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Research Article

On Nonseparated Three-Point Boundary Value Problems for Linear Functional Differential Equations

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For a system of linear functional differential equations, we consider a three-point problem with nonseparated boundary conditions determined by singular matrices. We show that, to investigate such a problem, it is often useful to reduce it to a parametric family of two-point boundary value problems for a suitably perturbed differential system. The auxiliary parametrised two-point problems are then studied by a method based upon a special kind of successive approximations constructed explicitly, whereas the values of the parameters that correspond to solutions of the original problem are found from certain numerical determining equations. We prove the uniform convergence of the approximations and establish some properties of the limit and determining functions.

1. Introduction

The aim of this paper is to show how a suitable parametrisation can help when dealing with nonseparated three-point boundary conditions determined by singular matrices. We construct a suitable numerical-analytic scheme allowing one to approach a three-point boundary value problem through a certain iteration procedure. To explain the term, we recall that, formally, the methods used in the theory of boundary value problems can be characterised as analytic, functional-analytic, numerical, or numerical-analytic ones.

While the analytic methods are generally used for the investigation of qualitative properties of solutions such as the existence, multiplicity, branching, stability, or dichotomy and generally use techniques of calculus (see, e.g., [1–11] and the references in [12]), the functional-analytic ones are based mainly on results of functional analysis and topological degree theory and essentially use various techniques related to operator equations in abstract spaces [13–26]. The numerical methods, under the assumption on the existence of solutions, provide practical numerical algorithms for their approximation [27, 28]. The numerical construction of approximate solutions is usually based on an idea of the shooting method and may face certain difficulties because, as a rule, this technique requires some global regularity conditions, which, however, are quite often satisfied only locally.

Methods of the so-called numerical-analytic type, in a sense, combine, advantages of the mentioned approaches and are usually based upon certain iteration processes constructed explicitly. Such an approach belongs to the few of them that offer constructive possibilities both for the investigation of the existence of a solution and its approximate construction. In the theory of nonlinear oscillations, numerical-analytic methods of this kind had apparently been first developed in [20, 29–31] for the investigation of periodic boundary value problems. Appropriate versions were later developed for handling more general types of nonlinear boundary value problems for ordinary and functional-differential equations. We refer, for example, to the books [12, 32–34], the handbook [35], the papers [36–50], and the survey [51–57] for related references.

For a boundary value problem, the numerical-analytic approach usually replaces the problem by the Cauchy problem for a suitably perturbed system containing some artificially introduced vector parameter z, which most often has the meaning of an initial value of the solution and the numerical value of which is to be determined later. The solution of Cauchy problem for the perturbed system is sought for in an analytic form by successive approximations. The functional "perturbation term," by which the modified equation differs from the original one, depends explicitly on the parameter z and generates a system of algebraic or transcendental "determining equations" from which the numerical values of z should be found. The solvability of the determining system, in turn, may by checked by studying some of its approximations that are constructed explicitly.

For example, in the case of the two-point boundary value problem

$$x'(t) = f(t, x(t)), \quad t \in [a, b], \tag{1.1}$$

$$Ax(a) + Dx(b) = d, \tag{1.2}$$

where $x : [a, b] \to \mathbb{R}^n, -\infty < a < b < +\infty$, $d \in \mathbb{R}^n$, $\det D \neq 0$, the corresponding Cauchy problem for the modified parametrised system of integrodifferential equations has the form [12]

$$\begin{gathered} x'(t) = f(t, x(t)) + \frac{1}{b-a}\left(D^{-1}d - \left(D^{-1}A + \mathbb{1}_n\right)z\right) - \frac{1}{b-a}\int_a^b f(s, x(s))ds, \quad t \in [a, b], \\ x(a) = z, \end{gathered} \tag{1.3}$$

where $\mathbb{1}_n$ is the unit matrix of dimension n and the parameter $z \in \mathbb{R}^n$ has the meaning of initial value of the solution at the point a. The expression

$$\frac{1}{b-a}\left(D^{-1}d - \left(D^{-1}A + \mathbb{1}_n\right)z\right) - \frac{1}{b-a}\int_a^b f(s, x(s))ds \tag{1.4}$$

