+1} u_n D^{m+2} u_n \right| dx + \int_{\Omega} \left| D^{m+2} u_n \right| dx \right). \tag{3.25}
\end{aligned}$$

From the conclusion of Lemma 3.3 for $k \leq m-1$, Young's inequality and (2.3), we have

$$\begin{aligned}
c(k_1, k_3, m) \int_{\Omega} \left| D^{m+3} u_n D^{m+2} u_n \right| dx &\leq \varepsilon \|D^{m+3} u_n\|^2 + c(k_1, k_3, m, \varepsilon) \|D^{m+2} u_n\|^2 \\
&\leq \varepsilon \|D^{m+3} u_n\|^2 + c(k_1, k_2, k_3, m, \varepsilon) \\
&\quad \times \|u_n\|_{H^{m+3}}^{2(m+2)/m+3} \|u_n\|^{2(1-(m+2)/(m+3))} \\
&\leq \varepsilon \|D^{m+3} u_n\|^2 + \varepsilon \|u_n\|_{H^{m+3}}^2 + c(k_1, k_2, k_3, m, \varepsilon) \|u_n\|^2 \\
&\leq 2\varepsilon \|D^{m+3} u_n\|^2 + \varepsilon \|D^{m+2} u_n\|^2 + c(k_1, k_2, k_3, m, \varepsilon), \\
c(k_1, k_3, m) \int_{\Omega} \left| (D^{m+2} u_n)^2 \right| dx &= c(k_1, k_3, m) \|D^{m+2} u_n\|^2 \\
&\leq c(k_1, k_2, k_3, m) \|u_n\|_{H^{m+3}}^{2(m+2)/(m+3)} \|u_n\|^{2(1-(m+2)/(m+3))} \\
&\leq \varepsilon \|D^{m+3} u_n\|^2 + \varepsilon \|D^{m+2} u_n\|^2 + c(k_1, k_2, k_3, m, \varepsilon),
\end{aligned}$$

$$\begin{aligned}
c(k_1, k_3, m) \int_{\Omega} \left| D^{m+1} u_n D^{m+2} u_n \right| dx &\leq \varepsilon \left\| D^{m+2} u_n \right\|^2 + c(k_1, k_3, m, \varepsilon) \left\| D^{m+1} u_n \right\|^2 \\
&\leq \varepsilon \left\| D^{m+2} u_n \right\|^2 + c(k_1, k_3, m, \varepsilon), \\
c(k_1, k_3, m) \int_{\Omega} \left| D^{m+2} u_n \right| dx &\leq \varepsilon \left\| D^{m+2} u_n \right\|^2 + c(k_1, k_3, m, \varepsilon, L).
\end{aligned} \tag{3.26}$$

Combining (3.25) and the above inequality, we can get

$$\left| \int_{\Omega} u_n u_{nxxx} D^{2(m+1)} u_n dx \right| \leq 3\varepsilon \left\| D^{m+3} u_n \right\|^2 + 4\varepsilon \left\| D^{m+2} u_n \right\|^2 + c(k_1, k_2, k_3, m, \varepsilon, L). \tag{3.27}$$

Similarly,

$$\begin{aligned}
\left| \int_{\Omega} u_{nx} u_{nxxx} D^{2(m+1)} u_n dx \right| &\leq \int_{\Omega} \left| u_{nx} D^{m+1} u_n D^{m+3} u_n \right| dx \\
&\quad + \int_{\Omega} \left| \left(\sum_{i=1}^{m-1} C_{m-1}^i D^{i+1} u_n D^{m+1-i} u_n \right) D^{m+3} u_n \right| dx \\
&\leq \|u_{nx}\|_{\infty} \int_{\Omega} \left| D^{m+1} u_n D^{m+3} u_n \right| dx \\
&\quad + \sum_{i=1}^{m-1} C_{m-1}^i \left\| D^{i+1} u_n \right\|_{\infty} \left\| D^{m+1-i} u_n \right\|_{\infty} \int_{\Omega} \left| D^{m+3} u_n \right| dx \\
&\leq 2\varepsilon \left\| D^{m+3} u_n \right\|^2 + c(m, k_1, \varepsilon, L).
\end{aligned} \tag{3.28}$$

