Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2011, Article ID 456547, 12 pages
doi:10.1155/2011/456547

Research Article

Regularity Criterion for Weak Solution to the 3D Micropolar Fluid Equations

Yu-Zhu Wang and Zigao Chen

School of Mathematics and Information Sciences, North China University of Water Resources and Electric Power, Zhengzhou 450011, China

Correspondence should be addressed to Yu-Zhu Wang, yuzhu108@163.com

Received 20 March 2011; Accepted 26 June 2011

Academic Editor: Ch Tsitouras

Regularity criterion for the 3D micropolar fluid equations is investigated. We prove that, for some $T > 0$, if $\int_0^T \|v_{x_3}\|_{L^\varrho}^{\rho} dt < \infty$, where $3/\varrho + 2/\rho \leq 1$ and $\varrho \geq 3$, then the solution (v, w) can be extended smoothly beyond $t = T$. The derivative v_{x_3} can be substituted with any directional derivative of v.

1. Introduction

In the paper, we investigate the initial value problem for the micropolar fluid equations in $\mathbb{R}^3$:

$$\begin{gathered}\partial_t v - (\nu + \kappa)\Delta v + v \cdot \nabla v + \nabla \pi - 2\kappa \nabla \times w = 0,\\ \partial_t w - \gamma \Delta w - (\alpha + \beta)\nabla\nabla \cdot w + 4\kappa w + v \cdot \nabla w - 2\kappa \nabla \times v = 0,\\ \nabla \cdot v = 0\end{gathered} \tag{1.1}$$

with the initial value

$$t = 0: \quad v = v_0(x), \quad w = w_0(x), \tag{1.2}$$

where $v(t, x)$, $w(t, x)$, and $\pi(t, x)$ stand for the divergence free velocity field, nondivergence free microrotation field (angular velocity of the rotation of the particles of the fluid), the scalar pressure, respectively $\nu > 0$ is the Newtonian kinetic viscosity, $\kappa > 0$ is the dynamics microrotation viscosity, and $\alpha, \beta, \gamma > 0$ are the angular viscosity (see, e.g., Lukaszewicz [1]).

The micropolar fluid equations was first proposed by Eringen [2]. It is a type of fluids which exhibits the microrotational effects and microrotational inertia and can be viewed as a non-Newtonian fluid. Physically, micropolar fluid may represent fluids that consists of rigid, randomly oriented (or spherical) particles suspended in a viscous medium, where the deformation of fluid particles is ignored. It can describe many phenomena appeared in a large number of complex fluids such as the suspensions, animal blood, and liquid crystals which cannot be characterized appropriately by the Navier-Stokes equations, and that is important to the scientists working with the hydrodynamic fluid problems and phenomena. For more background, we refer to [1] and references therein. Besides their physical applications, the micropolar fluid equations are also mathematically significant. The existences of weak and strong solutions for micropolar fluid equations were treated by Galdi and Rionero [3] and Yamaguchi [4], respectively. The convergence of weak solutions of the micropolar fluids in bounded domains of $\mathbb{R}^n$ was investigated (see [5]). When the viscosities tend to zero, in the limit, a fluid governed by an Euler-like system was found. Fundamental mathematical issues such as the global regularity of their solutions have generated extensive research, and many interesting results have been obtained (see [6–8]). A Beale-Kato-Madja criterion (see [9]) of smooth solutions to a related model with (1.1) was established in [10].

If $\kappa = 0$ and $w = 0$, then (1.1) reduces to be the Navier-Stokes equations. Besides its physical applications, the Navier-Stokes equations are also mathematically significant. In the last century, Leray [11] and Hopf [12] constructed weak solutions to the Navier-Stokes equations. The solution is called the Leray-Hopf weak solution. Later on, much effort has been devoted to establish the global existence and uniqueness of smooth solutions to the Navier-Stokes equations. Different criteria for regularity of the weak solutions have been proposed, and many interesting results are established (see [13–31]).

The purpose of this paper is to establish the regularity criteria of weak solutions to (1.1), (1.2) via the derivative of the velocity in one direction. It is proved that if $\int_0^T \|v_{x_3}\|_{L^{\varrho}}^{\rho} dt < \infty$ with

$$\frac{3}{\varrho} + \frac{2}{\rho} \leq 1, \quad \varrho \geq 3, \tag{1.3}$$

then the solution (v, w) can be extended smoothly beyond $t = T$.

The paper is organized as follows. We first state some important inequalities in Section 2, which play an important roles in the proof of our main result. Then, we give definition of weak solution and state main results in Section 3 and then prove main result in Section 4.

