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Research Article

The Existence of Solutions for a Nonlinear Fractional Multi-Point Boundary Value Problem at Resonance

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We discuss the existence of solution for a multipoint boundary value problem of fractional differential equation. An existence result is obtained with the use of the coincidence degree theory.

1. Introduction

In this paper, we study the multipoint boundary value problem

$$D_{0+}^{\alpha}u(t) = f\left(t, u(t), D_{0+}^{\alpha-1}u(t), D_{0+}^{\alpha-2}u(t)\right) + e(t), \quad 0 < t < 1, \tag{1.1}$$

$$I_{0+}^{3-\alpha}u(0) = 0, \qquad D_{0+}^{\alpha-2}u(0) = \sum_{j=1}^{n}\beta_j D_{0+}^{\alpha-2}u(\xi_j), \qquad u(1) = \sum_{i=1}^{m}\alpha_i u(\eta_i), \tag{1.2}$$

where $2 < \alpha \le 3,\ 0 < \xi_1 < \xi_2 < \cdots < \xi_n < 1,\ n \ge 1,\ 0 < \eta_1 < \cdots < \eta_m < 1,\ m \ge 2,\ \alpha_i, \beta_j \in \mathbb{R}$,

$$\sum_{i=1}^{m}\alpha_i\eta_i^{\alpha-1} = \sum_{i=1}^{m}\alpha_i\eta_i^{\alpha-2} = 1, \qquad \sum_{j=1}^{n}\beta_j\xi_j = 0, \quad \sum_{j=1}^{n}\beta_j = 1, \tag{1.3}$$

$f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ satisfying the Carathéodory conditions, $e \in L^1[0,1]$. D_{0+}^{α} and I_{0+}^{α} are the standard Riemann-Liouville derivative and integral, respectively. We assume, in addition, that

$$R = \frac{\Gamma(\alpha)^2\Gamma(\alpha-1)}{\Gamma(2\alpha)\Gamma(\alpha+1)}\sum_{j=1}^{n}\beta_j\xi_j^{\alpha}\left(1-\sum_{i=1}^{m}\alpha_i\eta_i^{2\alpha-1}\right) - \frac{\Gamma(\alpha)^2\Gamma(\alpha-1)}{\Gamma(\alpha+2)\Gamma(2\alpha-1)}\sum_{j=1}^{n}\beta_j\xi_j^{\alpha+1}\left(1-\sum_{i=1}^{m}\alpha_i\eta_i^{2\alpha-2}\right) \neq 0, \tag{1.4}$$

where Γ is the Gamma function. Due to condition (1.3), the fractional differential operator in (1.1), (1.2) is not invertible.

Fractional differential equation can describe many phenomena in various fields of science and engineering. Many methods have been introduced for solving fractional differential equations, such as the popular Laplace transform method, the iteration method. For details, see [1, 2] and the references therein.

Recently, there are some papers dealing with the solvability of nonlinear boundary value problems of fractional differential equation, by use of techniques of nonlinear analysis (fixed-point theorems, Leray-Schauder theory, etc.), see, for example, [3–6]. But there are few papers that consider the fractional-order boundary problems at resonance. Very recently [7], Y. H. Zhang and Z. B. Bai considered the existence of solutions for the fractional ordinary differential equation

$$D_{0+}^{\alpha}u(t) = f\left(t, u(t), D_{0+}^{\alpha-(n-1)}u(t), \ldots, D_{0+}^{\alpha-1}u(t)\right) + e(t), \quad 0 < t < 1, \tag{1.5}$$

subject to the following boundary value conditions:

$$I_{0+}^{n-\alpha}u(0) = D_{0+}^{\alpha-(n-1)}u(0) = \cdots = D_{0+}^{\alpha-2}u(0) = 0, \qquad u(1) = \sigma u(\eta), \tag{1.6}$$

where $n > 2$ is a natural number, $n-1 < \alpha \leq n$ is a real number, $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is continuous, and $e \in L^1[0,1]$, $\sigma \in (0,\infty)$, and $\eta \in (0,1)$ are given constants such that $\sigma\eta^{\alpha-1} = 1$. D_{0+}^{α} and I_{0+}^{α} are the standard Riemann-Liouville derivative and integral, respectively. By the conditions, the kernel of the linear operator is one dimensional.

Motivated by the above work and recent studies on fractional differential equations [8–18], in this paper, we consider the existence of solutions for multipoint boundary value problem (1.1), (1.2) at resonance. Note that under condition (1.3), the kernel of the linear operator in (1.1), (1.2) is two dimensional. Our method is based upon the coincidence degree theory of Mawhin [18].

Now, we will briefly recall some notation and abstract existence result.

Let Y, Z be real Banach spaces, let $L : \operatorname{dom}(L) \subset Y \to Z$ be a Fredholm map of index zero, and let $P : Y \to Y$, $Q : Z \to Z$ be continuous projectors such that $\operatorname{Im}(P) = \operatorname{Ker}(P)$, $\operatorname{Ker}(Q) = \operatorname{Im}(L)$, and $Y = \operatorname{Ker}(L) \oplus \operatorname{Ker}(P)$, $Z = \operatorname{Im}(L) \oplus \operatorname{Im}(Q)$. It follows that $L|_{\operatorname{dom}(L)\cap\operatorname{Ker}(P)} : \operatorname{dom}(L) \cap \operatorname{Ker}(P) \to \operatorname{Im}(L)$ is invertible. We denote the inverse of the map by

K_P. If Ω is an open-bounded subset of Y such that $\mathrm{dom}(L)\cap\Omega\neq\emptyset$, the map $N : Y \to Z$ will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N : \overline{\Omega} \to Y$ is compact.

