Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 385843, 9 pages
doi:10.1155/2011/385843

Research Article

On the Strong Convergence of Viscosity Approximation Process for Quasinonexpansive Mappings in Hilbert Spaces

Kanokwan Wongchan and Satit Saejung

Department of Mathematics, Khon Kaen University, Khon Kaen 40002, Thailand

Correspondence should be addressed to Satit Saejung, saejung@kku.ac.th

Received 8 March 2011; Accepted 2 May 2011

Academic Editor: Stefan Siegmund

We improve the viscosity approximation process for approximation of a fixed point of a quasi-nonexpansive mapping in a Hilbert space proposed by Maingé (2010). An example beyond the scope of the previously known result is given.

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$. In this paper, we denote the strong and weak convergence by $\to$ and $\rightharpoonup$, respectively. For a subset C of H, a mapping $T : C \to C$ is said to be *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$; and it is *quasinonexpansive* if its *fixed-point set* $\mathrm{Fix}(T) := \{x \in C : x = Tx\}$ is nonempty and $\|Tx - p\| \le \|x - p\|$ for all $x \in C$ and $p \in \mathrm{Fix}(T)$. It is clear that every nonexpansive mapping with a nonempty fixed-point set is quasinonexpansive, but the converse is not true. The process for approximation of a fixed point of a nonexpansive or quasinonexpansive mapping is one of interesting problems in mathematics and it has been investigated by many researchers. One of the effective processes for this problem is given by Moudafi [1]. Let C be a closed convex subset of H, and $T : C \to C$ is a nonexpansive mapping with a nonempty fixed-point set $\mathrm{Fix}(T)$. Moudafi proposed the following scheme which is known as Moudafi's viscosity approximation process:

$$\begin{aligned} &x_1 = x \in C \quad \text{arbitrarily chosen}, \\ &x_{n+1} = \frac{\varepsilon_n}{1+\varepsilon_n} f(x_n) + \frac{1}{1+\varepsilon_n} T x_n, \end{aligned} \tag{1.1}$$

where $f : C \to C$ is a *contraction*; that is, there exists an $\alpha \in [0,1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$ for all $x, y \in C$ and $\{\varepsilon_n\}$ is a sequence in $(0,1)$ satisfying

(M1) $\lim_{n\to\infty}\varepsilon_n = 0$,

(M2) $\sum_{n=1}^{\infty} \varepsilon_n = \infty$,

(M3) $\lim_{n\to\infty}(1/\varepsilon_n - 1/\varepsilon_{n+1}) = 0$.

It was also proved that $\{x_n\}$ converges to an element $z \in \operatorname{Fix}(T)$ satisfying the following inequality:

$$\langle f(z) - z, q - z\rangle \leq 0 \tag{1.2}$$

for all $q \in \operatorname{Fix}(T)$.

In the literature, Moudafi's scheme has been widely studied and extended (see [2–5] and references therein). For example, Xu [6] improved this result to a Banach space. The interesting improvement of this result given by Maingé [7] is our starting point. His result is given below.

Theorem 1.1. *Let C be a closed convex subset of a Hilbert space H, and $T : C \to C$ is a quasinonexpansive mapping such that $I - T$ is demiclosed at zero, that is, $z \in \operatorname{Fix}(T)$ whenever $\{z_n\}$ is a sequence in C such that $z_n \rightharpoonup z$ and $z_n - Tz_n \to 0$. Suppose that $f : C \to C$ is a contraction. Let $\{x_n\}$ be a sequence in C defined by*

$$\begin{aligned} x_1 &= x \in C \quad \textit{arbitrarily chosen}, \\ x_{n+1} &= \alpha_n f(x_n) + (1-\alpha_n)((1-\omega)I + \omega T)x_n, \end{aligned} \tag{1.3}$$

where $\omega \in (0,1)$, I is an identity mapping, and $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying

(C1) $\lim_{n\to\infty}\alpha_n = 0$,

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then the sequence $\{x_n\}$ converges to an element $z \in \operatorname{Fix}(T)$ and the following inequality holds

$$\langle f(z) - z, q - z\rangle \leq 0 \tag{1.4}$$

for all $q \in \operatorname{Fix}(T)$.