2. Preliminaries

In order to prove our main result, we need the following Lemma, which may be found in [32] (see also [33, 34]). For the convenience of the readers, the proof of the Lemmas are provided.

Lemma 2.1. *Assume that $\mu, \lambda, \iota \in \mathbb{R}$ and satisfy*

$$1 \leq \mu, \lambda < \infty, \qquad \frac{1}{\mu} + \frac{2}{\lambda} > 1, \qquad 1 + \frac{3}{\iota} = \frac{1}{\mu} + \frac{2}{\lambda}. \tag{2.1}$$

Assume that $f \in H^1(\mathbb{R}^3)$, $f_{x_1}, f_{x_2} \in L^\lambda(\mathbb{R}^3)$, and $f_{x_3} \in L^\mu(\mathbb{R}^3)$. Then, there exists a positive constant such that

$$\|f\|_{L^\iota} \le C\|f_{x_1}\|_{L^\lambda}^{1/3}\|f_{x_2}\|_{L^\lambda}^{1/3}\|f_{x_3}\|_{L^\mu}^{1/3}. \tag{2.2}$$

Especially, when $\lambda = 2$, there exists a positive constant $C = C(\mu)$ such that

$$\|f\|_{L^{3\mu}} \le C\|f_{x_1}\|_{L^2}^{1/3}\|f_{x_2}\|_{L^2}^{1/3}\|f_{x_3}\|_{L^\mu}^{1/3}, \tag{2.3}$$

which holds for any $f \in H^1(\mathbb{R}^3)$ and $f_{x_3} \in L^\mu(\mathbb{R}^3)$ with $1 \le \mu < \infty$.

Proof. It is not difficult to find

$$\begin{aligned}
\left|f(x_1,x_2,x_3)\right|^{1+(1-1/\lambda)\iota} &\le C\int_{-\infty}^{x_1}\left|f(x_1,x_2,x_3)\right|^{(1-(1/\lambda))\iota}\left|\partial_\tau f(\tau,x_2,x_3)\right|d\tau. \\
\left|f(x_1,x_2,x_3)\right|^{1+(1-1/\lambda)\iota} &\le C\int_{-\infty}^{x_2}\left|f(x_1,x_2,x_3)\right|^{(1-(1/\lambda))\iota}\left|\partial_\tau f(x_1,\tau,x_3)\right|d\tau, \\
\left|f(x_1,x_2,x_3)\right|^{1+(1-1/\mu)\iota} &\le C\int_{-\infty}^{x_3}\left|f(x_1,x_2,x_3)\right|^{(1-(1/\mu))\iota}\left|\partial_\tau f(x_1,x_2,\tau)\right|d\tau.
\end{aligned} \tag{2.4}$$

Then, we obtain

$$\begin{aligned}
\left|f(x_1,x_2,x_3)\right|^\iota \le C&\left[\int_{-\infty}^{\infty}\left|f(x_1,x_2,x_3)\right|^{(1-1/\lambda)\iota}\left|\partial_{x_1} f(x_1,x_2,x_3)\right|dx_1\right]^{1/2} \\
&\times\left[\int_{-\infty}^{\infty}\left|f(x_1,x_2,x_3)\right|^{(1-1/\lambda)\iota}\left|\partial_{x_2} f(x_1,x_2,x_3)\right|dx_2\right]^{1/2} \\
&\times\left[\int_{-\infty}^{\infty}\left|f(x_1,x_2,x_3)\right|^{(1-1/\mu)\iota}\left|\partial_{x_3} f(x_1,x_2,x_3)\right|dx_3\right]^{1/2}.
\end{aligned} \tag{2.5}$$

Integrating with respect to x_1 and using Hölder inequality, we have

$$\begin{aligned}
\int_{-\infty}^{\infty}\left|f(x_1,x_2,x_3)\right|^\iota dx_1 \le C&\left[\int_{-\infty}^{\infty}\left|f(x_1,x_2,x_3)\right|^{(1-1/\lambda)\iota}\left|\partial_{x_1} f(x_1,x_2,x_3)\right|dx_1\right]^{1/2} \\
&\times\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|f(x_1,x_2,x_3)\right|^{(1-1/\lambda)\iota}\left|\partial_{x_2} f(x_1,x_2,x_3)\right|dx_2dx_1\right]^{1/2} \\
&\times\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|f(x_1,x_2,x_3)\right|^{(1-1/\mu)\iota}\left|\partial_{x_3} f(x_1,x_2,x_3)\right|dx_3dx_1\right]^{1/2}.
\end{aligned} \tag{2.6}$$

