Hindawi Publishing Corporation
International Journal of Differential Equations
Volume 2011, Article ID 328394, 10 pages
doi:10.1155/2011/328394

Research Article

Existence of Positive Solutions for Fractional Differential Equation with Nonlocal Boundary Condition

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Received 29 May 2011; Revised 24 August 2011; Accepted 28 September 2011

Academic Editor: Om Agrawal

By using the fixed point theorem, existence of positive solutions for fractional differential equation with nonlocal boundary condition $D^{\alpha}_{0+}u(t) + a(t)f(t, u(t)) = 0$, $0 < t < 1$, $u(0) = 0$, $u(1) = \sum_{i=1}^{\infty}\alpha_i u(\xi_i)$ is considered, where $1 < \alpha \leq 2$ is a real number, D^{α}_{0+} is the standard Riemann-Liouville differentiation, and $\xi_i \in (0,1)$, $\alpha_i \in [0,\infty)$ with $\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1} < 1$, $a(t) \in C([0,1],[0,\infty))$, $f(t,u) \in C([0,1] \times [0,\infty),[0,\infty))$.

1. Introduction

Fractional differential equations have been of great interest recently. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications of such constructions in various sciences such as physics, mechanics, chemistry, and engineering. For details, see [1–6] and references therein.

It should be noted that most of papers and books on fractional calculus are devoted to the solvability of linear initial fractional differential equations in terms of special functions [6–8]. Recently, there are some papers that deal with the existence and multiplicity of solution (or positive solution) of nonlinear initial fractional differential equation by the use of techniques of nonlinear analysis (fixed-point theorems, Leray-Shauder theory, etc.); see [9–17].

Recently, Bai and Lü [15] studied the existence of positive solutions of nonlinear fractional differential equation

$$\begin{gathered} D^{\alpha}_{0+}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u(1) = 0, \end{gathered} \tag{1.1}$$

where $1 < \alpha \leq 2$ is a real number, D_{0+}^{α} is the standard Riemann-Liouville differentiation, and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous.

In this paper, we study the existence of positive solutions for fractional differential equation with nonlocal boundary condition

$$D_{0+}^{\alpha}u(t) + a(t)f(t,u(t)) = 0, \quad 0 < t < 1,$$
$$u(0) = 0, \qquad u(1) = \sum_{i=1}^{\infty}\alpha_i u(\xi_i), \tag{1.2}$$

where $1 < \alpha \leq 2$ is a real number, D_{0+}^{α} is the standard Riemann-Liouville differentiation, and $\xi_i \in (0,1)$, $\alpha_i \in [0,\infty)$ with $\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1} < 1$, $a(t) \in C([0,1],[0,\infty))$, $f(t,u) \in C([0,1] \times [0,\infty),[0,\infty))$.

We assume the following conditions hold throughout the paper:

(H1) $\xi_i \in (0,1)$, $\alpha_i \in [0,\infty)$ is both constants with $\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1} < 1$,

(H2) $a(t) \in C([0,1],[0,\infty))$, $a(t) \not\equiv 0$ on $[a,b] \subset (0,1)$,

(H3) $f(t,u) \in C([0,1] \times [0,\infty),[0,\infty))$.

Remark 1.1. To our knowledge, there are no results about the existence of positive solutions for problem (1.2).

2. The Preliminary Lemmas

For the convenience of the reader, we present here the necessary definitions from fractional calculus theory. These definitions can be found in the recent literature.

Definition 2.1. The fractional integral of order $\alpha > 0$ of a function $y : (0,\infty) \to R$ is given by

$$I_{0+}^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds, \tag{2.1}$$

provided the right side is pointwise defined on $(0,\infty)$.

Definition 2.2. The fractional derivative of order $\alpha > 0$ of a function $y : (0,\infty) \to R$ is given by

$$D_{0+}^{\alpha}y(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}}ds, \tag{2.2}$$

where $n = [\alpha] + 1$, provided the right side is pointwise defined on $(0,\infty)$.