It should be noted that Maingé's result is more widely applicable than Moudafi's. However, after a careful reading, we find that there is a small mistake in Maingé's proof. The following fact (see [7, Remark 2.1(i3)]) is used: if $T : C \to C$ is quasinonexpansive and $T_\omega := (1-\omega)I + \omega T$ where $\omega \in (0,1]$, then

$$\langle x - T_\omega x, x - q\rangle \geq \omega\|x - Tx\|^2 \tag{1.5}$$

for all $x \in C$ and $q \in \operatorname{Fix}(T)$. Note that the inequality above is equivalent to

$$\langle x - Tx, x - q\rangle \geq \|x - Tx\|^2. \tag{1.6}$$

But this fails; for example, let us consider the nonexpansive mapping $T : \mathbb{R} \to \mathbb{R}$ defined by $Tx = -x$ for all $x \in \mathbb{R}$. It is clear that $\mathrm{Fix}(T) = \{0\}$ and $\langle x - Tx, x - q\rangle = 2x^2 \not\geq 4x^2 = \|x - Tx\|^2$.

Recall the following identities in a Hilbert space H: for $x, y \in H$, $\omega \in [0,1]$

(i) $\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$;

(ii) $\|(1-\omega)x + \omega y\|^2 = (1-\omega)\|x\|^2 + \omega\|y\|^2 - (1-\omega)\omega\|x - y\|^2$.

The correction of Maingé's result is as follows.

Proposition 1.2. *Let C be a subset of a Hilbert space and $T : C \to C$ be a mapping with a nonempty fixed-point set* $\mathrm{Fix}(T)$. *Suppose that $T_\omega := (1-\omega)I + \omega T$ where $\omega \in (0,1]$. Then T is quasinonexpansive if and only if*

$$\langle x - T_\omega x, x - q\rangle \geq \frac{\omega}{2}\|x - Tx\|^2 \tag{1.7}$$

for all $x \in C$ and $q \in$ $\mathrm{Fix}(T)$.

Proof. Notice that $x - T_\omega x = \omega(x - Tx)$ and

$$\begin{aligned}\|T_\omega x - q\|^2 &= \|(T_\omega x - x) + (x - q)\|^2\\ &= \|T_\omega x - x\|^2 + 2\langle T_\omega x - x, x - q\rangle + \|x - q\|^2\\ &= \omega^2\|x - Tx\|^2 + 2\langle T_\omega x - x, x - q\rangle + \|x - q\|^2.\end{aligned} \tag{1.8}$$

On the other hand,

$$\begin{aligned}\|T_\omega x - q\|^2 &= \|(1-\omega)(x - q) + \omega(Tx - q)\|^2\\ &= (1-\omega)\|x - q\|^2 + \omega\|Tx - q\|^2 - (1-\omega)\omega\|x - Tx\|^2.\end{aligned} \tag{1.9}$$

Hence

$$2\langle x - T_\omega x, x - q\rangle = \omega\left(\|x - q\|^2 - \|Tx - q\|^2\right) + \omega\|x - Tx\|^2. \tag{1.10}$$

□

Remark 1.3. Unfortunately, this effects the main result (see [7, Theorem 3.1]) in Maingé's paper. More precisely, inequality (32) of its proof (page 78, line 22) should read

$$\begin{aligned}&\frac{1}{2}\|x_{n+1} - z\|^2 - \frac{1}{2}\|x_n - z\|^2 + \left(\frac{1}{2} - \omega\right)\omega(1-\alpha_n)\|x_n - Tx_n\|^2\\ &\quad\leq \alpha_n\left(\alpha_n\|f(x_n) - x_n\|^2 - \langle x_n - f(x_n), x_n - z\rangle\right)\end{aligned} \tag{1.11}$$

rather than

$$\frac{1}{2}\|x_{n+1}-z\|^2-\frac{1}{2}\|x_n-z\|^2+(1-\omega)\omega(1-\alpha_n)\|x_n-Tx_n\|^2 \\ \leq \alpha_n\Big(\alpha_n\big\|f(x_n)-x_n\big\|^2-\langle x_n-f(x_n),x_n-z\rangle\Big). \tag{1.12}$$

Therefore, Theorem 1.1 above is valid for only $\omega \in (0,1/2)$ under the same technique.

The purpose of this paper is to simultaneously present a correction of the proof of Theorem 1.1 which is valid *for all* $\omega \in (0,1)$, and extend his scheme to a wider class of mappings including average mappings, that is, mappings of the form $(1-\omega)I+\omega T$. Our result is more general than Maingé's theorem. An example of a quasinonexpansive mapping which is not applied by Maingé's theorem but applied by our result is given.