Integrating with respect to x_2, x_3 and using Hölder inequality, we obtain

$$\begin{aligned}\int_{\mathbb{R}^3} |f(x_1,x_2,x_3)|^{\iota} dx \le C \left[\int_{-\infty}^{\infty} |f(x_1,x_2,x_3)|^{(1-1/\lambda)\iota} |\partial_{x_1} f(x_1,x_2,x_3)| dx\right]^{1/2} \\ \times \left[\iint_{\mathbb{R}^3} |f(x_1,x_2,x_3)|^{(1-1/\lambda)\iota} |\partial_{x_2} f(x_1,x_2,x_3)| dx\right]^{1/2} \\ \times \left[\iint_{\mathbb{R}^3} |f(x_1,x_2,x_3)|^{(1-1/\mu)\iota} |\partial_{x_3} f(x_1,x_2,x_3)| dx\right]^{1/2}.\end{aligned} \tag{2.7}$$

It follows from Hölder inequality that

$$\|f\|_{L^\iota}^{\iota} \le C \|f\|_{L^\iota}^{(1-1/\lambda)\iota/2} \|\partial_{x_1} f\|_{L^\lambda}^{1/2} \|f\|_{L^\iota}^{(1-1/\lambda)\iota/2} \|\partial_{x_2} f\|_{L^\lambda}^{1/2} \|f\|_{L^\iota}^{(1-1/\mu)\iota/2} \|\partial_{x_3} f\|_{L^\mu}^{1/2}. \tag{2.8}$$

By the above inequality, we get (2.2). □

Lemma 2.2. *Let $2 \le q \le 6$ and assume that $f \in H^1(\mathbb{R}^3)$. Then, there exists a positive constant $C = C(q)$ such that*

$$\|f\|_{L^q} \le C \|f\|_{L^2}^{(6-q)/2q} \|\partial_{x_1} f\|_{L^2}^{(q-2)/2q} \|\partial_{x_2} f\|_{L^2}^{(q-2)/2q} \|\partial_{x_3} f\|_{L^2}^{(q-2)/2q}. \tag{2.9}$$

Proof. Using the interpolating inequality, we obtain

$$\|f\|_{L^q} \le C \|f\|_{L^2}^{(6-q)/2q} \|f\|_{L^6}^{(3q-6)/2q}. \tag{2.10}$$

By (2.3) with $\mu = 2$, we have

$$\|f\|_{L^6} \le C \|\partial_{x_1} f\|_{L^2}^{1/3} \|\partial_{x_2} f\|_{L^2}^{1/3} \|\partial_{x_3} f\|_{L^2}^{1/3}. \tag{2.11}$$

Combining (2.10) and (2.11) yields (2.9). □

3. Main Results

Before stating our main results, we introduce some function spaces. Let

$$C_{0,\sigma}^{\infty}\left(\mathbb{R}^3\right) = \left\{\varphi \in \left(C^{\infty}\left(\mathbb{R}^3\right)\right)^3 : \nabla \cdot \varphi = 0\right\} \subset \left(C^{\infty}\left(\mathbb{R}^3\right)\right)^3. \tag{3.1}$$

The subspace

$$L_\sigma^2 = \overline{C_{0,\sigma}^{\infty}(\mathbb{R}^3)}^{\|\cdot\|_{L^2}} = \left\{\varphi \in L^2\left(\mathbb{R}^3\right) : \nabla \cdot \varphi = 0\right\} \tag{3.2}$$

is obtained as the closure of $C_{0,\sigma}^{\infty}$ with respect to L^2-norm $\|\cdot\|_{L^2}$. H_{σ}^{r} is the closure of $C_{0,\sigma}^{\infty}$ with respect to the H^r-norm

$$\|\varphi\|_{H^r} = \left\|(I-\Delta)^{r/2}\varphi\right\|_{L^2}, \quad r \geq 0. \tag{3.3}$$

Before stating our main results, we give the definition of weak solution to (1.1), (1.2) (see [6]).

Definition 3.1 (Weak solutions). Let $T > 0$, $v_0 \in L_{\sigma}^2(\mathbb{R}^3)$, and $w_0 \in L^2(\mathbb{R}^3)$. A measurable $\mathbb{R}^3$-valued triple (v, w) is said to be a weak solution to (1.1), (1.2) on $[0, T]$ if the following conditions hold the following.