The theorem that we used is Theorem 2.4 of [18].

Theorem 1.1. *Let L be a Fredholm operator of index zero and N be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

(i) $Lx \neq \lambda Nx$ *for every* $(x,\lambda) \in [(\mathrm{dom}(L) \setminus \mathrm{Ker}(L)) \cap \partial\Omega] \times (0,1)$,

(ii) $Nx \notin \mathrm{Im}(L)$ *for every* $x \in \mathrm{Ker}(L) \cap \partial\Omega$,

(iii) $\deg(JQN|_{\mathrm{Ker}(L)}, \Omega \cap \mathrm{Ker}(L), 0) \neq 0$, *where* $Q : Z \to Z$ *is a projection as above with* $\mathrm{Im}(L) = \mathrm{Ker}(Q)$, *and* $J : \mathrm{Im}(Q) \to \mathrm{Ker}(L)$ *is any isomorphism,*

then the equation $Lx = Nx$ *has at least one solution in* $\mathrm{dom}(L) \cap \overline{\Omega}$.

The rest of this paper is organized as follows. In Section 2, we give some notation and Lemmas. In Section 3, we establish an existence theorem of a solution for the problem (1.1), (1.2).

2. Background Materials and Preliminaries

For the convenience of the reader, we present here some necessary basic knowledge and definitions for fractional calculus theory, and these definitions can be found in the recent literature [1, 2].

Definition 2.1. The fractional integral of order $\alpha > 0$ of a function $y : (0,\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds, \tag{2.1}$$

provided the right side is pointwise defined on $(0,\infty)$. And we let $I_{0+}^{0} y(t) = y(t)$ for every continuous $y : (0,\infty) \to \mathbb{R}$.

Definition 2.2. The fractional derivative of order $\alpha > 0$ of a function $y : (0,\infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}} ds, \tag{2.2}$$

where $n = [\alpha] + 1$, provided the right side is pointwise defined on $(0,\infty)$.

Lemma 2.3 (see [3]). *Assume that $u \in C(0,1) \cap L^1[0,1]$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L^1[0,1]$, then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}, \tag{2.3}$$

for some $C_i \in \mathbb{R}$, $i = 1,2,\ldots,N$, where N is the smallest integer greater than or equal to α.

We use the classical space $C[0,1]$ with the norm $\|x\|_\infty = \max_{t\in[0,1]}|x(t)|$. Given $\mu > 0$ and $N = [\mu] + 1$, one can define a linear space

$$C^\mu[0,1] := \left\{u \mid u(t) = I_{0+}^\mu x(t) + c_1 t^{\mu-1} + c_2 t^{\mu-2} + \cdots + c_{N-1} t^{\mu-(N-1)},\ t \in [0,1]\right\}, \tag{2.4}$$

where $x \in C[0,1]$ and $c_i \in \mathbb{R}$, $i = 1,2,\ldots,N-1$. By means of the linear function analysis theory, one can prove that with the norm $\|u\|_{C^\mu} = \|D_{0+}^\mu u\|_\infty + \cdots + \|D_{0+}^{\mu-(N-1)} u\|_\infty + \|u\|_\infty$, $C^\mu[0,1]$ is a Banach space.

Lemma 2.4 (see [7]). *$F \subset C^\mu[0,1]$ is a sequentially compact set if and only if F is uniformly bounded and equicontinuous. Here, uniformly bounded means that there exists $M > 0$ such that for every $u \in F$,*

$$\|u\|_{C^\mu} = \left\|D_{0+}^\mu u\right\|_\infty + \cdots + \left\|D_{0+}^{\mu-(N-1)} u\right\|_\infty + \|u\|_\infty < M, \tag{2.5}$$

and equicontinuous means that for all $\varepsilon > 0$, $\exists\delta > 0$ such that

$$\begin{gathered} |u(t_1) - u(t_2)| < \varepsilon, \quad (\forall t_1, t_2 \in [0,1],\ |t_1 - t_2| < \delta,\ \forall u \in F), \\ \left|D_{0+}^{\alpha-i} u(t_1) - D_{0+}^{\alpha-i} u(t_2)\right| < \varepsilon, \quad (t_1, t_2 \in [0,1], |t_1 - t_2| < \delta,\ \forall u \in F,\ \forall i \in \{0,\ldots,N-1\}). \end{gathered} \tag{2.6}$$

Let $Z = L^1[0,1]$ with the norm $\|g\|_1 = \int_0^1 |g(s)|ds$. $Y = C^{\alpha-1}[0,1] = \{u \mid u(t) = I_{0+}^{\alpha-1} x(t) + ct^{\alpha-2}, t \in [0,1]\}$, where $x \in C[0,1]$, $c \in \mathbb{R}$, with the norm $\|u\|_{C^{\alpha-1}} = \|D_{0+}^{\alpha-1} u\|_\infty + \|D_{0+}^{\alpha-2} u\|_\infty + \|u\|_\infty$, and Y is a Banach space.

Definition 2.5. By a solution of the boundary value problem (1.1), (1.2), we understand a function $u \in C^{\alpha-1}[0,1]$ such that $D_{0+}^{\alpha-1} u$ is absolutely continuous on $(0,1)$ and satisfies (1.1), (1.2).

Definition 2.6. We say that the map $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions with respect to $L^1[0,1]$ if the following conditions are satisfied:

(i) for each $z \in \mathbb{R}$, the mapping $t \to f(t,z)$ is Lebesgue measurable,

(ii) for almost every $t \in [0,1]$, the mapping $z \to f(t,z)$ is continuous on $\mathbb{R}$,

(iii) for each $r > 0$, there exists $\rho_r \in L^1([0,1],\mathbb{R})$ such that, for a.e., $t \in [0,1]$ and every $|z| \le r$, we have $|f(t,z)| \le \rho_r(t)$.