2. Result

First, let us recall some lemmas which are needed for proving the main result.

Lemma 2.1 (see [8, Lemma 2.3]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ a sequence of $(0,1)$ with $\sum_{n=1}^{\infty}\alpha_n=\infty$, $\{\beta_n\}$ a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty}\beta_n<\infty$, and $\{\gamma_n\}$ a sequence of real numbers with $\limsup_{n\to\infty}\gamma_n \leq 0$. Suppose that*

$$s_{n+1} \leq (1-\alpha_n)s_n+\alpha_n\gamma_n+\beta_n \tag{2.1}$$

for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty}s_n=0$.

The following nice result was proved by Maingé (see [7, Lemma 2.1]).

Lemma 2.2. *Let $\{s_n\}$ be a sequence of nonnegative real numbers. If there exists a subsequence $\{s_{n_j}\}$ of $\{s_n\}$ such that $s_{n_j}<s_{n_j+1}$ for all $j \in \mathbb{N}$, then there exists a subsequence $\{s_{m_k}\}$ of $\{s_n\}$ such that*

$$s_{m_k} \leq s_{m_k+1}, \qquad s_k \leq s_{m_k+1} \tag{2.2}$$

for all $k \in \mathbb{N}$.

For a closed convex subset C of a Hilbert space H, the *metric projection* $P_C : H \to C$ is defined for each $x \in H$ as the unique element $P_Cx \in C$ such that

$$\|x-P_Cx\| = \inf\{\|x-z\| : z \in C\}. \tag{2.3}$$

It is well known that (see, e.g., [9]) for $x \in H$ and $y \in C$

$$y=P_Cx \Longleftrightarrow \langle x-y,y-z\rangle \geq 0, \quad \forall z \in C. \tag{2.4}$$

For $x,y \in H$, the following inequality is known as the *subdifferential inequality*:

$$\big\|x+y\big\|^2 \leq \|x\|^2+2\langle y,x+y\rangle. \tag{2.5}$$

A mapping $T : C \to C$ is said to be *strongly quasinonexpansive* [10] if it is quasinonexpansive and $z_n - Tz_n \to 0$ whenever $\{z_n\}$ is a bounded sequence in C such that $\lim_{n\to\infty}(\|z_n - p\| - \|Tz_n - p\|) = 0$ for some $p \in \mathrm{Fix}(T)$. It is known that every metric projection is strongly quasinonexpansive.

We are now ready to present our main result.

Theorem 2.3. *Let C be a closed convex subset of a Hilbert space H and $T : C \to C$ is a strongly quasinonexpansive mapping such that $I - T$ is demiclosed at zero. Suppose that $f : C \to C$ is a contraction. Let $\{x_n\}$ be a sequence in C defined by*

$$\begin{aligned} &x_1 = x \in C \quad \textit{arbitrarily chosen}, \\ &x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \end{aligned} \tag{2.6}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying

(C1) $\lim_{n\to\infty}\alpha_n = 0$,

(C2) $\sum_{n=1}^{\infty}\alpha_n = \infty$.

Then the sequence $\{x_n\}$ converges to an element $z \in \mathrm{Fix}(T)$ and the following inequality holds

$$\langle f(z) - z, q - z\rangle \le 0 \tag{2.7}$$

for all $q \in \mathrm{Fix}(T)$.

Before we give the proof, we note that $\mathrm{Fix}(T)$ is closed and convex (see [11] for more general setting). Hence the mapping $P_{\mathrm{Fix}(T)} \circ f : C \to C$ is a contraction. Then it follows from the well-known Banach's contraction principle that there exists a unique element $z \in C$ such that $z = P_{\mathrm{Fix}(T)} \circ f(z)$. In particular, $z \in \mathrm{Fix}(T)$ and $\langle f(z) - z, q - z\rangle \le 0$ for all $q \in \mathrm{Fix}(T)$.

Let us assume that $\|f(x) - f(y)\| \le \alpha\|x - y\|$ for all $x, y \in C$ where α is a real number in $[0,1)$.