in (1.3) plays the role of a "perturbation term" and its choice is, of course, not unique. The solution of problem (1.3) is sought for in an analytic form by the method of successive approximations similar to the Picard iterations. According to the formulas

$$x_{m+1}(t,z) := z + \int_a^t \left(f(s, x_m(s,z))ds - \frac{1}{b-a} \int_a^b f(\tau, x_m(\tau,z))d\tau \right) ds \\ + \frac{t-a}{b-a}\left(D^{-1}d - \left(D^{-1}A + \mathbb{1}_n\right)z\right), \quad m = 0,1,2,\ldots, \tag{1.5}$$

where $x_0(t,z) := z$ for all $t \in [a,b]$ and $z \in \mathbb{R}^n$, one constructs the iterations $x_m(\cdot,z)$, $m = 1,2,\ldots$, which depend upon the parameter z and, for arbitrary its values, satisfy the given boundary conditions (1.2): $Ax_m(a,z) + Dx_m(b,z) = d$, $z \in \mathbb{R}^n$, $m = 1,2,\ldots$. Under suitable assumptions, one proves that sequence (1.5) converges to a limit function $x_\infty(\cdot,z)$ for any value of z.

The procedure of passing from the original differential system (1.1) to its "perturbed" counterpart and the investigation of the latter by using successive approximations (1.5) leads one to the system of determining equations

$$D^{-1}d - \left(D^{-1}A + \mathbb{1}_n\right)z - \int_a^b f(s, x_\infty(s,z))ds = 0, \tag{1.6}$$

which gives those numerical values $z = z_*$ of the parameter that correspond to solutions of the given boundary value problem (1.1), (1.10). The form of system (1.6) is, of course, determined by the choice of the perturbation term (1.4); in some other related works, auxiliary equations are constructed in a different way (see, e.g., [42]). It is clear that the complexity of the given equations and boundary conditions has an essential influence both on the possibility of an efficient construction of approximate solutions and the subsequent solvability analysis.

The aim of this paper is to extend the techniques used in [46] for the system of n linear functional differential equations of the form

$$x'(t) = P_0(t)x(t) + P_1(t)x(\beta(t)) + f(t), \quad t \in [0,T], \tag{1.7}$$

subjected to the inhomogeneous three-point Cauchy-Nicoletti boundary conditions

$$\begin{gathered} x_1(0) = x_{10}, \ldots, \ x_p(0) = x_{p0}, \\ x_{p+1}(\xi) = d_{p+1}, \ldots, \ x_{p+q}(\xi) = d_{p+q}, \\ x_{p+q+1}(T) = d_{p+q+1}, \ldots, \ x_n(T) = d_n, \end{gathered} \tag{1.8}$$

with $\xi \in (0,T)$ is given and $x = \operatorname{col}(x_1,\ldots,x_n)$, to the case where the system of linear functional differential equations under consideration has the general form

$$x'(t) = (lx)(t) + f(t), \quad t \in [a,b], \tag{1.9}$$

and the three-point boundary conditions are non-separated and have the form

$$Ax(a) + Bx(\xi) + Cx(b) = d, \tag{1.10}$$

where A, B, and C are singular matrices, $d = \operatorname{col}(d_1, \ldots, d_n)$. Here, $l = (l_k)_{k=1}^{n} : C([a,b], \mathbb{R}^n) \to L_1([a,b], \mathbb{R}^n)$ is a bounded linear operator and $f \in L_1([a,b], \mathbb{R}^n)$ is a given function.

It should be noted that, due to the singularity of the matrices that determine the boundary conditions (1.10), certain technical difficulties arise which complicate the construction of successive approximations.

The following notation is used in the sequel:

$C([a,b], \mathbb{R}^n)$ is the Banach space of the continuous functions $[a,b] \to \mathbb{R}^n$ with the standard uniform norm;

$L_1([a,b], \mathbb{R}^n)$ is the usual Banach space of the vector functions $[a,b] \to \mathbb{R}^n$ with Lebesgue integrable components;

$\mathcal{L}(\mathbb{R}^n)$ is the algebra of all the square matrices of dimension n with real elements;

$r(Q)$ is the maximal, in modulus, eigenvalue of a matrix $Q \in \mathcal{L}(\mathbb{R}^n)$;

$\mathbb{1}_k$ is the unit matrix of dimension k;

$\mathbb{0}_{i,j}$ is the zero matrix of dimension $i \times j$;

$\mathbb{0}_i = \mathbb{0}_{i,i}$.