Taking (3.22)–(3.24) and (3.27)–(3.28) into account, we can deduce that

$$\begin{aligned}
\frac{d}{dt} \left(\left\| D^{m+1} u_n \right\|^2 + \left\| D^{m+2} u_n \right\|^2 \right) + 2\gamma \left(\left\| D^{m+1} u_n \right\|^2 + \left\| D^{m+2} u_n \right\|^2 \right) \\
\leq 16\varepsilon \left\| D^{m+3} u_n \right\|^2 + 14\varepsilon \left\| D^{m+2} u_n \right\|^2 + c(k_1, k_2, k_3, m, \varepsilon, f, L).
\end{aligned} \tag{3.29}$$

From the above relation, we can infer

$$\frac{d}{dt} \left(\left\| D^{m+1} u_n \right\|^2 + \left\| D^{m+2} u_n \right\|^2 \right) + d_3 \left(\left\| D^{m+1} u_n \right\|^2 + \left\| D^{m+2} u_n \right\|^2 \right) \leq c(k_1, k_2, k_3, m, \varepsilon, f, L), \tag{3.30}$$

where $d_3 = \{2\gamma - 16\varepsilon\lambda_n, 2\gamma - 14\varepsilon\} > 0$.

Integrating (3.30) about t from 0 to ω , there exists $t^* \in [0, \omega)$ such that

$$\|D^{m+1}u_n(t^*)\|^2 + \|D^{m+2}u_n(t^*)\|^2 \leq \frac{c(k_1, k_2, k_3, m, \varepsilon, f, L)}{d_3}. \quad (3.31)$$

From (3.30), we have

$$\frac{d}{dt} \left(\|D^{m+1}u_n\|^2 + \|D^{m+2}u_n\|^2 \right) \leq c(k_1, k_2, k_3, m, \varepsilon, f, L). \quad (3.32)$$

Integrating the above inequality from t^* to $t \in [t^*, t^* + \omega]$ and with (3.31), we can easily obtain

$$\sup_{0 \leq t \leq \omega} \left(\|D^{m+1}u_n\|^2 + \|D^{m+2}u_n\|^2 \right) \leq \left(\frac{1}{d_3} + \omega \right) c(k_1, k_2, k_3, m, \varepsilon, f, L) \triangleq c. \quad (3.33)$$

The proof is completed. \square

Lemma 3.4. For any $k \geq 0$, if $f \in C^1(\omega; H^{k+1}(\Omega))$, then

$$\sup_{0 \leq t \leq \omega} \left(\|D^k u_{nt}\|^2 + \|D^{k+1} u_{nt}\|^2 \right) \leq c \quad (3.34)$$

where c is a constant which only depends on $L, \omega, \varepsilon, \gamma, \lambda_n, k, k_1, k_2, k_3$, and f .

Proof. We first prove the conclusion of Lemma 3.4 holds for $k = 0$. Multiplying (3.1) by $a'_{jn}(t)$ and summing up over j from 1 to n , we have

$$\|u_{nt}\|^2 + \|u_{nxt}\|^2 = (Nu_n + f - \gamma(u_n - u_{nxx}), u_{nt}). \quad (3.35)$$

By Lemma 3.3, if $f \in C^1(\omega; H^1(\Omega))$, then we have $\|u_n\|_{H^4}^2 \leq c$. Hence,

$$|(Nu_n + f - \gamma(u_n - u_{nxx}), u_{nt})| \leq \|Nu_n + f - \gamma(u_n - u_{nxx})\| \|u_{nt}\| \leq c \|u_{nt}\|. \quad (3.36)$$

Therefore, from (3.35) and (3.36), it is easy to know that

$$\sup_{0 \leq t \leq \omega} \left(\|u_{nt}\|^2 + \|u_{nxt}\|^2 \right) \leq c. \quad (3.37)$$

Assume that the conclusion of Lemma 3.4 holds for $k \leq m (m \geq 1)$, we want to prove that the conclusion of Lemma 3.4 also holds for $k = m + 1$.