Lemma 2.4. *The sequence $\{x_n\}$ is bounded.*

Proof. We consider the following inequality:

$$\begin{aligned} \|x_{n+1} - z\| &\le \alpha_n \|f(x_n) - z\| + (1-\alpha_n)\|Tx_n - z\| \\ &\le \alpha_n \|f(x_n) - f(z)\| + \alpha_n\|f(z) - z\| + (1-\alpha_n)\|Tx_n - z\| \\ &\le (\alpha_n\alpha + 1 - \alpha_n)\|x_n - z\| + \alpha_n\|f(z) - z\| \\ &= (1 - \alpha_n(1-\alpha))\|x_n - z\| + \alpha_n(1-\alpha)\frac{\|f(z) - z\|}{1-\alpha} \\ &\le \max\left\{\|x_n - z\|, \frac{\|f(z) - z\|}{1-\alpha}\right\}. \end{aligned} \tag{2.8}$$

By induction, we conclude that the sequence $\{\|x_n - z\|\}$ is bounded and hence so is the sequence $\{x_n\}$. ☐

Lemma 2.5. *The following inequality holds for all $n \in \mathbb{N}$:*

$$\|x_{n+1}-z\|^2 \leq (1-\alpha_n)^2\|x_n-z\|^2 + 2\alpha_n\alpha\|x_n-z\|\|x_{n+1}-z\| + 2\alpha_n\langle f(z)-z, x_{n+1}-z\rangle. \tag{2.9}$$

Proof. It follows from the subdifferential inequality that

$$\begin{aligned}
\|x_{n+1}-z\|^2 &= \left\|\alpha_n(f(x_n)-z)+(1-\alpha_n)(Tx_n-z)\right\|^2 \\
&\leq (1-\alpha_n)^2\|Tx_n-z\|^2 + 2\alpha_n\langle f(x_n)-z, x_{n+1}-z\rangle \\
&\leq (1-\alpha_n)^2\|x_n-z\|^2 + 2\alpha_n\langle f(x_n)-f(z), x_{n+1}-z\rangle \\
&\quad + 2\alpha_n\langle f(z)-z, x_{n+1}-z\rangle \\
&\leq (1-\alpha_n)^2\|x_n-z\|^2 + 2\alpha_n\alpha\|x_n-z\|\|x_{n+1}-z\| \\
&\quad + 2\alpha_n\langle f(z)-z, x_{n+1}-z\rangle.
\end{aligned} \tag{2.10}$$

□

Lemma 2.6. *If there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that* $\liminf_{k\to\infty}(\|x_{n_k+1}-z\|-\|x_{n_k}-z\|) \geq 0$, *then* $\limsup_{k\to\infty}\langle f(z)-z, x_{n_k+1}-z\rangle \leq 0$.

Proof. First, we note that $\alpha_{n_k} \to 0$ and let us consider the following inequality:

$$\begin{aligned}
0 &\leq \liminf_{k\to\infty}(\|x_{n_k+1}-z\|-\|x_{n_k}-z\|) \\
&\leq \liminf_{k\to\infty}\left(\alpha_{n_k}\left\|f(x_{n_k})-z\right\|+(1-\alpha_{n_k})\|Tx_{n_k}-z\|-\|x_{n_k}-z\|\right) \\
&= \liminf_{k\to\infty}(\|Tx_{n_k}-z\|-\|x_{n_k}-z\|) \\
&\leq \limsup_{k\to\infty}(\|Tx_{n_k}-z\|-\|x_{n_k}-z\|) \\
&\leq 0.
\end{aligned} \tag{2.11}$$

This implies that $\lim_{k\to\infty}(\|x_{n_k}-z\|-\|Tx_{n_k}-z\|)=0$. Since T is a strongly quasinonexpansive mapping, $x_{n_k}-Tx_{n_k} \to 0$. In particular, $x_{n_k}-x_{n_k+1} \to 0$. Because $\{x_{n_k}\}$ is bounded, so there exists a subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_l}} \rightharpoonup q$ and

$$\lim_{l\to\infty}\left\langle f(z)-z, x_{n_{k_l}}-z\right\rangle = \limsup_{k\to\infty}\langle f(z)-z, x_{n_k}-z\rangle. \tag{2.12}$$

It follows from the demiclosedness of $I-T$ at zero that $q \in \mathrm{Fix}(T)$. Then

$$\lim_{l\to\infty}\left\langle f(z)-z, x_{n_{k_l}}-z\right\rangle = \langle f(z)-z, q-z\rangle \leq 0. \tag{2.13}$$

Hence $\limsup_{k\to\infty}\langle f(z)-z, x_{n_k+1}-z\rangle = \limsup_{k\to\infty}\langle f(z)-z, x_{n_k}-z\rangle \leq 0$, as desired. □

Proof of Theorem 2.3. Let us consider the following two cases.