2. Problem Setting and Freezing Technique

We consider the system of n linear functional differential equations (1.9) subjected to the nonseparated inhomogeneous three-point boundary conditions of form (1.10). In the boundary value problem (1.1), (1.10), we suppose that $-\infty < a < b < \infty$, $l = (l_k)_{k=1}^{n} : C([a,b], \mathbb{R}^n) \to L_1([a,b], \mathbb{R}^n)$ is a bounded linear operator, $f : [a,b] \to \mathbb{R}^n$ is an integrable function, $d \in \mathbb{R}^n$ is a given vector, A, B, and C are singular square matrices of dimension n, and C has the form

$$C = \begin{pmatrix} V & W \\ \mathbb{0}_{n-q,q} & \mathbb{0}_{n-q} \end{pmatrix}, \tag{2.1}$$

where V is nonsingular square matrix of dimension $q < n$ and W is an arbitrary matrix of dimension $q \times (n-q)$. The singularity of the matrices determining the boundary conditions (1.10) causes certain technical difficulties. To avoid dealing with singular matrices in the boundary conditions and simplify the construction of a solution in an analytic form, we use a two-stage parametrisation technique. Namely, we first replace the three-point boundary conditions by a suitable parametrised family of two-point inhomogeneous conditions, after which one more parametrisation is applied in order to construct an auxiliary perturbed differential system. The presence of unknown parameters leads one to a certain system of determining equations, from which one finds those numerical values of the parameters that correspond to the solutions of the given three-point boundary value problem.

We construct the auxiliary family of two-point problems by "freezing" the values of certain components of x at the points ξ and b as follows:

$$\begin{gathered}\operatorname{col}(x_1(\xi),\ldots,x_n(\xi)) = \lambda,\\ \operatorname{col}\bigl(x_{q+1}(b),\ldots,x_n(b)\bigr) = \eta,\end{gathered} \tag{2.2}$$

where $\lambda = \operatorname{col}(\lambda_1,\ldots,\lambda_n) \in \mathbb{R}^n$ and $\eta = col(\eta_1,\ldots,\eta_{n-q}) \in \mathbb{R}^{n-q}$ are vector parameters. This leads us to the parametrised two-point boundary condition

$$Ax(a) + Dx(b) = d - B\lambda - N_q\eta, \tag{2.3}$$

where

$$N_q := \begin{pmatrix} \mathbb{0}_{q,n-q} \\ \mathbb{1}_{n-q} \end{pmatrix} \tag{2.4}$$

and the matrix D is given by the formula

$$D := \begin{pmatrix} V & W \\ \mathbb{0}_{n-q,q} & \mathbb{1}_{n-q} \end{pmatrix} \tag{2.5}$$

with a certain rectangular matrix W of dimension $q \times (n-q)$. It is important to point out that the matrix D appearing in the two-point condition (2.3) is non-singular.

It is easy to see that the solutions of the original three-point boundary value problem (1.1), (1.10) coincide with those solutions of the two-point boundary value problem (1.1), (2.3) for which the additional condition (2.2) is satisfied.

Remark 2.1. The matrices A and B in the boundary conditions (1.10) are arbitrary and, in particular, may be singular. If the number r of the linearly independent boundary conditions in (1.10) is less than n, that is, the rank of the $(n \times 3n)$-dimensional matrix $[A, B, C]$ is equal to r, then the boundary value problem (1.1), (1.10) may have an $(n-r)$-parametric family of solutions.