Definition 2.3. The map θ is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E, provided that $\theta : P \to [0,\infty)$ is continuous and

$$\theta(tx + (1-t)y) \geq t\theta(x) + (1-t)\theta(y), \tag{2.3}$$

for all $x, y \in P$ and $0 \leq t \leq 1$.

Remark 2.4. As a basic example, we quote for $\lambda > -1$,

$$D_{0+}^{\alpha} t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} t^{\lambda-\alpha}, \tag{2.4}$$

giving in particular $D_{0+}^{\alpha} t^{\alpha-m} = 0,\ m = 1,2,\ldots,N$, where N is the smallest integer greater than or equal to α.

From Definition 2.2 and Remark 2.4, we then obtain the following.

Lemma 2.5. *Let $\alpha > 0$. If one assumes $u \in C(0,1) \cap L(0,1)$, then the fractional differential equation*

$$D_{0+}^{\alpha} u(t) = 0 \tag{2.5}$$

has $u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N},\ C_i \in R,\ i = 1,2,\ldots,N$, where N is the smallest integer greater than or equal to α, as unique solutions.

Lemma 2.6. *Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $u \in C(0,1) \cap L(0,1)$. Then,*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}, \tag{2.6}$$

for some $C_i \in R,\ i = 1,2,\ldots,N$.

Lemma 2.7 (see [15]). *Given $y \in C[0,1]$ and $1 < \alpha \le 2$, the unique solution of*

$$\begin{gathered} D_{0+}^{\alpha} u(t) + y(t) = 0, \quad 0 < t < 1, \\ u(0) = u(1) = 0 \end{gathered} \tag{2.7}$$

is

$$u(t) = \int_0^1 G(t,s) y(s) ds, \tag{2.8}$$

where

$$G(t,s) = \begin{cases} \dfrac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \dfrac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1. \end{cases} \tag{2.9}$$

Lemma 2.8. *Suppose $(H1)$ holds. Given $y \in C[0,1]$ and $1 < \alpha \le 2$, the unique solution of*

$$\begin{gathered} D_{0+}^{\alpha} u(t) + y(t) = 0, \quad 0 < t < 1, \\ u(0) = 0, \qquad u(1) = \sum_{i=1}^{\infty} \alpha_i u(\xi_i) \end{gathered} \tag{2.10}$$

is

$$u(t) = \int_0^1 G(t,s)y(s)ds + B(y)t^{\alpha-1}, \tag{2.11}$$

where

$$G(t,s) = \begin{cases} \dfrac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \dfrac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1, \end{cases}$$
$$B(y) = \frac{\sum_{i=1}^{\infty} \alpha_i \int_0^1 G(\xi_i, s)y(s)ds}{1 - \sum_{i=1}^{\infty} \alpha_i \xi_i^{\alpha-1}}. \tag{2.12}$$

Proof. By applying Lemmas 2.6 and 2.7, we have

$$u(t) = \int_0^1 G(t,s)y(s)ds + C_1 t^{\alpha-1} + C_2 t^{\alpha-2}. \tag{2.13}$$

Because

$$\sum_{i=1}^{\infty} \alpha_i \int_0^1 G(\xi_i, s)ds = \frac{\sum_{i=1}^{\infty} \alpha_i \xi_i^{\alpha-1}(1-\xi_i)}{\alpha\Gamma(\alpha)}, \quad \alpha_i \xi_i^{\alpha-1}(1-\xi_i) < \alpha_i \xi_i^{\alpha-1}, \tag{2.14}$$

by (H1), $\sum_{i=1}^{\infty} \alpha_i \xi_i^{\alpha-1}(1-\xi_i)$ is converge. Therefore, $\sum_{i=1}^{\infty} \alpha_i \int_0^1 G(\xi_i, s)ds$ is converge. $y(t)$ is continuous function on $[0,1]$, so $\sum_{i=1}^{\infty} \alpha_i \int_0^1 G(\xi_i, s)y(s)ds$ is converge.