Case 1. There exists an $N \in \mathbb{N}$ such that $\|x_{n+1} - z\| \leq \|x_n - z\|$ for all $n \geq N$. It follows then that $\lim_{n\to\infty}\|x_n - z\|$ exists and hence $\liminf_{n\to\infty}(\|x_{n+1} - z\| - \|x_n - z\|) = 0$. This implies that $\limsup_{n\to\infty}\langle f(z) - z, x_{n+1} - z\rangle \leq 0$. By Lemma 2.5, for all $n \geq N$,

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\leq (1-\alpha_n)^2\|x_n - z\|^2 + 2\alpha_n\alpha\|x_n - z\|\|x_{n+1} - z\| \\
&\quad + 2\alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\
&\leq (1 - 2\alpha_n + 2\alpha_n\alpha)\|x_n - z\|^2 + \alpha_n^2\|x_n - z\|^2 \\
&\quad + 2\alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\
&= (1 - 2\alpha_n(1-\alpha))\|x_n - z\|^2 \\
&\quad + 2\alpha_n(1-\alpha)\left(\frac{\alpha_n\|x_n - z\|^2}{2(1-\alpha)} + \frac{\langle f(z) - z, x_{n+1} - z\rangle}{1-\alpha}\right).
\end{aligned} \tag{2.14}$$

Notice that $\sum_{n=N}^{\infty} 2\alpha_n(1-\alpha) = \infty$ and

$$\limsup_{n\to\infty}\left(\frac{\alpha_n\|x_n - z\|^2}{2(1-\alpha)} + \frac{\langle f(z) - z, x_{n+1} - z\rangle}{1-\alpha}\right) \leq 0. \tag{2.15}$$

By Lemma 2.1, we have $\lim_{n\to\infty}\|x_n - z\|^2 = 0$.

Case 2. There exists a subsequence $\{\|x_{n_j} - z\|\}$ of $\{\|x_n - z\|\}$ such that $\|x_{n_j} - z\| < \|x_{n_j+1} - z\|$ for all $j \in \mathbb{N}$. In this case, it follows from Lemma 2.2 that there exists a subsequence $\{\|x_{m_k} - z\|\}$ of $\{\|x_n - z\|\}$ such that

$$\|x_{m_k} - z\| \leq \|x_{m_k+1} - z\|, \qquad \|x_k - z\| \leq \|x_{m_k+1} - z\| \tag{2.16}$$

for all $k \in \mathbb{N}$. It follows from $\liminf_{k\to\infty}(\|x_{m_k+1} - z\| - \|x_{m_k} - z\|) \geq 0$ that $\limsup_{k\to\infty}\langle f(z) - z, x_{m_k+1} - z\rangle \leq 0$. Moreover, by Lemma 2.5, we have

$$\begin{aligned}
\|x_{m_k+1} - z\|^2 &\leq (1-\alpha_{m_k})^2\|x_{m_k} - z\|^2 + 2\alpha_{m_k}\alpha\|x_{m_k} - z\|\|x_{m_k+1} - z\| \\
&\quad + 2\alpha_{m_k}\langle f(z) - z, x_{m_k+1} - z\rangle \\
&\leq (1-\alpha_{m_k})^2\|x_{m_k+1} - z\|^2 + 2\alpha_{m_k}\alpha\|x_{m_k+1} - z\|^2 \\
&\quad + 2\alpha_{m_k}\langle f(z) - z, x_{m_k+1} - z\rangle.
\end{aligned} \tag{2.17}$$

In particular, it follows that

$$(2 - \alpha_{m_k} - 2\alpha)\|x_{m_k+1} - z\|^2 \leq 2\langle f(z) - z, x_{m_k+1} - z\rangle. \tag{2.18}$$

This implies that

$$(2-2\alpha)\limsup_{k\to\infty}\|x_{m_k+1}-z\|^2 \le \limsup_{k\to\infty} 2\langle f(z)-z, x_{m_k+1}-z\rangle \le 0. \tag{2.19}$$

Hence

$$\limsup_{k\to\infty}\|x_k-z\|^2 \le \limsup_{k\to\infty}\|x_{m_k+1}-z\|^2 = 0. \tag{2.20}$$

Then $\lim_{k\to\infty}\|x_k-z\|^2 = 0$. This completes the proof. □

Remark 2.7. If C is a convex subset of a Hilbert space and $T : C \to C$ is a quasinonexpansive mapping, then the mapping $T_\omega := (1-\omega)I + \omega T$ is strongly quasinonexpansive whenever $\omega \in (0,1)$ (see [10]). This means that Maingé's result is included in ours as a special case.