(1)

$$\begin{aligned} &v \in L^{\infty}\left(0, T; L_{\sigma}^2\left(\mathbb{R}^3\right)\right) \bigcap L^2\left(0, T; H_{\sigma}^1\left(\mathbb{R}^3\right)\right), \\ &w \in L^{\infty}\left(0, T; L^2\left(\mathbb{R}^3\right)\right) \bigcap L^2\left(0, T; H^1\left(\mathbb{R}^3\right)\right). \end{aligned} \tag{3.4}$$

(2) Equations (1.1), (1.2) are satisfied in the sense of distributions; that is, for every $\varphi \in H^1((0,T); H_{\sigma}^1)$ and $\psi \in H^1((0,T); H^1)$ with $\varphi(T) = \psi(T) = 0$, hold

$$\begin{aligned} &\int_0^T \{-\langle v, \partial_\tau \varphi\rangle + \langle v\cdot\nabla v, \varphi\rangle + (\nu+\kappa)\langle \nabla v, \nabla\varphi\rangle\}d\tau - \int_0^T \{2\kappa\langle \nabla\times w, \varphi\rangle\}d\tau \\ &\quad = \langle v_0, \varphi(0)\rangle, \end{aligned} \tag{3.5}$$

$$\begin{aligned} &\int_0^T \{-\langle w, \partial_\tau \psi\rangle\} + \gamma\langle \nabla w, \nabla\psi\rangle + (\alpha+\beta)\langle \nabla\cdot w, \nabla\psi\rangle + 4\kappa\langle w, \psi\rangle d\tau \\ &\qquad + \int_0^T \{\langle v\cdot\nabla w, \psi\rangle - 2\kappa\langle \nabla\times v, \psi\rangle\}d\tau \\ &\quad = \langle w_0, \psi(0)\rangle. \end{aligned} \tag{3.6}$$

(3) The energy inequality, that is,

$$\begin{aligned} &\|v(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2 + 2\int_0^t \left(\nu\|\nabla v(\tau)\|_{L^2}^2 + \gamma\|\nabla w(\tau)\|_{L^2}^2\right)d\tau + 2(\alpha+\beta)\int_0^t \|\nabla\cdot w(\tau)\|_{L^2}^2 d\tau \\ &\quad \leq \|v_0\|_{L^2}^2 + \|w_0\|_{L^2}^2. \end{aligned} \tag{3.7}$$

Theorem 3.2. *Let* $v_0 \in H_{\sigma}^1(\mathbb{R}^3)$ *with* $w_0 \in H^1(\mathbb{R}^3)$. *Assume that* (v, w) *is a weak solution to* (1.1), (1.2) *on some interval* $[0, T]$. *If*

$$\Theta(T) \equiv \int_0^T \|v_{x_3}\|_{L^{\varrho}}^{\rho} dt < \infty, \tag{3.8}$$

where

$$\frac{3}{\varrho}+\frac{2}{\rho}\leq 1,\quad \varrho\geq 3, \tag{3.9}$$

then the solution (v, w) *can be extended smoothly beyond* $t = T$*.*

4. Proof of Theorem 3.2

Proof. Multiplying the first equation of (1.1) by v and integrating with respect to x on $\mathbb{R}^3$, using integration by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|_{L^2}^2+(\nu+\kappa)\|\nabla v(t)\|_{L^2}^2=2\kappa\int_{\mathbb{R}^3}(\nabla\times w)\cdot v dx. \tag{4.1}$$

Similarly, we get

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_{L^2}^2+\gamma\|\nabla w(t)\|_{L^2}^2+(\alpha+\beta)\|\nabla\cdot w\|_{L^2}^2+4\kappa\|w\|_{L^2}^2=2\kappa\int_{\mathbb{R}^3}(\nabla\times v)\cdot w dx. \tag{4.2}$$

Summing up (4.1)-(4.2), we deduce that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|v(t)\|_{L^2}^2+\|w(t)\|_{L^2}^2\right)+(\nu+\kappa)\|\nabla v(t)\|_{L^2}^2\\&\quad+\gamma\|\nabla w(t)\|_{L^2}^2+(\alpha+\beta)\|\nabla\cdot w\|_{L^2}^2+4\kappa\|w\|_{L^2}^2\\&=2\kappa\int_{\mathbb{R}^3}(\nabla\times w)\cdot v dx+2\kappa\int_{\mathbb{R}^3}(\nabla\times v)\cdot w dx.\end{aligned} \tag{4.3}$$

By integration by parts and Cauchy inequality, we obtain

$$2\kappa\int_{\mathbb{R}^3}(\nabla\times w)\cdot v dx+2\kappa\int_{\mathbb{R}^3}(\nabla\times v)\cdot w dx\leq\kappa\|\nabla v\|_{L^2}^2+4\kappa\|w\|_{L^2}^2. \tag{4.4}$$