Multiplying (3.1) by $(-1)^{m+1} \lambda_j^{m+1} a'_{jn}(t)$ and summing up over j from 1 to n , we have

$$(-1)^{m+1} \left(\|D^{m+1}u_{nt}\|^2 + \|D^{m+2}u_{nt}\|^2 \right) = (Nu_n + f - \gamma(u_n - u_{nxx}), D^{2(m+1)}u_{nt}). \quad (3.38)$$

By Lemma 3.3, if $f \in C^1(\omega; H^{m+2}(\Omega))$, then $\|D^k u_n\|^2 \leq c$ for $k \leq m + 5$. Hence,

$$\begin{aligned} \left| \left(Nu_n + f - \gamma(u_n - u_{nxx}), D^{2(m+1)} u_{nt} \right) \right| &\leq \|D^{m+1} [Nu_n + f - \gamma(u_n - u_{nxx})]\| \|D^{m+1} u_{nt}\| \\ &\leq c \|D^{m+1} u_{nt}\|. \end{aligned} \quad (3.39)$$

Taking (3.38) and (3.39) into account, it follows

$$\sup_{0 \leq t \leq \omega} \left(\|D^{m+1} u_{nt}\|^2 + \|D^{m+2} u_{nt}\|^2 \right) \leq c. \quad (3.40)$$

This completes the proof of Lemma 3.4 by an inductive argument. \square

4. Existence and Uniqueness of Time-Periodic Solution

We have proved that (1.6)–(1.8) have a sequence of approximate solutions $\{u_n\}_{n=1}^\infty$. In this section, we want to prove that the sequence converges and the limit is a solution of (1.6)–(1.8).

By Lemmas 3.1–3.4 and standard compactness arguments, we conclude that there is a subsequence which we denote also by $\{u_n\}$ such that for any $K \geq 0$, if $f \in C^1(\omega; H^{k+1}(\Omega))$, we have

$$\begin{aligned} u_n(t) &\longrightarrow u(t), \text{ weakly } * \text{ in } L^\infty(\omega; H^{k+4}(\Omega)), \\ u_n(t) &\longrightarrow u(t), \text{ strongly in } L^\infty(\omega; H^{k+3}(\Omega)), \\ u_{nt}(t) &\longrightarrow u_t(t), \text{ weakly } * \text{ in } L^\infty(\omega; H^{k+1}(\Omega)), \\ u_{nt}(t) &\longrightarrow u_t(t), \text{ strongly in } L^\infty(\omega; H^k(\Omega)). \end{aligned} \quad (4.1)$$

From the above lemmas, we know that the nonlinear terms are well defined

$$\begin{aligned} \|u_n u_{nx} - u u_x\| &\leq \|u_n(u_{nx} - u_x)\| + \|u_x(u_n - u)\| \\ &\leq \|u_n\|_\infty \|u_{nx} - u_x\| + \|u_x\|_\infty \|u_n - u\| \longrightarrow 0, \end{aligned} \quad (4.2)$$

as $n \rightarrow \infty$, uniformly in t ,

$$\begin{aligned} \|u_{nx} u_{nxx} - u_x u_{xx}\| &\leq \|u_{nx}(u_{nxx} - u_{xx})\| + \|u_{xx}(u_{nx} - u_x)\| \\ &\leq \|u_{nx}\|_\infty \|u_{nxx} - u_{xx}\| + \|u_{xx}\|_\infty \|u_{nx} - u_x\| \longrightarrow 0, \end{aligned} \quad (4.3)$$

as $n \rightarrow \infty$, uniformly in t ,

$$\begin{aligned} \|u_n u_{nxxx} - u u_{xxx}\| &\leq \|u_n(u_{nxxx} - u_{xxx})\| + \|u_{xxx}(u_n - u)\| \\ &\leq \|u_n\|_\infty \|u_{nxxx} - u_{xxx}\| + \|u_n - u\|_\infty \|u_{xxx}\| \\ &\leq \|u_n\|_\infty \|u_{nxxx} - u_{xxx}\| + k_1 \|u_n - u\|_{H^1} \|u_{xxx}\| \rightarrow 0, \end{aligned} \quad (4.4)$$

as $n \rightarrow \infty$, uniformly in t .