We assume that throughout the paper the operator l determining the system of equations (1.9) is represented in the form

$$l = l^0 - l^1, \tag{2.6}$$

where $l^j = (l_k^j)_{k=1}^n : C([a,b],\mathbb{R}^n) \to L_1([a,b],\mathbb{R}^n), j = 0,1$, are bounded linear operators posi-tive in the sense that $(l_k^j u)(t) \ge 0$ for a.e. $t \in [a,b]$ and any $k = 1,2,\ldots,n$, $j = 0,1$, and $u \in C([a,b],\mathbb{R}^n)$ such that $\min_{t\in[a,b]} u_k(t) \ge 0$ for all $k = 1,2,\ldots,n$. We also put $\widehat{l}_k := l_k^0 + l_k^1$, $k = 1,2,\ldots,n$, and

$$\widehat{l} := l^0 + l^1. \tag{2.7}$$

3. Auxiliary Estimates

In the sequel, we will need several auxiliary statements.

Lemma 3.1. *For an arbitrary essentially bounded function $u : [a,b] \to \mathbb{R}$, the estimates*

$$\left|\int_a^t \left(u(\tau) - \frac{1}{b-a}\int_a^b u(s)ds\right)d\tau\right| \le \alpha(t)\left(\operatorname*{ess\,sup}_{s\in[a,b]} u(s) - \operatorname*{ess\,inf}_{s\in[a,b]} u(s)\right) \tag{3.1}$$

$$\le \frac{b-a}{4}\left(\operatorname*{ess\,sup}_{s\in[a,b]} u(s) - \operatorname*{ess\,inf}_{s\in[a,b]} u(s)\right) \tag{3.2}$$

are true for all $t \in [a,b]$, where

$$\alpha(t) := (t-a)\left(1 - \frac{t-a}{b-a}\right), \quad t \in [a,b]. \tag{3.3}$$

Proof. Inequality (3.1) is established similarly to [58, Lemma 3] (see also [12, Lemma 2.3]), whereas (3.2) follows directly from (3.1) if the relation

$$\max_{t\in[a,b]} \alpha(t) = \frac{1}{4}(b-a) \tag{3.4}$$

is taken into account. □

Let us introduce some notation. For any $k = 1,2,\ldots,n$, we define the n-dimensional row-vector e_k by putting

$$e_k := (\underbrace{0,0,\ldots,0}_{k-1},1,0,\ldots,0). \tag{3.5}$$

Using operators (2.7) and the unit vectors (3.5), we define the matrix-valued function $K_l : [a,b] \to \mathcal{L}(\mathbb{R}^n)$ by setting

$$K_l := \left[\widehat{l}e_1^*, \widehat{l}e_2^*, \ldots, \widehat{l}e_n^*\right]. \tag{3.6}$$

Note that, in (3.6), $\widehat{l}e_i^*$ means the value of the operator $\widehat{l}$ on the constant vector function is equal identically to e_i^*, where e_i^* is the vector transpose to e_i. It is easy to see that the components of K_l are Lebesgue integrable functions.

Lemma 3.2. *The componentwise estimate*

$$|(lx)(t)| \le K_l(t)\max_{s\in[a,b]}|x(s)|, \quad t \in [a,b], \tag{3.7}$$

is true for any $x \in C([a,b],\mathbb{R}^n)$, where $K_l : [a,b] \to \mathcal{L}(\mathbb{R}^n)$ is the matrix-valued function given by formula (3.6).

Proof. Let $x = (x_k)_{k=1}^n$ be an arbitrary function from $C([a,b],\mathbb{R}^n)$. Then, recalling the notation for the components of l, we see that

$$lx = \sum_{i=1}^n e_i^* l_i x. \tag{3.8}$$

On the other hand, due to (3.5), we have $x = \sum_{k=1}^n e_k^* x_k$ and, therefore, by virtue of (3.8) and (2.6),

$$lx = \sum_{i=1}^n e_i^* l_i x = \sum_{i=1}^n e_i^* l_i \left(\sum_{k=1}^n e_k^* x_k \right) = \sum_{i=1}^n e_i^* \left(\sum_{k=1}^n \left(l_i^0 e_k^* x_k - l_i^1 e_k^* x_k \right) \right). \tag{3.9}$$