Combining (4.3)-(4.4) yields

$$\frac{1}{2}\frac{d}{dt}\left(\|v(t)\|_{L^2}^2+\|w(t)\|_{L^2}^2\right)+\nu\|\nabla v(t)\|_{L^2}^2+\gamma\|\nabla w(t)\|_{L^2}^2+(\alpha+\beta)\|\nabla\cdot w\|_{L^2}^2\leq 0. \tag{4.5}$$

Integrating with respect to t, we have

$$\begin{aligned}&\|v(t)\|_{L^2}^2+\|w(t)\|_{L^2}^2+2\int_0^t\left(\nu\|\nabla v(\tau)\|_{L^2}^2+\gamma\|\nabla w(\tau)\|_{L^2}^2\right)d\tau+2(\alpha+\beta)\int_0^t\|\nabla\cdot w(\tau)\|_{L^2}^2 d\tau\\&\quad\leq\|v_0\|_{L^2}^2+\|w_0\|_{L^2}^2.\end{aligned} \tag{4.6}$$

Differentiating (1.1) with respect to x_3, we obtain

$$\begin{gathered}\partial_t v_{x_3} - (\nu+\kappa)\Delta v_{x_3} + v_{x_3}\cdot\nabla v + v\cdot\nabla v_{x_3} + \nabla\pi_{x_3} - 2\kappa\nabla\times w_{x_3} = 0,\\ \partial_t w_{x_3} - \gamma\Delta w_{x_3} - (\alpha+\beta)\nabla\nabla\cdot w_{x_3} + 4\kappa w_{x_3} + v_{x_3}\cdot\nabla w + v\cdot\nabla w_{x_3} - 2\kappa\nabla\times v_{x_3} = 0.\end{gathered}\tag{4.7}$$

Taking the inner product of v_{x_3} with the first equation of (4.7) and using integration by parts yields

$$\frac{1}{2}\frac{d}{dt}\|v_{x_3}(t)\|_{L^2}^2 + (\nu+\kappa)\|\nabla v_{x_3}(t)\|_{L^2}^2 = -\int_{\mathbb{R}^3} v_{x_3}\cdot\nabla v\cdot v_{x_3}dx + 2\kappa\int_{\mathbb{R}^3}(\nabla\times w_{x_3})\cdot v_{x_3}dx.\tag{4.8}$$

Similarly, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|w_{x_3}(t)\|_{L^2}^2 + \gamma\|\nabla w_{x_3}(t)\|_{L^2}^2 + (\alpha+\beta)\|\nabla\cdot w_{x_3}\|_{L^2}^2 + 4\kappa\|w_{x_3}\|_{L^2}^2\\ &\quad= -\int_{\mathbb{R}^3} v_{x_3}\cdot\nabla w\cdot w_{x_3}dx + 2\kappa\int_{\mathbb{R}^3}(\nabla\times v_{x_3})\cdot w_{x_3}dx.\end{aligned}\tag{4.9}$$

Combining (4.8)–(4.9) yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|v_{x_3}(t)\|_{L^2}^2 + \|w_{x_3}(t)\|_{L^2}^2\right) + (\nu+\kappa)\|\nabla v_{x_3}(t)\|_{L^2}^2\\ &\qquad+\gamma\|\nabla w_{x_3}(t)\|_{L^2}^2 + (\alpha+\beta)\|\nabla\cdot w_{x_3}\|_{L^2}^2 + 4\kappa\|w_{x_3}\|_{L^2}^2\\ &\quad= -\int_{\mathbb{R}^3} v_{x_3}\cdot\nabla v\cdot v_{x_3}dx + 2\kappa\int_{\mathbb{R}^3}(\nabla\times w_{x_3})\cdot v_{x_3}dx\\ &\qquad-\int_{\mathbb{R}^3} v_{x_3}\cdot\nabla w\cdot w_{x_3}dx + 2\kappa\int_{\mathbb{R}^3}(\nabla\times v_{x_3})\cdot w_{x_3}dx.\end{aligned}\tag{4.10}$$

Using integration by parts and Cauchy inequality, we obtain

$$2\kappa\int_{\mathbb{R}^3}(\nabla\times w_{x_3})\cdot v_{x_3}dx + 2\kappa\int_{\mathbb{R}^3}(\nabla\times v_{x_3})\cdot w_{x_3}dx \le \kappa\|\nabla v_{x_3}\|_{L^2}^2 + 4\kappa\|w_{x_3}\|_{L^2}^2.\tag{4.11}$$