Consequently, it follows that

$$(u_t - u_{xxt} + \gamma(u - u_{xx}), \eta) = (Nu + f, \eta), \eta \in L^2_{\text{per}}. \quad (4.5)$$

Thanks to the estimates obtained in the previous section, we have

$$u_t - u_{xxt} + \gamma(u - u_{xx}) = Nu + f, \quad (4.6)$$

a.e. on $\mathbb{R}^1 \times \Omega$.

So we obtain that the existence of time periodic solution for (1.6)–(1.8), which is the following theorem.

Theorem 4.1. *Given $f \in C^1(\omega; H^{k+1}(\Omega))$, $k \geq 0$, there exists a time periodic solution $u(t, x)$ to (1.6)–(1.8), such that $u(t, x) \in L^\infty(\omega; H^{k+4}(\Omega)) \cap W^{1,\infty}(\omega; H^k(\Omega))$.*

Under the assumption of Theorem 4.1, we are unable to prove the uniqueness of the solution for (1.6)–(1.8). But if we assume that M is sufficiently small, then the result can be obtained.

Theorem 4.2. *Suppose that the assumption in Theorem 4.1 holds. If M is sufficiently small, then the time periodic solution of (1.6)–(1.8) in Theorem 4.1 is unique.*

Proof. Let u and \bar{u} be any two time periodic solutions of (1.6)–(1.8). With $v = u - \bar{u}$, we can get from (1.6) that

$$v_t - v_{xxt} + \gamma(v - v_{xx}) = Nu - N\bar{u}. \quad (4.7)$$

Taking the inner product of (4.7) with v , we have

$$\frac{1}{2} \frac{d}{dt} (\|v\|^2 + \|v_x\|^2) + \gamma (\|v\|^2 + \|v_x\|^2) = (Nu - N\bar{u}, v). \quad (4.8)$$

Since,

$$\begin{aligned} |(-3uu_x + 3\bar{u}\bar{u}_x, v)| &\leq |(-3uv_x, v)| + |(-3\bar{u}_x v, v)| \\ &\leq \frac{3}{2} \|u\|_\infty (\|v\|^2 + \|v_x\|^2) + 3\|\bar{u}_x\|_\infty \|v\|^2 \\ &\leq \left(\frac{3k_1}{2} c_1^{1/2} + 3k_1 c_2^{1/2}\right) \|v\|^2 + \frac{3k_1}{2} c_1^{1/2} \|v_x\|^2, \end{aligned} \quad (4.9)$$

$$\begin{aligned} |(2u_x u_{xx} - 2\bar{u}_x \bar{u}_{xx}, v)| &\leq |(2u_x v_{xx}, v)| + |(2\bar{u}_{xx} v_x, v)| \\ &\leq \|u_{xx}\|_\infty (\|v\|^2 + \|v_x\|^2) + 2\|u_x\|_\infty \|v_x\|^2 + \|\bar{u}_{xx}\|_\infty (\|v\|^2 + \|v_x\|^2) \\ &\leq 2k_1 (c_2 + c)^{1/2} \|v\|^2 + \left[2k_1 (c_2 + c)^{1/2} + 2k_1 c_2^{1/2}\right] \|v_x\|^2, \end{aligned} \quad (4.10)$$