By $u(0) = 0$, $u(1) = \sum_{i=1}^{\infty} \alpha_i u(\xi_i)$, there are $C_2 = 0$, $C_1 = (\sum_{i=1}^{\infty} \alpha_i \int_0^1 G(\xi_i, s)y(s)ds)/(1 - \sum_{i=1}^{\infty} \alpha_i \xi_i^{\alpha-1})$. Therefore,

$$u(t) = \int_0^1 G(t,s)y(s)ds + B(y)t^{\alpha-1},$$
$$B(y) = \frac{\sum_{i=1}^{\infty} \alpha_i \int_0^1 G(\xi_i, s)y(s)ds}{1 - \sum_{i=1}^{\infty} \alpha_i \xi_i^{\alpha-1}}. \tag{2.15}$$

□

Lemma 2.9 (see [15]). *The function $G(t,s)$ defined by* (2.9) *satisfies the following conditions:*

(1) $G(t,s) > 0$, *for* $t, s \in (0,1)$,

(2) *there exists a positive function* $\gamma \in C(0,1)$ *such that*

$$\min_{(1/4)\le t \le (3/4)} G(t,s) \ge \gamma(s) \max_{0\le t\le 1} G(t,s) = \gamma(s)G(s,s), \quad 0 < s < 1. \tag{2.16}$$

Lemma 2.10 (see [18]). *Let E be a Banach space, $P \subseteq E$ a cone and Ω_1, Ω_2 two bounded open sets of E with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. Suppose that $A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ is a completely continuous operator such that either*

(i) $\|Ax\| \le \|x\|$, $x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \ge \|x\|, x \in P \cap \partial\Omega_2$, *or*

(ii) $\|Ax\| \ge \|x\|$, $x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \le \|x\|, x \in P \cap \partial\Omega_2$

holds. Then, A has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Lemma 2.11 (see [19]). *Let P be a cone in real Banach space $E, P_c = \{x \in P \mid \|x\| \le c\}, \theta$ a nonnegative continuous concave functional on P such that $\theta(x) \le \|x\|$, for all $x \in \overline{P}_c$, and $P(\theta, b, d) = \{x \in P \mid b \le \theta(x),\ \|x\| \le d\}$. Suppose that $A : \overline{P}_c \to \overline{P}_c$ is completely continuous, and there exist constants $0 < a < b < d \le c$ such that*

(C1) $\{x \in P(\theta, b, d) \mid \theta(x) > b\} \ne \emptyset$, *and* $\theta(Ax) > b$, $x \in P(\theta, b, d)$,

(C2) $\|Ax\| \le a$, *for* $x \le a$,

(C3) $\theta(Ax) > b$ *for* $x \in P(\theta, b, c)$ *with* $\|Ax\| > d$.

Then, A has at least three fixed points x_1, x_2, x_3 with

$$\|x_1\| < a, \qquad b < \theta(x_2), \qquad a < \|x_3\|, \qquad \theta(x_3) < b. \tag{2.17}$$

Remark 2.12. If there holds $d = c$, then condition (C1) of Lemma 2.11 implies condition (C3) of Lemma 2.11.

3. The Main Results

Let $E = C[0,1]$ be endowed with the ordering $u \le v$ if $u(t) \le v(t)$ for all $t \in [0,1]$, and the maximum norm, $\|u\| = \max_{0 \le t \le 1}|u(t)|$. Define the cone $P \subset E$ by $P = \{u \in E \mid u(t) \ge 0\}$.

Let the nonnegative continuous concave functional θ on the cone P be defined by $\theta(u) = \min_{(1/4) \le t \le (3/4)}|u(t)|$.