Define L to be the linear operator from $\mathrm{dom}(L) \cap Y$ to Z with

$$\begin{gathered} \mathrm{dom}(L) = \left\{u \in C^{\alpha-1}[0,1] \mid D_{0+}^\alpha u \in L^1[0,1],\ u \text{ satisfies } (1.2)\right\}, \\ Lu = D_{0+}^\alpha u, \quad u \in \mathrm{dom}(L). \end{gathered} \tag{2.7}$$

We define $N : Y \to Z$ by setting

$$Nu(t) = f\left(t, u(t), D_{0+}^{\alpha-1} u(t), D_{0+}^{\alpha-2} u(t)\right) + e(t). \tag{2.8}$$

Then boundary value problem (1.1), (1.2) can be written as

$$Lu = Nu. \tag{2.9}$$

Lemma 2.7. *Let condition* (1.3) *and* (1.4) *hold, then* $L : \mathrm{dom}(L) \cap Y \to Z$ *is a Fredholm map of index zero.*

Proof. It is clear that $\mathrm{Ker}(L) = \{at^{\alpha-1} + bt^{\alpha-2} \mid a, b \in \mathbb{R}\} \cong \mathbb{R}^2$.

Let $g \in Z$ and

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, \tag{2.10}$$

then $D_{0+}^{\alpha} u(t) = g(t)$ a.e., $t \in (0,1)$ and, if

$$\begin{gathered} \int_0^1 (1-s)^{\alpha-1} g(s)ds - \sum_{i=1}^{m} \alpha_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} g(s)ds = 0, \\ \sum_{j=1}^{n} \beta_j \int_0^{\xi_j} (\xi_j - s) g(s)ds = 0 \end{gathered} \tag{2.11}$$

hold. Then $u(t)$ satisfies the boundary conditions (1.2), that is, $u \in \mathrm{dom}(L)$, and we have

$$\{g \in Z \mid g \text{ satisfies } (2.11)\} \subseteq \mathrm{Im}(L). \tag{2.12}$$

Let $u \in \mathrm{dom}(L)$, then for $D_{0+}^{\alpha} u \in \mathrm{Im}(L)$, we have

$$u(t) = I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3}, \tag{2.13}$$

which, due to the boundary value condition (1.2), implies that $D_{0+}^{\alpha} u$ satisfies (2.11). In fact, from $I_{0+}^{3-\alpha} u(0) = 0$, we have $c_3 = 0$, from $u(1) = \sum_{i=1}^{m} \alpha_i u(\eta_i)$, we have

$$\int_0^1 (1-s)^{\alpha-1} D_{0+}^{\alpha} u(s)ds - \sum_{i=1}^{m} \alpha_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} D_{0+}^{\alpha} u(s)ds = 0, \tag{2.14}$$

and from $D_{0+}^{\alpha-2} u(0) = \sum_{j=1}^{n} \beta_j D_{0+}^{\alpha-2} u(\xi_j)$, we have

$$\sum_{j=1}^{n} \beta_j \int_0^{\xi_j} (\xi_j - s) D_{0+}^{\alpha} u(s)ds = 0. \tag{2.15}$$

Hence,

$$\mathrm{Im}(L) \subseteq \{g \in Z \mid g \text{ satisfies } (2.11)\}. \tag{2.16}$$

Therefore,

$$\operatorname{Im}(L) = \{g \in Z \mid g \text{ satisfies } (2.11)\}. \tag{2.17}$$

Consider the continuous linear mapping $Q_1 : Z \to Z$ and $Q_2 : Z \to Z$ defined by

$$\begin{gathered}
Q_1 g = \int_0^1 (1-s)^{\alpha-1} g(s)ds - \sum_{i=1}^{m} \alpha_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} g(s)ds, \\
Q_2 g = \sum_{j=1}^{n} \beta_j \int_0^{\xi_j} (\xi_j - s) g(s)ds.
\end{gathered} \tag{2.18}$$

Using the above definitions, we construct the following auxiliary maps $R_1,\ R_2 : Z \to Z$:

$$\begin{gathered}
R_1 g = \frac{1}{R}\left[\frac{\Gamma(\alpha-1)}{\Gamma(\alpha+1)} \sum_{j=1}^{n} \beta_j \xi_j^{\alpha} Q_1 g(t) - \frac{\Gamma(\alpha)\Gamma(\alpha-1)}{\Gamma(2\alpha-1)} \left(1 - \sum_{i=1}^{m} \alpha_i \eta_i^{2\alpha-2}\right) Q_2 g(t)\right], \\
R_2 g = -\frac{1}{R}\left[\frac{\Gamma(\alpha)}{\Gamma(\alpha+2)} \sum_{j=1}^{n} \beta_j \xi_j^{\alpha+1} Q_1 g(t) - \frac{(\Gamma(\alpha))^2}{\Gamma(2\alpha)} \left(1 - \sum_{i=1}^{m} \alpha_i \eta_i^{2\alpha-1}\right) Q_2 g(t)\right].
\end{gathered} \tag{2.19}$$

Since the condition (1.4) holds, the mapping $Q : Z \to Z$ defined by

$$(Qy)(t) = (R_1 g(t)) t^{\alpha-1} + (R_2 g(t)) t^{\alpha-2} \tag{2.20}$$

is well defined.