$$\begin{aligned} |(uu_{xxx} - \bar{u}\bar{u}_{xxx}, v)| &\leq |(uv_{xxx}, v)| + |(\bar{u}_{xxx} v, v)| \\ &\leq \int_\Omega |u_{xx} v v_x| dx + \frac{3}{2} \int_\Omega |u_x v_x^2| dx + 2 \int_\Omega |\bar{u}_{xx} v v_x| dx \\ &\leq \frac{1}{2} \|u_{xx}\|_\infty (\|v\|^2 + \|v_x\|^2) + \frac{3}{2} \|u_x\|_\infty \|v_x\|^2 + \|\bar{u}_{xx}\|_\infty (\|v\|^2 + \|v_x\|^2) \\ &\leq \frac{3k_1}{2} (c_2 + c)^{1/2} \|v\|^2 + \left[\frac{3k_1}{2} (c_2 + c)^{1/2} + \frac{3k_1}{2} c_2^{1/2}\right] \|v_x\|^2. \end{aligned} \quad (4.11)$$

Hence, if M is sufficient small such that $2\gamma \geq 3k_1 c_1^{1/2} + 6k_1 c_2^{1/2} + 7k_1 (c_2 + c)^{1/2}$, $2\gamma \geq 3k_1 c_1^{1/2} + 7k_1 c_2^{1/2} + 7k_1 (c_2 + c)^{1/2}$, then it follows from (4.8)–(4.11), we get

$$\frac{d}{dt} (\|v\|^2 + \|v_x\|^2) + \rho (\|v\|^2 + \|v_x\|^2) \leq 0, \quad (4.12)$$

where $\rho \geq 0$ is suitable constant.

Applying Gronwall's inequality, we derive that

$$\left(\|v(t)\|^2 + \|v_x(t)\|^2\right) \leq \left(\|v(0)\|^2 + \|v_x(0)\|^2\right) e^{-\rho t}, \quad \text{for any } t \geq 0. \quad (4.13)$$

Since v is ω -periodic in t , then for any positive integer m we have

$$\|v(t)\|^2 + \|v_x(t)\|^2 = \|v(t + m\omega)\|^2 + \|v_x(t + m\omega)\|^2. \quad (4.14)$$

Then we can infer that

$$\left(\|v(t)\|^2 + \|v_x(t)\|^2\right) \leq \left(\|v(0)\|^2 + \|v_x(0)\|^2\right) e^{-\rho(t+m\omega)}. \quad (4.15)$$

It follows from $v(0) = v_x(0) = 0$ that $u(t, x) = \bar{u}(t, x)$, which completes the proof of Theorem 4.2. \square