Combining (4.10)–(4.11) yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|v_{x_3}(t)\|_{L^2}^2 + \|w_{x_3}(t)\|_{L^2}^2\right) + \nu\|\nabla v_{x_3}(t)\|_{L^2}^2\\ &\qquad+\gamma\|\nabla w_{x_3}(t)\|_{L^2}^2 + (\alpha+\beta)\|\nabla\cdot w_{x_3}\|_{L^2}^2\\ &\quad\le -\int_{\mathbb{R}^3} v_{x_3}\cdot\nabla v\cdot v_{x_3}dx - \int_{\mathbb{R}^3} v_{x_3}\cdot\nabla w\cdot w_{x_3}dx\\ &\quad\triangleq I_1 + I_2.\end{aligned}\tag{4.12}$$

In what follows, we estimate I_j $(j = 1, 2 \ldots, 5)$. By integration by parts and Hölder inequality, we obtain

$$I_1 \le C\|\nabla v_{x_3}\|_{L^2}\|v_{x_3}\|_{L^\sigma}\|v\|_{L^{3\varrho}}, \tag{4.13}$$

where

$$\frac{1}{\sigma}+\frac{1}{3\varrho}=\frac{1}{2}, \quad 2\le\sigma\le 6. \tag{4.14}$$

It follows from the interpolating inequality that

$$\|v_{x_3}\|_{L^\sigma} \le C\|v_{x_3}\|_{L^2}^{1-3(1/2-1/\sigma)}\|\nabla v_{x_3}\|_{L^2}^{3(1/2-1/\sigma)}. \tag{4.15}$$

From (2.3), we get

$$\begin{aligned} I_1 &\le C\|\nabla v_{x_3}\|_{L^2}\|v_{x_3}\|_{L^2}^{1-3(1/2-1/\sigma)}\|\nabla v_{x_3}\|_{L^2}^{3(1/2-1/\sigma)}\|\nabla v\|_{L^2}^{2/3}\|v_{x_3}\|_{L^\varrho}^{1/3} \\ &\le C\|\nabla v_{x_3}\|_{L^2}^{1+3(1/2-1/\sigma)}\|v_{x_3}\|_{L^2}^{1-3(1/2-1/\sigma)}\|\nabla v\|_{L^2}^{2/3}\|v_{x_3}\|_{L^\varrho}^{1/3} \\ &\le \frac{\nu}{2}\|\nabla v_{x_3}\|_{L^2}^2 + C\|v_{x_3}\|_{L^2}^2\|\nabla v\|_{L^2}^{2q}\|v_{x_3}\|_{L^2}^{q}, \end{aligned} \tag{4.16}$$

where

$$q=\frac{2}{3-9(1/2-1/\sigma)}=\frac{2}{3(1-1/\varrho)}. \tag{4.17}$$

When $\varrho \ge 3$, we have $2q \le 2$ and application of Young inequality yields

$$I_1 \le \frac{\nu}{2}\|\nabla v_{x_3}\|_{L^2}^2 + C\|v_{x_3}\|_{L^2}^2\left(\|\nabla v\|_{L^2}^2+\|v_{x_3}\|_{L^\varrho}^{\delta}\right), \tag{4.18}$$

where

$$\frac{3}{\varrho}+\frac{2}{\delta}=1. \tag{4.19}$$

From Hölder inequality, we obtain

$$\begin{aligned} I_2 &\le C\|\nabla w\|_{L^2}\|w_{x_3}\|_{L^{2\varrho/(\varrho-2)}}\|v_{x_3}\|_{L^\varrho} \\ &\le C\|\nabla w\|_{L^2}\|v_{x_3}\|_{L^\varrho}\|w_{x_3}\|_{L^2}^{1-3/\varrho}\|\nabla w_{x_3}\|_{L^2}^{3/\varrho} \\ &\le C\|\nabla w_{x_3}\|_{L^2}^2 + \|\nabla w\|_{L^2}^{2\varrho/(2\varrho-3)}\|v_{x_3}\|_{L^\varrho}^{2\varrho/(2\varrho-3)}\ \|w_{x_3}\|_{L^2}^{(2\varrho-6)/(2\varrho-3)} \\ &\le \frac{\gamma}{2}\|\nabla w_{x_3}\|_{L^2}^2 + C\left(\|\nabla w\|_{L^2}^2+\|v_{x_3}\|_{L^\varrho}^{\delta}\right)\|w_{x_3}\|_{L^2}^{(2\varrho-6)/(2\varrho-3)}, \end{aligned} \tag{4.20}$$