Recall (1.4) and note that

$$\begin{aligned}
R_1\left(R_1 g t^{\alpha-1}\right) &= \frac{1}{R}\left[\frac{\Gamma(\alpha-1)}{\Gamma(\alpha+1)} \sum_{j=1}^{n} \beta_j \xi_j^{\alpha} Q_1\left(R_1 g t^{\alpha-1}\right)\right. \\
&\qquad \left. - \frac{\Gamma(\alpha)\Gamma(\alpha-1)}{\Gamma(2\alpha-1)} \left(1 - \sum_{i=1}^{m} \alpha_i \eta_i^{2\alpha-2}\right) Q_2\left(R_1 g t^{\alpha-1}\right)\right] \\
&= R_1 g \frac{1}{R}\left[\frac{\Gamma(\alpha-1)\Gamma(\alpha^2)}{\Gamma(\alpha+1)\Gamma(2\alpha)} \sum_{j=1}^{n} \beta_j \xi_j^{\alpha} \left(1 - \sum_{i=1}^{m} \alpha_i \eta_i^{2\alpha-1}\right)\right. \\
&\qquad \left. - \frac{\Gamma(\alpha-1)\Gamma(\alpha^2)}{\Gamma(2\alpha-1)\Gamma(\alpha+2)} \left(1 - \sum_{i=1}^{m} \alpha_i \eta_i^{2\alpha-2}\right) \sum_{j=1}^{n} \beta_j \xi_j^{\alpha+1}\right] \\
&= R_1 g,
\end{aligned} \tag{2.21}$$

and similarly we can derive that

$$R_1\left(R_2 g t^{\alpha-2}\right) = 0,$$
$$R_2\left(R_1 g t^{\alpha-1}\right) = 0, \tag{2.22}$$
$$R_2\left(R_2 g t^{\alpha-2}\right) = R_2 g.$$

So, for $g \in Z$, it follows from the four relations above that

$$\begin{aligned} Q^2 g &= R_1\left(R_1 g t^{\alpha-1} + R_2 g t^{\alpha-2}\right)t^{\alpha-1} + R_2\left(R_1 g t^{\alpha-1} + R_2 g t^{\alpha-2}\right)t^{\alpha-2} \\ &= R_1\left(R_1 g t^{\alpha-1}\right)t^{\alpha-1} + R_1\left(R_2 g t^{\alpha-2}\right)t^{\alpha-1} + R_2\left(R_1 g t^{\alpha-1}\right)t^{\alpha-2} + R_2\left(R_2 g t^{\alpha-2}\right)t^{\alpha-2} \\ &= R_1 g t^{\alpha-1} + R_2 g t^{\alpha-2} \\ &= Qg, \end{aligned} \tag{2.23}$$

that is, the map Q is idempotent. In fact, Q is a continuous linear projector.

Note that $g \in \operatorname{Im}(L)$ implies $Qg = 0$. Conversely, if $Qg = 0$, then we must have $R_1 g = R_2 g = 0$; since the condition (1.4) holds, this can only be the case if $Q_1 g = Q_2 g = 0$, that is, $g \in \operatorname{Im}(L)$. In fact, $\operatorname{Im}(L) = \operatorname{Ker}(Q)$.

Take $g \in Z$ in the form $g = (g - Qg) + Qg$, so that $g - Qg \in \operatorname{Im}(L) = \operatorname{Ker}(Q)$ and $Qg \in \operatorname{Im}(Q)$. Thus, $Z = \operatorname{Im}(L) + \operatorname{Im}(Q)$. Let $g \in \operatorname{Im}(L) \cap \operatorname{Im}(Q)$ and assume that $g(s) = as^{\alpha-1} + bs^{\alpha-2}$ is not identically zero on $[0,1]$, then, since $g \in \operatorname{Im}(L)$, from (2.11) and the condition (1.4), we derive $a = b = 0$, which is a contradiction. Hence, $\operatorname{Im}(L) \cap \operatorname{Im}(Q) = \{0\}$; thus, $Z = \operatorname{Im}(L) \oplus \operatorname{Im}(Q)$.

Now, $\dim \operatorname{Ker}(L) = 2 = \operatorname{co}\dim \operatorname{Im}(L)$, and so L is a Fredholm operator of index zero. □

Let $P : Y \to Y$ be defined by

$$Pu(t) = \frac{1}{\Gamma(\alpha)} D_{0+}^{\alpha-1} u(0) t^{\alpha-1} + \frac{1}{\Gamma(\alpha-1)} D_{0+}^{\alpha-2} u(0) t^{\alpha-2}, \quad t \in [0,1]. \tag{2.24}$$

Note that P is a continuous linear projector and

$$\operatorname{Ker}(P) = \left\{u \in Y \mid D_{0+}^{\alpha-1} u(0) = D_{0+}^{\alpha-2} u(0) = 0\right\}. \tag{2.25}$$

It is clear that $Y = \operatorname{Ker}(L) \oplus \operatorname{Ker}(P)$.