Lemma 3.1 (see [15]). *Let $T : P \to E$ be the operator defined by $Tu(t) := \int_0^1 G(t,s) f(s, u(s)) ds$, then $T : P \to P$ is completely continuous.*

Lemma 3.2. *Let $A : P \to E$ be the operator defined by*

$$Au(t) := \int_0^1 G(t,s) a(s) f(s, u(s)) ds + B\big(a(\cdot) f(\cdot, u(\cdot))\big) t^{\alpha - 1}, \tag{3.1}$$

then $A : P \to P$ is completely continuous.

Proof. The proof is similar to Lemma 3.1, so we omit.

Denote

$$M = \left(\int_0^1 G(s,s)a(s)ds + \frac{\sum_{i=1}^{\infty} \alpha_i \int_0^1 G(\xi_i, s)a(s)ds}{1 - \sum_{i=1}^{\infty} \alpha_i \xi_i^{\alpha-1}} \right)^{-1},$$
$$N = \left(\int_{1/4}^{3/4} \gamma(s)G(s,s)a(s)ds \right)^{-1}. \tag{$*$}$$

□

Theorem 3.3. *Assume (H1)–(H3) hold, and there exist two positive constants $r_2 > r_1 > 0$ such that*

(1) $f(t,u) \leq Mr_2$, *for all* $t \in [0,1]$, $u \in [0, r_2]$,

(2) $f(t,u) \geq Nr_1$, *for all* $t \in [0,1]$, $u \in [0, r_1]$, *where* M, N *is defined in* $(*)$,

then problem (1.2) *has at least one positive solution* u *such that* $r_1 \leq \|u\| \leq r_2$.

Proof. By Lemmas 2.8 and 3.2, we know $A : P \to P$ is completely continuous, and problem (1.2) has a solution $u = u(t)$ if and only if u solves the operator equation $u = Au$. In order to apply Lemma 2.10, we separate the proof into the following two steps.

Step 1. Let $\Omega_2 = \{u \in P \mid \|u\| \leq r_2\}$. For $u \in \partial\Omega_2$, we have $0 \leq u(t) \leq r_2$ for all $t \in [0,1]$. It follows from (1) that for $t \in [0,1]$,

$$\begin{aligned} \|Au\| &\leq \int_0^1 G(s,s)a(s)f(s,u(s))ds + \frac{\sum_{i=1}^{\infty} \alpha_i \int_0^1 G(\xi_i, s)a(s)f(s,u(s))ds}{1 - \sum_{i=1}^{\infty} \alpha_i \xi_i^{\alpha-1}} \\ &\leq Mr_2 \left[\int_0^1 G(s,s)a(s)ds + \frac{\sum_{i=1}^{\infty} \alpha_i \int_0^1 G(\xi_i, s)a(s)ds}{1 - \sum_{i=1}^{\infty} \alpha_i \xi_i^{\alpha-1}} \right] \\ &= r_2 = \|u\|. \end{aligned} \tag{3.2}$$

Therefore,

$$\|Au\| \leq \|u\|, \quad u \in P \cap \partial\Omega_2. \tag{3.3}$$

Step 2. Let $\Omega_1 = \{u \in P \mid \|u\| \leq r_1\}$. For $u \in \partial\Omega_1$, we have $0 \leq u(t) \leq r_1$ for all $t \in [0,1]$. By assumption (2), for $t \in [1/4, 3/4]$, there is

$$\begin{aligned} (Au)(t) &= \int_0^1 G(t,s)a(s)f(s,u(s))ds + \frac{t^{\alpha-1}\sum_{i=1}^{\infty} \alpha_i \int_0^1 G(\xi_i, s)a(s)f(s,u(s))ds}{1 - \sum_{i=1}^{\infty} \alpha_i \xi_i^{\alpha-1}} \\ &\geq \int_0^1 \gamma(s)G(s,s)a(s)f(s,u(s))ds \\ &\geq Nr \int_{1/4}^{3/4} \gamma(s)G(s,s)a(s)ds \\ &= r_1 = \|u\|. \end{aligned} \tag{3.4}$$