References

- [1] R. Camassa and D. D. Holm, "An integrable shallow water equation with peaked solitons," *Physical Review Letters*, vol. 71, no. 11, pp. 1661–1664, 1993.
- [2] R. Camassa, D. D. Holm, and J. M. Hyman, "A new integrable shallow water equation," *Advances in Applied Mechanics*, vol. 31, pp. 1–33, 1994.
- [3] A. Constantin and W. A. Strauss, "Stability of a class of solitary waves in compressible elastic rods," *Physics Letters A*, vol. 270, no. 3-4, pp. 140–148, 2000.
- [4] H.-H. Dai, "Model equations for nonlinear dispersive waves in a compressible Mooney-Rivlin rod," *Acta Mechanica*, vol. 127, no. 1–4, pp. 193–207, 1998.
- [5] J. Lenells, "Conservation laws of the Camassa-Holm equation," *Journal of Physics A*, vol. 38, no. 4, pp. 869–880, 2005.
- [6] B. Fuchssteiner and A. S. Fokas, "Symplectic structures, their Bäcklund transformations and hereditary symmetries," *Physica D*, vol. 4, no. 1, pp. 47–66, 1981/82.
- [7] A. Constantin, "On the scattering problem for the Camassa-Holm equation," *The Royal Society of London. Proceedings. Series A*, vol. 457, no. 2008, pp. 953–970, 2001.
- [8] A. Constantin, "On the inverse spectral problem for the Camassa-Holm equation," *Journal of Functional Analysis*, vol. 155, no. 2, pp. 352–363, 1998.
- [9] A. Constantin, V. S. Gerdjikov, and R. I. Ivanov, "Inverse scattering transform for the Camassa-Holm equation," *Inverse Problems*, vol. 22, no. 6, pp. 2197–2207, 2006.
- [10] A. Boutet de Monvel and D. Shepelsky, "Riemann-Hilbert approach for the Camassa-Holm equation on the line," *Comptes Rendus Mathématique. Académie des Sciences. Paris*, vol. 343, no. 10, pp. 627–632, 2006.
- [11] A. Constantin and H. P. McKean, "A shallow water equation on the circle," *Communications on Pure and Applied Mathematics*, vol. 52, no. 8, pp. 949–982, 1999.
- [12] A. Constantin and B. Kolev, "Geodesic flow on the diffeomorphism group of the circle," *Commentarii Mathematici Helvetici*, vol. 78, no. 4, pp. 787–804, 2003.
- [13] A. Constantin, T. Kappeler, B. Kolev, and P. Topalov, "On geodesic exponential maps of the Virasoro group," *Annals of Global Analysis and Geometry*, vol. 31, no. 2, pp. 155–180, 2007.
- [14] R. Beals, D. H. Sattinger, and J. Szmigielski, "Multi-peakons and a theorem of Stieltjes," *Inverse Problems*, vol. 15, no. 1, pp. L1–L4, 1999.
- [15] A. Constantin and W. A. Strauss, "Stability of peakons," *Communications on Pure and Applied Mathematics*, vol. 53, no. 5, pp. 603–610, 2000.
- [16] A. Constantin and W. A. Strauss, "Stability of the Camassa-Holm solitons," *Journal of Nonlinear Science*, vol. 12, no. 4, pp. 415–422, 2002.
- [17] A. Constantin, "The trajectories of particles in Stokes waves," *Inventiones Mathematicae*, vol. 166, no. 3, pp. 523–535, 2006.
- [18] A. Constantin and J. Escher, "Particle trajectories in solitary water waves," *American Mathematical Society*, vol. 44, no. 3, pp. 423–431, 2007.
- [19] J. F. Toland, "Stokes waves," *Topological Methods in Nonlinear Analysis*, vol. 7, no. 1, pp. 1–48, 1996.
- [20] A. Constantin, "Existence of permanent and breaking waves for a shallow water equation: a geometric approach," *Université de Grenoble. Annales de l'Institut Fourier*, vol. 50, no. 2, pp. 321–362, 2000.
- [21] A. Constantin and J. Escher, "Wave breaking for nonlinear nonlocal shallow water equations," *Acta Mathematica*, vol. 181, no. 2, pp. 229–243, 1998.
- [22] Y. A. Li and P. J. Olver, "Well-posedness and blow-up solutions for an integrable nonlinearly dispersive model wave equation," *Journal of Differential Equations*, vol. 162, no. 1, pp. 27–63, 2000.
- [23] G. Rodríguez-Blanco, "On the Cauchy problem for the Camassa-Holm equation," *Nonlinear Analysis*, vol. 46, no. 3, pp. 309–327, 2001.
- [24] Z. Yin, "Well-posedness, blowup, and global existence for an integrable shallow water equation," *Discrete and Continuous Dynamical Systems. Series A*, vol. 11, no. 2-3, pp. 393–411, 2004.
- [25] Z. Yin, "Well-posedness, global solutions and blowup phenomena for a nonlinearly dispersive wave equation," *Journal of Evolution Equations*, vol. 4, no. 3, pp. 391–419, 2004.
- [26] A. Constantin and J. Escher, "Global existence and blow-up for a shallow water equation," *Annali della Scuola Normale Superiore di Pisa. Classe di Scienze. Serie IV*, vol. 26, no. 2, pp. 303–328, 1998.
- [27] A. Constantin and J. Escher, "Well-posedness, global existence, and blowup phenomena for a periodic quasi-linear hyperbolic equation," *Communications on Pure and Applied Mathematics*, vol. 51, no. 5, pp. 475–504, 1998.