Note that the projectors P and Q are exact. Define $K_P : \operatorname{Im}(L) \to \operatorname{dom}(L) \cap \operatorname{Ker}(P)$ by

$$K_P g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\,ds = I_{0+}^{\alpha} g(t). \tag{2.26}$$

Hence, we have

$$D_{0+}^{\alpha-1}(K_P g)(t) = \int_0^t g(s)ds, \qquad D_{0+}^{\alpha-2}(K_P g)(t) = \int_0^t (t-s)g(s)ds, \tag{2.27}$$

then $\|K_P g\|_\infty \le (1/\Gamma(\alpha))\|g\|_1$, $\|D_{0+}^{\alpha-1}(K_P g)\|_\infty \le \|g\|_1$, $\|D_{0+}^{\alpha-2}(K_P g)\|_\infty \le \|g\|_1$, and thus

$$\|K_P g\|_{C^{\alpha-1}} \le \left(2 + \frac{1}{\Gamma(\alpha)}\right)\|g\|_1. \tag{2.28}$$

In fact, if $g \in \operatorname{Im}(L)$, then $(LK_P)g = D_{0+}^{\alpha} I_{0+}^{\alpha} g = g$. Also, if $u \in \operatorname{dom}(L) \cap \operatorname{Ker}(P)$, then

$$(K_P L g)(t) = I_{0+}^{\alpha} D_{0+}^{\alpha} g(t) = g(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3}, \tag{2.29}$$

from boundary value condition (1.2) and the fact that $u \in \operatorname{dom}(L) \cap \operatorname{Ker}(P)$, we have $c_1 = c_2 = c_3 = 0$. Thus,

$$K_P = \left(L|_{\operatorname{dom}(L)\cap\operatorname{Ker}(P)}\right)^{-1}. \tag{2.30}$$

Using (2.19), we write

$$\begin{gathered} QNu(t) = (R_1 Nu)t^{\alpha-1} + (R_2 Nu)t^{\alpha-2}, \\ K_P(I-Q)Nu(t) = \frac{1}{\Gamma(\alpha)} \int_0^1 (t-s)^{\alpha-1}[Nu(s) - QNu(s)]ds. \end{gathered} \tag{2.31}$$

With arguments similar to those of [7], we obtain the following Lemma.

Lemma 2.8. *$K_{P(I-Q)}N : Y \to Y$ is completely continuous.*

3. The Main Results

Assume that the following conditions on the function $f(t,x,y,z)$ are satisfied:

(H1) there exists a constant $A > 0$, such that for $u \in \operatorname{dom}(L) \setminus \operatorname{Ker}(L)$ satisfying $|D_{0+}^{\alpha-1}u(t)| + |D_{0+}^{\alpha-2}u(t)| > A$ for all $t \in [0,1]$, we have

$$Q_1 Nu(t) \neq 0 \qquad \text{or } Q_2 Nu(t) \neq 0, \tag{3.1}$$

(H2) there exist functions $a, b, c, d, r \in L^1[0,1]$ and a constant $\theta \in [0,1]$ such that for all $(x, y, z) \in \mathbb{R}^3$ and a.e., $t \in [0,1]$, one of the following inequalities is satisfied:

$$
\begin{aligned}
|f(t,x,y,z)| &\le a(t)|x| + b(t)|y| + c(t)|z| + d(t)|z|^{\theta} + r(t), \\
|f(t,x,y,z)| &\le a(t)|x| + b(t)|y| + c(t)|z| + d(t)|y|^{\theta} + r(t), \\
|f(t,x,y,z)| &\le a(t)|x| + b(t)|y| + c(t)|z| + d(t)|x|^{\theta} + r(t),
\end{aligned}
\tag{3.2}
$$

(H3) there exists a constant $B > 0$ such that for every $a, b \in \mathbb{R}$ satisfying $a^2 + b^2 > B$, then either

$$
aR_1N\left(at^{\alpha-1} + bt^{\alpha-2}\right) + bR_2N\left(at^{\alpha-1} + bt^{\alpha-2}\right) < 0, \tag{3.3}
$$

or else

$$
aR_1N\left(at^{\alpha-1} + bt^{\alpha-2}\right) + bR_2N\left(at^{\alpha-1} + bt^{\alpha-2}\right) > 0. \tag{3.4}
$$

Remark 3.1. $R_1N(at^{\alpha-1} + bt^{\alpha-2})$ and $R_2N(at^{\alpha-1} + bt^{\alpha-2})$ from (H3) stand for the images of $u(t) = at^{\alpha-1} + bt^{\alpha-2}$ under the maps R_1N and R_2N, respectively.

Theorem 3.2. *If (H1)–(H3) hold, then boundary value problem* (1.1)-(1.2) *has at least one solution provided that*

$$
\|a\|_1 + \|b\|_1 + \|c\|_1 < \frac{1}{\tau}, \tag{3.5}
$$

where $\tau = 5 + 2/\Gamma(\alpha) + 1/\Gamma(\alpha - 1)$.

Proof. Set

$$
\Omega_1 = \{u \in \operatorname{dom}(L) \setminus \operatorname{Ker}(L) \mid Lu = \lambda Nu \text{ for some } \lambda \in [0,1]\}, \tag{3.6}
$$

then for $u \in \Omega_1$, $Lu = \lambda Nu$; thus, $\lambda \neq 0$, $Nu \in \operatorname{Im}(L) = \operatorname{Ker}(Q)$, and hence $QNu(t) = 0$ for all $t \in [0,1]$. By the definition of Q, we have $Q_1Nu(t) = Q_2Nu(t) = 0$. It follows from (H1) that there exists $t_0 \in [0,1]$ such that $|D_{0+}^{\alpha-1}u(t_0)| + |D_{0+}^{\alpha-2}u(t_0)| \le A$. Now,

$$
\begin{aligned}
D_{0+}^{\alpha-1}u(t) &= D_{0+}^{\alpha-1}u(t_0) + \int_{t_0}^{t} D_{0+}^{\alpha}u(s)\,ds, \\
D_{0+}^{\alpha-2}u(t) &= D_{0+}^{\alpha-2}u(t_0) + \int_{t_0}^{t} D_{0+}^{\alpha-1}u(s)\,ds,
\end{aligned}
\tag{3.7}
$$