So,

$$\|Au\| \geq \|u\|, \quad u \in P \cap \partial\Omega_1. \tag{3.5}$$

Therefore, by (ii) of Lemma 2.10, we complete the proof. □

Example 3.4. Consider the problem

$$D_{0+}^{3/2}u(t) + u^2 + \frac{\sin t}{4} + \frac{1}{5} = 0, \quad 0 < t < 1,$$
$$u(0) = 0, \qquad u(1) = \sum_{i=1}^{\infty}\alpha_i u(\xi_i), \tag{3.6}$$

where $\sum_{i=1}^{\infty}\alpha_i\xi_i^{1/2} = 1/5$.

A simple computation showed $M \geq 1.4, N \approx 13.6649$. Choosing $r_1 = (1/70), r_2 = (9/10)$, we have

$$f(t,u) = u^2 + \frac{\sin t}{4} + \frac{1}{5} \leq 1.2207 \leq Mr_2, \quad (t,u) \in [0,1] \times \left[0, \frac{9}{10}\right],$$
$$f(t,u) = u^2 + \frac{\sin t}{4} + \frac{1}{5} \geq \frac{1}{5} \geq Nr_1, \quad (t,u) \in [0,1] \times \left[0, \frac{1}{70}\right]. \tag{3.7}$$

With the use of Theorem 3.3, problem (3.6) has at least one positive solutions u such that $(1/70) \leq \|u\| \leq (9/10)$.

Theorem 3.5. *Assume (H1)–(H3) hold, and there exist constants $0 < a < b < c$ such that the following assumptions hold:*

(A1) $f(t,u) < Ma$ *for* $(t,u) \in [0,1] \times [0,a]$,

(A2) $f(t,u) \geq Nb$ *for* $(t,u) \in [1/4, 3/4] \times [b,c]$,

(A3) $f(t,u) \leq Mc$ *for* $(t,u) \in [0,1] \times [0,c]$, *where* M, N *is defined in* $(*)$.

Then, the boundary value problem (1.2) *has at least three positive solutions* u_1, u_2, u_3 *with*

$$\|u_1\| < a, \qquad b < \min_{(1/4)\leq t\leq(3/4)}|u_2| < \|u_2\| \leq c,$$
$$a < \|u_3\| \leq c, \qquad \min_{(1/4)\leq t\leq(3/4)}|u_3| < b. \tag{3.8}$$

Proof. We show that all the conditions of Lemma 2.9 are satisfied.

If $u \in \overline{P}_c$, then $\|u\| \leq c$. Assumption (A3) implies $f(t,u(t)) \leq Mc$ for $0 \leq t \leq 1$. Consequently,

$$\begin{aligned}\|Au\| &= \max_{0\leq t\leq 1}\left|\int_0^1 G(t,s)a(s)f(s,u(s))ds + \frac{t^{\alpha-1}\sum_{i=1}^{\infty}\alpha_i\int_0^1 G(\xi_i,s)a(s)f(s,u(s))ds}{1-\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}}\right| \\ &\leq \int_0^1 G(s,s)a(s)f(s,u(s))ds + \frac{t^{\alpha-1}\sum_{i=1}^{\infty}\alpha_i\int_0^1 G(\xi_i,s)a(s)f(s,u(s))ds}{1-\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}} \\ &\leq \left[\int_0^1 G(s,s)a(s)ds + \frac{\sum_{i=1}^{\infty}\alpha_i\int_0^1 G(\xi_i,s)a(s)ds}{1-\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}}\right]\|u\| \\ &\leq \|u\|.\end{aligned} \tag{3.9}$$

Hence, $A : \overline{P}_c \to \overline{P}_c$. In the same way, if $u \in \overline{P}_a$, then assumption (A1) yields $f(t,u(t)) < Ma$, $0 \leq t \leq 1$. Therefore, condition (C2) of Lemma 2.11 is satisfied.