- [28] A. Constantin and J. Escher, "On the blow-up rate and the blow-up set of breaking waves for a shallow water equation," *Mathematische Zeitschrift*, vol. 233, no. 1, pp. 75–91, 2000.
- [29] R. Danchin, "A few remarks on the Camassa-Holm equation," *Differential and Integral Equations*, vol. 14, no. 8, pp. 953–988, 2001.
- [30] A. Bressan and A. Constantin, "Global conservative solutions of the Camassa-Holm equation," *Archive for Rational Mechanics and Analysis*, vol. 183, no. 2, pp. 215–239, 2007.
- [31] A. Constantin and L. Molinet, "Global weak solutions for a shallow water equation," *Communications in Mathematical Physics*, vol. 211, no. 1, pp. 45–61, 2000.
- [32] L. Molinet, "On well-posedness results for Camassa-Holm equation on the line: a survey," *Journal of Nonlinear Mathematical Physics*, vol. 11, no. 4, pp. 521–533, 2004.
- [33] E. Wahlén, "Global existence of weak solutions to the Camassa-Holm equation," *International Mathematics Research Notices*, vol. 2006, Article ID 28976, 12 pages, 2006.
- [34] Z. Xin and P. Zhang, "On the weak solutions to a shallow water equation," *Communications on Pure and Applied Mathematics*, vol. 53, no. 11, pp. 1411–1433, 2000.
- [35] A. Bressan and A. Constantin, "Global dissipative solutions of the Camassa-Holm equation," *Analysis and Applications*, vol. 5, no. 1, pp. 1–27, 2007.
- [36] J. Escher and Z. Yin, "Initial boundary value problems of the Camassa-Holm equation," *Communications in Partial Differential Equations*, vol. 33, no. 1–3, pp. 377–395, 2008.
- [37] J. Escher and Z. Yin, "Initial boundary value problems for nonlinear dispersive wave equations," *Journal of Functional Analysis*, vol. 256, no. 2, pp. 479–508, 2009.
- [38] E. Ott and R. N. Sudan, "Damping of solitary waves," *Physics of Fluids*, vol. 13, no. 6, pp. 1432–1434, 1970.
- [39] J.-M. Ghidaglia, "Weakly damped forced Korteweg-de Vries equations behave as a finite-dimensional dynamical system in the long time," *Journal of Differential Equations*, vol. 74, no. 2, pp. 369–390, 1988.
- [40] S. Y. Wu and Z. Y. Yin, "Blowup and decay of solutions to the weakly dissipative Camassa-Holm equation," *Acta Mathematicae Applicatae Sinica*, vol. 30, no. 6, pp. 996–1003, 2007.
- [41] S. Wu and Z. Yin, "Blow-up, blow-up rate and decay of the solution of the weakly dissipative Camassa-Holm equation," *Journal of Mathematical Physics*, vol. 47, no. 1, Article ID 013504, 12 pages, 2006.
- [42] S. Wu and Z. Yin, "Global existence and blow-up phenomena for the weakly dissipative Camassa-Holm equation," *Journal of Differential Equations*, vol. 246, no. 11, pp. 4309–4321, 2009.
- [43] H. Kato, "Existence of periodic solutions of the Navier-Stokes equations," *Journal of Mathematical Analysis and Applications*, vol. 208, no. 1, pp. 141–157, 1997.
- [44] B. Wang, "Existence of time periodic solutions for the Ginzburg-Landau equations of superconductivity," *Journal of Mathematical Analysis and Applications*, vol. 232, no. 2, pp. 394–412, 1999.
- [45] Y. Fu and B. Guo, "Time periodic solution of the viscous Camassa-Holm equation," *Journal of Mathematical Analysis and Applications*, vol. 313, no. 1, pp. 311–321, 2006.
- [46] L. C. Evans, *Partial Differential Equations*, vol. 19 of *Graduate Studies in Mathematics*, American Mathematical Society, Providence, RI, USA, 1998.
- [47] R. A. Adams, *Sobolev Spaces*, vol. 6, Academic Press, New York, NY, USA, 1975.