so

$$
\begin{aligned}
\left|D_{0+}^{\alpha-1}u(0)\right| &\le \left\|D_{0+}^{\alpha-1}u(t)\right\|_\infty \\
&\le \left|D_{0+}^{\alpha-1}u(t_0)\right| + \left\|D_{0+}^{\alpha}u\right\|_1 \\
&\le A + \|Lu\|_1 \\
&\le A + \|Nu\|_1, \\
\left|D_{0+}^{\alpha-2}u(0)\right| &\le \left\|D_{0+}^{\alpha-2}u(t)\right\|_\infty \\
&\le \left|D_{0+}^{\alpha-2}u(t_0)\right| + \left\|D_{0+}^{\alpha-1}u\right\|_\infty \\
&\le \left|D_{0+}^{\alpha-2}u(t_0)\right| + \left|D_{0+}^{\alpha-1}u(t_0)\right| + \left\|D_{0+}^{\alpha}u\right\|_1 \\
&\le A + \|Lu\|_1 \\
&\le A + \|Nu\|_1.
\end{aligned}
\tag{3.8}
$$

Now by (3.8), we have

$$
\begin{aligned}
\|Pu\|_{C^{\alpha-1}} &= \left\|\frac{1}{\Gamma(\alpha)}D_{0+}^{\alpha-1}u(0)t^{\alpha-1} + \frac{1}{\Gamma(\alpha-1)}D_{0+}^{\alpha-2}u(0)t^{\alpha-2}\right\|_{C^{\alpha-1}} \\
&= \left\|\frac{1}{\Gamma(\alpha)}D_{0}^{\alpha-1}u(0)t^{\alpha-1} + \frac{1}{\Gamma(\alpha-1)}D_{0}^{\alpha-2}u(0)t^{\alpha-2}\right\|_\infty \\
&\quad + \left\|D_{0+}^{\alpha-1}u(0)\right\|_\infty + \left\|D_{0+}^{\alpha-1}u(0)t + D_{0+}^{\alpha-2}u(0)\right\|_\infty \\
&\le \left(2 + \frac{1}{\Gamma(\alpha)}\right)\left|D_{0+}^{\alpha-1}u(0)\right| + \left(1 + \frac{1}{\Gamma(\alpha-1)}\right)\left|D_{0+}^{\alpha-2}u(0)\right| \\
&\le \left(2 + \frac{1}{\Gamma(\alpha)}\right)(A + \|Nu\|_1) + \left(1 + \frac{1}{\Gamma(\alpha-1)}\right)(A + \|Nu\|_1).
\end{aligned}
\tag{3.9}
$$

Note that $(I-P)u \in \operatorname{Im}(K_P) = \operatorname{dom}(L) \cap \operatorname{Ker}(P)$ for $u \in \Omega_1$, then, by (2.28) and (2.30),

$$
\begin{aligned}
\|(I-P)u\|_{C^{\alpha-1}} &= \|K_P L(I-P)\|_{C^{\alpha-1}} \\
&\le \left(2 - \frac{1}{\Gamma(\alpha)}\right)\|L(I-P)u\|_1 \\
&= \left(2 - \frac{1}{\Gamma(\alpha)}\right)\|Lu\|_1 \\
&\le \left(2 - \frac{1}{\Gamma(\alpha)}\right)\|Nu\|_1.
\end{aligned}
\tag{3.10}
$$

Using (3.9) and (3.10), we obtain

$$\begin{aligned}
\|u\|_{C^{\alpha-1}} &= \|Pu + (I-P)u\|_{C^{\alpha-1}} \\
&\leq \|Pu\|_{C^{\alpha-1}} + \|(I-P)u\|_{C^{\alpha-1}} \\
&\leq \left(2+\frac{1}{\Gamma(\alpha)}\right)(A+\|Nu\|_1) + \left(1+\frac{1}{\Gamma(\alpha-1)}\right)(A+\|Nu\|_1) + \left(2+\frac{1}{\Gamma(\alpha)}\right)\|Nu\|_1 \\
&= \left(5+\frac{2}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha-1)}\right)\|Nu\|_1 + \left(3+\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha-1)}\right)A \\
&= \tau\|Nu\|_1 + C_1,
\end{aligned} \tag{3.11}$$

where $C_1 = (3+1/\Gamma(\alpha)+1/\Gamma(\alpha-1))A$ is a constant. This is for all $u \in \Omega_1$,

$$\|u\|_{C^{\alpha-1}} \leq \tau\|Nu\|_1 + C_1. \tag{3.12}$$

If the first condition of (H2) is satisfied, then we have

$$\begin{aligned}
&\max\left\{\|u\|_\infty, \left\|D_{0+}^{\alpha-1}u\right\|_\infty, \left\|D_{0+}^{\alpha-2}u\right\|_\infty\right\} \\
&\quad\leq \|u\|_{C^{\alpha-1}} \leq \tau\Bigg(\|a\|_1\|u\|_\infty + \|b\|_1\left\|D_{0+}^{\alpha-1}u\right\|_\infty + \|c\|_1\left\|D_{0+}^{\alpha-2}u\right\|_\infty \\
&\qquad + \|d\|_1\left\|D_{0+}^{\alpha-2}u\right\|_\infty^\theta + \|r\|_1 + \|e\|_1\Bigg) + C_1,
\end{aligned} \tag{3.13}$$