To check condition (C1) of Lemma 2.11, we choose $u(t) = (b+c)/2$, $0 \leq t \leq 1$. It is easy to see that $u(t) = (b+c)/2 \in P(\theta,b,c)$, $\theta(u) = (\theta(b+c))/2 > b$, and consequently, $\{u \in P(\theta,b,d) \mid \theta(u) > b\} \neq \emptyset$ Hence, if $u \in P(\theta,b,c)$, then $b \leq u(t) \leq c$ for $(1/4) \leq t \leq (3/4)$. From assumption (A2), we have $f(t,u(t)) \geq Nb$ for $(1/4) \leq t \leq (3/4)$. So,

$$\begin{aligned}\theta(Au) &= \min_{(1/4)\leq t\leq(3/4)}|(Au)(t)| \\ &\geq \int_0^1 \gamma(s)G(s,s)a(s)f(s,u(s))ds \\ &> Nb\int_{1/4}^{3/4}\gamma(s)G(s,s)a(s)ds \\ &= b = \|u\|\end{aligned} \tag{3.10}$$

$\theta(Au) > b$, for all $u \in P(\theta,b,c)$.

This shows that condition (C1) of Lemma 2.11 is also satisfied.

By Lemma 2.11 and Remark 2.12, the boundary value problem (1.2) has at least three positive solutions $u_1, u_2,$ and u_3 with

$$\begin{aligned}&\|u_1\| < a, \qquad b < \min_{(1/4)\leq t\leq(3/4)}|u_2|, \\ &a < \|u_3\|, \qquad \min_{(1/4)\leq t\leq(3/4)}|u_3| < b.\end{aligned} \tag{3.11}$$

The proof is complete. □

Example 3.6. Consider the problem

$$D_{0+}^{3/2}u(t) + f(t,u) = 0, \quad 0 < t < 1,$$
$$u(0) = 0, \qquad u(1) = \sum_{i=1}^{\infty}\alpha_i u(\xi_i), \tag{3.12}$$

where $\sum_{i=1}^{\infty}\alpha_i\xi_i^{1/2} = (1/5)$,

$$f(t,u) = \begin{cases} \left(\dfrac{t}{40}\right) + 14u^2, & u \le 1, \\ 13 + \left(\dfrac{t}{40}\right) + u, & u > 1. \end{cases} \tag{3.13}$$

We have $M \ge 1.4, N \approx 13.6649$. Choosing $a = (1/14), b = 1, c = 36$, there hold

$$f(t,u) = \frac{t}{40} + 14u^2 \le 0.097 \le Ma, \quad (t,u) \in [0,1] \times \left[0, \frac{1}{14}\right],$$
$$f(t,u) = 13 + \frac{t}{40} + u \ge 14.025 \ge Nb \approx 13.7, \quad (t,u) \in \left[\frac{1}{4}, \frac{3}{4}\right] \times [1,36], \tag{3.14}$$
$$f(t,u) \le 13 + \frac{t}{40} + u \le 48.025 \le Mc \approx 50.4, \quad (t,u) \in [0,1] \times [0,36].$$

With the use of Theorem 3.5, problem (3.12) has at least three positive solutions u_1, u_2, and u_3 with

$$\max_{0\le t\le 1}|u_1(t)| < \frac{1}{14}, \qquad 1 < \min_{(1/4)\le t\le (3/4)}|u_2(t)| < \max_{0\le t\le 1}|u_2(t)| \le 36,$$
$$\frac{1}{14} < \max_{0\le t\le 1}|u_3(t)| \le 36, \qquad \min_{(1/4)\le t\le (3/4)}|u_3(t)| < 1. \tag{3.15}$$

Acknowledgment

This work is sponsored by the NSFC (11101335, 11026060, Gansu Provincial Department of Education Fund (1101-02), NWNU-KJCXGC (03-69).

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