Remark 2.8. There is a strongly quasinonexpansive mapping S such that S is not of the form $(1-\omega)I + \omega T$ where $\alpha \in (0,1/2)$ and T is a quasinonexpansive mapping. This means that there is an example which is beyond the scope of Maingé's result (see Remark 1.3, Theorem 1.1 with his old proof is valid *for only* $\alpha \in (0,1/2)$).

Example 2.9. Let $A = \{(x,x) : x \in \mathbb{R}\}$. It is clear that A is a closed and convex subset of $\mathbb{R}^2$. Notice that $S := P_A$ is a strongly quasinonexpansive mapping and $(0,0) \in \mathrm{Fix}(S)$. Suppose that $S = (1-\omega)I + \omega T$ where $\omega \in (0,1/2)$ and T is a quasinonexpansive mapping. Then, by Proposition 1.2, we have

$$\langle (2,0) - S(2,0), (2,0) - (0,0)\rangle \ge \frac{\omega}{2}\|(2,0) - T(2,0)\|^2 = \frac{1}{2\omega}\|(2,0) - S(2,0)\|^2. \tag{2.21}$$

It is easy to see that $S(2,0) = (1,1)$. In particular,

$$2 = \langle (2,0) - (1,1), (2,0)\rangle \ge \frac{1}{2\omega}\|(2,0) - (1,1)\|^2 = \frac{1}{\omega}. \tag{2.22}$$

That is $\omega \ge 1/2$, a contradiction.

3. Conclusion

We propose a viscosity approximation process for approximation of a fixed point of a quasinonexpansive mapping. This not only corrects Maingé's result but also essentially improves his result to a more general relaxation.

Acknowledgments

The first author is supported by grant fund under the program Strategic Scholarships for Frontier Research Network for the Ph.D. Program Thai Doctoral degree from the Office of the Commission on Higher Education. The second author is supported by Thailand Research Fund, the Commission on Higher Education and Khon Kaen University under Grant RMU5380039.

References

[1] A. Moudafi, "Viscosity approximation methods for fixed-points problems," *Journal of Mathematical Analysis and Applications*, vol. 241, no. 1, pp. 46–55, 2000.

[2] T. Suzuki, "Moudafi's viscosity approximations with Meir-Keeler contractions," *Journal of Mathematical Analysis and Applications*, vol. 325, no. 1, pp. 342–352, 2007.

[3] F. Cianciaruso, G. Marino, L. Muglia, and H. Zhou, "Strong convergence of viscosity methods for continuous pseudocontractions in Banach spaces," *Journal of Applied Mathematics and Stochastic Analysis*, vol. 2008, Article ID 149483, 11 pages, 2008.

[4] J. W. Peng and J. C. Yao, "A viscosity approximation scheme for system of equilibrium problems, nonexpansive mappings and monotone mappings," *Nonlinear Analysis*, vol. 71, no. 12, pp. 6001–6010, 2009.

[5] S. Saejung, "Halpern's iteration in Banach spaces," *Nonlinear Analysis*, vol. 73, no. 10, pp. 3431–3439, 2010.

[6] H. K. Xu, "Viscosity approximation methods for nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 298, no. 1, pp. 279–291, 2004.

[7] P. E. Maingé, "The viscosity approximation process for quasi-nonexpansive mappings in Hilbert spaces," *Computers & Mathematics with Applications*, vol. 59, no. 1, pp. 74–79, 2010.

[8] K. Aoyama, Y. Kimura, W. Takahashi, and M. Toyoda, "Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space," *Nonlinear Analysis*, vol. 67, no. 8, pp. 2350–2360, 2007.

[9] W. Takahashi, *Nonlinear Functional Analysis*, vol. 2 of *Fixed Point Theory and Its Applications*, Yokohama Publishers, Yokohama, Japan, 2000.

[10] S. Reich, "A limit theorem for projections," *Linear and Multilinear Algebra*, vol. 13, no. 3, pp. 281–290, 1983.

[11] S. Itoh and W. Takahashi, "The common fixed point theory of singlevalued mappings and multivalued mappings," *Pacific Journal of Mathematics*, vol. 79, no. 2, pp. 493–508, 1978.