and consequently,

$$\begin{aligned}
\|u\|_\infty \leq{}& \frac{\tau}{1-\|a\|_1\tau}\left(\|b\|_1\left\|D_{0+}^{\alpha-1}u\right\|_\infty + \|c\|_1\left\|D_{0+}^{\alpha-2}u\right\|_\infty + \|d\|_1\left\|D_{0+}^{\alpha-2}u\right\|_\infty^\theta + \|r\|_1 + \|e\|_1\right) \\
&+ \frac{C_1}{1-\|a\|_1\tau},
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
\left\|D_{0+}^{\alpha-1}u\right\|_\infty \leq{}& \frac{\tau}{1-\|a\|_1\tau-\|b\|_1\tau}\left(\|c\|_1\left\|D_{0+}^{\alpha-2}u\right\|_\infty + \|d\|_1\left\|D_{0+}^{\alpha-2}u\right\|_\infty^\theta + \|r\|_1 + \|e\|_1\right) \\
&+ \frac{C_1}{1-\|a\|_1\tau-\|b\|_1\tau},
\end{aligned} \tag{3.15}$$

$$\left\|D_{0+}^{\alpha-1}u\right\|_\infty \leq \frac{\tau\|d\|_1\left\|D_{0+}^{\alpha-2}u\right\|_\infty^\theta}{1-\|a\|_1\tau-\|b\|_1\tau-\|c\|_1\tau} + \frac{\tau(\|r\|_1+\|e\|_1)+C_1}{1-\|a\|_1\tau-\|b\|_1\tau-\|c\|_1\tau}. \tag{3.16}$$

Note that $\theta \in [0,1)$ and $\|a\|_1 + \|b\|_1 + \|c\|_1 < 1/\tau$, so there exists $M_1 > 0$ such that $\|D_{0+}^{\alpha-1}u\|_\infty \le M_1$ for all $u \in \Omega_1$. The inequalities (3.14) and (3.15) show that there exist $M_2, M_3 > 0$ such that $\|D_{0+}^{\alpha-1}u\|_\infty \le M_2$, $\|u\|_\infty \le M_3$ for all $u \in \Omega_1$. Therefore, for all $u \in \Omega_1$, $\|u\|_{C^{\alpha-1}} = \|u\|_\infty + \|D_{0+}^{\alpha-1}u\|_\infty + \|D_{0+}^{\alpha-2}u\|_\infty \le M_1 + M_2 + M_3$, that is, Ω_1 is bounded given the first condition of (H2). If the other conditions of (H2) hold, by using an argument similar to the above, we can prove that Ω_1 is also bounded.

Let

$$\Omega_2 = \{u \in \operatorname{Ker}(L) \mid Nu \in \operatorname{Im}(L)\}. \tag{3.17}$$

For $u \in \Omega_2$, $u \in \operatorname{Ker}(L) = \{u \in \operatorname{dom}(L) \mid u = at^{\alpha-1} + bt^{\alpha-2}, a, b \in \mathbb{R}, t \in [0,1]\}$, and $QN(at^{\alpha-1} + bt^{\alpha-2}) = 0$; thus, $R_1N(at^{\alpha-1} + bt^{\alpha-2}) = R_2N(at^{\alpha-1} + bt^{\alpha-2}) = 0$. By (H3), $a^2 + b^2 \le B$, that is, Ω_2 is bounded.

We define the isomorphism $J : \operatorname{Im}(Q) \to \operatorname{Ker}(L)$ by

$$J\left(at^{\alpha-1} + bt^{\alpha-2}\right) = at^{\alpha-1} + bt^{\alpha-2}, \quad a, b \in \mathbb{R}. \tag{3.18}$$

If the first part of (H3) is satisfied, let

$$\Omega_3 = \left\{u \in \operatorname{Ker} L : -\lambda J^{-1}u + (1-\lambda)QNu = 0,\ \lambda \in [0,1]\right\}. \tag{3.19}$$

For every $at^{\alpha-1} + bt^{\alpha-2} \in \Omega_3$,

$$\lambda\left(at^{\alpha-1} + bt^{\alpha-2}\right) = (1-\lambda)\left[\left(R_1N\left(at^{\alpha-1} + bt^{\alpha-2}\right)\right)t^{\alpha-1} + \left(R_2N\left(at^{\alpha-1} + bt^{\alpha-2}\right)\right)t^{\alpha-2}\right]. \tag{3.20}$$

If $\lambda = 1$, then $a = b = 0$, and if $a^2 + b^2 > B$, then by (H3),

$$\lambda\left(a^2 + b^2\right) = (1-\lambda)\left[aR_1N\left(at^{\alpha-1} + bt^{\alpha-2}\right) + bR_2N\left(at^{\alpha-1} + bt^{\alpha-2}\right)\right] < 0, \tag{3.21}$$

which, in either case, obtain a contradiction. If the other part of (H3) is satisfied, then we take

$$\Omega_3 = \left\{u \in \operatorname{Ker} L : \lambda J^{-1}u + (1-\lambda)QNu = 0,\ \lambda \in [0,1]\right\}, \tag{3.22}$$

and, again, obtain a contradiction. Thus, in either case,

$$\begin{aligned}
\|u\|_{C^{\alpha-1}} &= \|u\|_\infty + \left\|D_{0+}^{\alpha-1}u\right\|_\infty + \left\|D_{0+}^{\alpha-2}u\right\|_\infty \\
&= \left\|at^{\alpha-1} + bt^{\alpha-2}\right\|_{C^{\alpha-1}} \\
&= \left\|at^{\alpha-1} + bt^{\alpha-2}\right\|_\infty + \|a\Gamma(\alpha)\|_\infty + \|a\Gamma(\alpha)t + b\Gamma(\alpha-1)\|_\infty \\
&\le (1 + 2\Gamma(\alpha))|a| + (1 + \Gamma(\alpha-1))|b| \\
&\le (2 + 2\Gamma(\alpha) + \Gamma(\alpha-1))|a|,
\end{aligned} \tag{3.23}$$

for all $u \in \Omega_3$, that is, Ω_3 is bounded.