where

$$\frac{3}{\varrho} + \frac{2}{\delta} = 1. \tag{4.21}$$

Combining (4.12)–(4.20) yields

$$\begin{aligned} &\frac{d}{dt}\left(\|v_{x_3}\|_{L^2}^2 + \|w_{x_3}\|_{L^2}^2\right) + \nu\|\nabla v_{x_3}\|_{L^2}^2 + \gamma\|\nabla w_{x_3}\|_{L^2}^2 + (\alpha+\beta)\|\nabla\cdot w_{x_3}\|_{L^2}^2 \\ &\quad \le C\|v_{x_3}\|_{L^2}^2\left(\|\nabla v\|_{L^2}^2 + \|v_{x_3}\|_{L^\varrho}^{\delta}\right) + C\left(\|\nabla w\|_{L^2}^2 + \|v_{x_3}\|_{L^\varrho}^{\delta}\right)\|w_{x_3}\|_{L^2}^{(2\varrho-6)/(2\varrho-3)}. \end{aligned} \tag{4.22}$$

From Gronwall inequality, we get

$$\begin{aligned} &\|v_{x_3}\|_{L^2}^2 + \|w_{x_3}\|_{L^2}^2 + \nu\int_0^t \|\nabla v_{x_3}\|_{L^2}^2 d\tau + \int_0^t\left(\gamma\|\nabla w_{x_3}\|_{L^2}^2 + (\alpha+\beta)\|\nabla\cdot w_{x_3}\|_{L^2}^2\right)d\tau \\ &\quad \le Ce^{(\|v_0\|_{L^2}^2+\|w_0\|_{L^2}^2)}e^{\Theta(t)}\left[\|v_0\|_{H^1}^2 + \|w_0\|_{H^1}^2 + C\left(\|v_0\|_{L^2}^2 + \|w_0\|_{L^2}^2 + \Theta(t)\right)^{2\varrho-3/\varrho}\right]. \end{aligned} \tag{4.23}$$

Multiplying the first equation of (1.1) by $-\Delta v$ and integrating with respect to x on $\mathbb{R}^3$, then using integration by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla v(t)\|_{L^2}^2 + (\nu+\kappa)\|\Delta v\|_{L^2}^2 = \int_{\mathbb{R}^3} v\cdot\nabla v\cdot\Delta v dx - 2\kappa\int_{\mathbb{R}^3}(\nabla\times w)\cdot\Delta v dx. \tag{4.24}$$

Similarly, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\nabla w(t)\|_{L^2}^2 + \gamma\|\Delta w\|_{L^2}^2 + (\alpha+\beta)\|\nabla\nabla\cdot w\|_{L^2}^2 + 4\kappa\|\nabla w\|_{L^2}^2 \\ &\quad = \int_{\mathbb{R}^3} v\cdot\nabla w\cdot\Delta w dx - 2\kappa\int_{\mathbb{R}^3}(\nabla\times v)\cdot\Delta w dx. \end{aligned} \tag{4.25}$$

Collecting (4.24) and (4.25) yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\nabla v(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2\right) + (\nu+\kappa)\|\Delta v\|_{L^2}^2 \\ &\qquad + \gamma\|\Delta w\|_{L^2}^2 + (\alpha+\beta)\|\nabla\nabla\cdot w\|_{L^2}^2 + 4\kappa\|\nabla w\|_{L^2}^2 \\ &\quad = \int_{\mathbb{R}^3} v\cdot\nabla v\cdot\Delta v dx - 2\kappa\int_{\mathbb{R}^3}(\nabla\times w)\cdot\Delta v dx \\ &\qquad + \int_{\mathbb{R}^3} v\cdot\nabla w\cdot\Delta w dx - 2\kappa\int_{\mathbb{R}^3}(\nabla\times v)\cdot\Delta w dx. \end{aligned} \tag{4.26}$$

Thanks to integration by parts and Cauchy inequality, we get

$$-2\kappa\int_{\mathbb{R}^3}(\nabla\times w)\cdot\Delta v dx-2\kappa\int_{\mathbb{R}^3}(\nabla\times v)\cdot\Delta w dx\le\kappa\|\Delta v\|_{L^2}^2+4\kappa\|\nabla w\|_{L^2}^2. \tag{4.27}$$

It follows from (4.26)-(4.27) and integration by parts that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla v(t)\|_{L^2}^2+\|\nabla w(t)\|_{L^2}^2\right)+\nu\|\Delta v\|_{L^2}^2+\gamma\|\Delta w\|_{L^2}^2+(\alpha+\beta)\|\nabla\nabla\cdot w\|_{L^2}^2\\
&\quad\le-\int_{\mathbb{R}^3}\nabla v\cdot\nabla v\cdot\nabla v dx-\int_{\mathbb{R}^3}\nabla v\cdot\nabla w\cdot\nabla w dx\\
&\quad\triangleq J_1+J_2.
\end{aligned} \tag{4.28}$$

In what follows, we estimate $J_i(i=1,2)$.