In the following, we will prove that all the conditions of Theorem 1.1 are satisfied. Set Ω to be a bounded open set of Y such that $U_{i=1}^{3}\overline{\Omega_i} \subset \Omega$. by Lemma 2.8, the operator $K_P(I-Q)N : \overline{\Omega} \to Y$ is compact; thus, N is L-compact on $\overline{\Omega}$, then by the above argument, we have

(i) $Lu \neq \lambda Nx$, for every $(u,\lambda) \in [(\operatorname{dom}(L) \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0,1)$,

(ii) $Nu \notin \operatorname{Im}(L)$, for every $u \in \operatorname{Ker}(L) \cap \partial\Omega$.

Finally, we will prove that (iii) of Theorem 1.1 is satisfied. Let $H(u,\lambda) = \pm Iu + (1-\lambda)JQNu$, where I is the identity operator in the Banach space Y. According to the above argument, we know that

$$H(u,\lambda) \neq 0, \quad \forall u \in \partial\Omega \cap \operatorname{Ker}(L), \tag{3.24}$$

and thus, by the homotopy property of degree,

$$\begin{aligned}\deg&\Big(JQN|_{\operatorname{Ker}(L)}, \Omega \cap \operatorname{Ker}(L), 0\Big)\\ &= \deg(H(\ldots,0), \Omega \cap \operatorname{Ker}(L), 0)\\ &= \deg(H(\ldots,1), \Omega \cap \operatorname{Ker}(L), 0)\\ &= \deg(\pm I, \Omega \cap \operatorname{Ker}(L), 0)\\ &= \pm 1 \neq 0,\end{aligned} \tag{3.25}$$

then by Theorem 1.1, $Lu = Nu$ has at least one solution in $\operatorname{dom}(L) \cap \overline{\Omega}$, so boundary problem (1.1), (1.2) has at least one solution in the space $C^{\alpha-1}[0,1]$. The proof is finished. □

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References

[1] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, The Netherlands, 2006.

[2] K. S. Miller, "Fractional differential equations," *Journal of Fractional Calculus*, vol. 3, pp. 49–57, 1993.

[3] Z. Bai and H. Lü, "Positive solutions for boundary value problem of nonlinear fractional differential equation," *Journal of Mathematical Analysis and Applications*, vol. 311, no. 2, pp. 495–505, 2005.

[4] A. Babakhani and V. Daftardar-Gejji, "Existence of positive solutions of nonlinear fractional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 278, no. 2, pp. 434–442, 2003.

[5] V. Daftardar-Gejji and A. Babakhani, "Analysis of a system of fractional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 293, no. 2, pp. 511–522, 2004.

[6] F. Meng and Z. Du, "Solvability of a second-order multi-point boundary value problem at resonance," *Applied Mathematics and Computation*, vol. 208, no. 1, pp. 23–30, 2009.

[7] Y. H. Zhang and Z. B. Bai, "Exitence of solutions for nonlinear fractional three-point boundary value problem at resonance," *Journal of Applied Mathematics and Computing*, vol. 36, no. 1-2, pp. 417–440, 2011.

[8] H. Jafari and V. Daftardar-Gejji, "Positive solutions of nonlinear fractional boundary value problems using Adomian decomposition method," *Applied Mathematics and Computation*, vol. 180, no. 2, pp. 700–706, 2006.

[9] C. Yu and G. Z. Gao, "On the solution of nonlinear fractional order differential equation," *Nonlinear Analysis*, vol. 63, no. 5–7, pp. e971–e976, 1998.

[10] N. Kosmatov, "A multi-point boundary value problem with two critical conditions," *Nonlinear Analysis*, vol. 65, no. 3, pp. 622–633, 2006.

[11] B. Liu and Z. Zhao, "A note on multi-point boundary value problems," *Nonlinear Analysis*, vol. 67, no. 9, pp. 2680–2689, 2007.

[12] R. P. Agarwal, V. Lakshmikantham, and J. J. Nieto, "On the concept of solution for fractional differential equations with uncertainty," *Nonlinear Analysis*, vol. 72, no. 6, pp. 2859–2862, 2010.

[13] Y.-K. Chang and J. J. Nieto, "Some new existence results for fractional differential inclusions with boundary conditions," *Mathematical and Computer Modelling*, vol. 49, no. 3-4, pp. 605–609, 2009.

[14] A. M. A. El-Sayed, "Nonlinear functional-differential equations of arbitrary orders," *Nonlinear Analysis*, vol. 33, no. 2, pp. 181–186, 1998.

[15] D. Guo and J. Sun, *Nonlinear Integral Equations*, Shandong Science and Technology Press, Jinan, China, 1987.

[16] V. Lakshmikantham and S. Leela, "Nagumo-type uniqueness result for fractional differential equations," *Nonlinear Analysis*, vol. 71, no. 7-8, pp. 2886–2889, 2009.

[17] V. Lakshmikantham and S. Leela, "A Krasnoselskii-Krein-type uniqueness result for fractional differential equations," *Nonlinear Analysis*, vol. 71, no. 7-8, pp. 3421–3424, 2009.

[18] J. Mawhin, "Topological degree and boundary value problems for nonlinear differential equations," in *Topological Methods for Ordinary Differential Equations*, P. M. Fitzpatrick, M. Martelli, J. Mawhin, and R. Nussbaum, Eds., vol. 1537 of *Lecture Notes in Mathematics*, pp. 74–142, Springer, Berlin, Germany, 1991.