By (2.9) and Young inequality, we deduce that

$$\begin{aligned}
J_1&\le C\|\nabla v\|_{L^3}^3\\
&\le C\|\nabla v\|_{L^2}^{3/2}\|\nabla_{\tilde{x}}\nabla v\|_{L^2}\|\nabla v_{x_3}\|_{L^2}^{1/2}\\
&\le\frac{\nu}{4}\|\nabla_{\tilde{x}}\nabla v\|_{L^2}^2+C\|\nabla v\|_{L^2}^3\|\nabla v_{x_3}\|_{L^2}\\
&\le\frac{\nu}{4}\|\nabla_{\tilde{x}}\nabla v\|_{L^2}^2+C\left(\|\nabla v\|_{L^2}^2+\|\nabla v_{x_3}\|_{L^2}^2\right)\|\nabla v\|_{L^2}^2,
\end{aligned} \tag{4.29}$$

where $\nabla_{\tilde{x}}=(\partial_{x_1},\partial_{x_2})$.

By (2.9) and Young inequality, we have

$$\begin{aligned}
J_2&\le\|\nabla v\|_{L^3}\|\nabla w\|_{L^3}^2\\
&\le C\|\nabla v\|_{L^2}^{1/2}\|\nabla_{\tilde{x}}\nabla v\|_{L^2}^{1/3}\|\nabla v_{x_3}\|_{L^2}^{1/6}\|\nabla w\|_{L^2}\|\nabla_{\tilde{x}}\nabla w\|_{L^2}^{2/3}\|\nabla w_{x_3}\|_{L^2}^{1/3}\\
&\le\frac{\nu}{4}\|\nabla_{\tilde{x}}\nabla v\|_{L^2}^2+C\|\nabla v\|_{L^2}^{3/5}\|\nabla v_{x_3}\|_{L^2}^{1/5}\|\nabla w\|_{L^2}^{6/5}\|\nabla\nabla_{\tilde{x}}w\|_{L^2}^{4/5}\|\nabla w_{x_3}\|_{L^2}^{2/5}\\
&\le\frac{\nu}{4}\|\nabla_{\tilde{x}}\nabla v\|_{L^2}^2+\frac{\gamma}{2}\|\nabla_{\tilde{x}}\nabla w\|_{L^2}^2+C\|\nabla v\|_{L^2}\|\nabla v_{x_3}\|_{L^2}^{1/3}\|\nabla w\|_{L^2}^2\|\nabla w_{x_3}\|_{L^2}^{2/3}\\
&\le\frac{\nu}{4}\|\nabla_{\tilde{x}}\nabla v\|_{L^2}^2+\frac{\gamma}{2}\|\nabla_{\tilde{x}}\nabla w\|_{L^2}^2+C\|\nabla w\|_{L^2}^2\left(\|\nabla v\|_{L^2}^2+\|\nabla v_{x_3}\|_{L^2}^2+\|\nabla w_{x_3}\|_{L^2}^2\right),
\end{aligned} \tag{4.30}$$

where $\nabla_{\tilde{x}}=(\partial_{x_1},\partial_{x_2})$.

Combining (4.28)–(4.30) yields

$$\begin{aligned}
&\frac{d}{dt}\left(\|\nabla v(t)\|_{L^2}^2+\|\nabla w(t)\|_{L^2}^2\right)+\nu\|\Delta v\|_{L^2}^2+\gamma\|\Delta v\|_{L^2}^2+(\alpha+\beta)\|\nabla\nabla\cdot w\|_{L^2}^2\\
&\qquad\le C\left(\|\nabla v\|_{L^2}^2+\|\nabla w\|_{L^2}^2\right)\left(\|\nabla v\|_{L^2}^2+\|\nabla v_{x_3}\|_{L^2}^2+\|\nabla w_{x_3}\|_{L^2}^2\right).
\end{aligned} \tag{4.31}$$

From (4.31), Gronwall inequality, (4.6), and (4.23), we know that $(v, w) \in L^\infty(0, T; H^1(\mathbb{R}^3))$. Thus, (v, w) can be extended smoothly beyond $t = T$. We have completed the proof of Theorem 3.2. □

Acknowledgments

This work was supported in part by the NNSF of China (Grant no. 10971190) and the Research Initiation Project for High-level Talents (201031) of the North China University of Water Resources and Electric Power.

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