On the other hand, the obvious estimate

$$\sigma x_k(t) \le \max_{s\in[a,b]} |x_k(s)|, \quad t \in [a,b],\ k = 1,2,\ldots,n,\ \sigma \in \{-1,1\}, \tag{3.10}$$

and the positivity of the operators l^j, $j = 0,1$, imply

$$l_i^j(\sigma x_k)(t) = \sigma\left(l_i^j x_k\right)(t) \le l_i^j \max_{s\in[a,b]} |x_k(s)| \tag{3.11}$$

for a.e. $t \in [a,b]$ and any $k, j = 1,2,\ldots,n$, $\sigma \in \{-1,1\}$. This, in view of (2.7) and (3.9), leads us immediately to estimate (3.7). □

4. Successive Approximations

To study the solution of the auxiliary two-point parametrised boundary value problem (1.9), (2.3) let us introduce the sequence of functions $x_m : [a,b] \times \mathbb{R}^{3n-q} \to \mathbb{R}^n$, $m \ge 0$, by putting

$$\begin{aligned} x_{m+1}(t,z,\lambda,\eta) &:= \varphi_{z,\lambda,\eta}(t) + \int_a^t \left((lx_m(\cdot,z,\lambda,\eta))(s) + f(s) \right) ds \\ &\quad - \frac{t-a}{b-a} \int_a^b \left((lx_m(\cdot,z,\lambda,\eta))(s) + f(s) \right) ds, \quad m = 0,1,2,\ldots, \\ x_0(t,z,\lambda,\eta) &:= \varphi_{z,\lambda,\eta}(t) \end{aligned} \tag{4.1}$$

for all $t \in [a,b]$, $z \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n$, and $\eta \in \mathbb{R}^{n-q}$, where

$$\varphi_{z,\lambda,\eta}(t) := z + \frac{t-a}{b-a}\left(D^{-1}(d - B\lambda + N_q\eta) - \left(D^{-1}A + \mathbb{1}_n \right) z \right). \tag{4.2}$$

In the sequel, we consider x_m as a function of t and treat the vectors z, λ, and η as parameters.

Lemma 4.1. *For any $m \geq 0$, $t \in [a,b]$, $z \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n$, and $\eta \in \mathbb{R}^{n-q}$, the equalities*

$$\begin{gathered} x_m(a,z,\lambda,\eta) = z, \\ Ax_m(a,z,\lambda,\eta) + Dx_m(b,z,\lambda,\eta) = d - B\lambda + N_q\eta, \end{gathered} \tag{4.3}$$

are true.

The proof of Lemma 4.1 is carried out by straightforward computation. We emphasize that the matrix D appearing in the two-point condition (2.3) is non-singular. Let us also put

$$(\mathcal{M}y)(t) := \left(1 - \frac{t-a}{b-a}\right)\int_a^t y(s)ds + \frac{t-a}{b-a}\int_t^b y(s)ds, \quad t \in [a,b], \tag{4.4}$$

for an arbitrary $y \in L_1([a,b],\mathbb{R}^n)$. It is clear that $\mathcal{M} : L_1([a,b],\mathbb{R}^n) \to C([a,b],\mathbb{R}^n)$ is a positive linear operator. Using the operator $\mathcal{M}$, we put

$$Q_l := \left[\mathcal{M}(K_l e_1^*), \mathcal{M}(K_l e_2^*), \ \ldots, \mathcal{M}(K_l e_n^*)\right], \tag{4.5}$$

where K_l is given by formula (3.6). Finally, define a constant square matrix Q_l of dimension n by the formula

$$Q_l := \max_{t\in[a,b]} Q_l(t). \tag{4.6}$$

We point out that, as before, the maximum in (4.6) is taken componentwise (one should remember that, for $n > 1$, a point $t_* \in [a,b]$ such that $Q_l = Q_l(t_*)$ may not exist).

Theorem 4.2. *If the spectral radius of the matrix Q_l satisfies the inequality*

$$r(Q_l) < 1, \tag{4.7}$$

then, for arbitrary fixed $z \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n$, and $\eta \in \mathbb{R}^{n-q}$:

(1) *the sequence of functions* (4.1) *converges uniformly in $t \in [a,b]$ for any fixed $(z,\lambda,\eta) \in \mathbb{R}^{3n-q}$ to a limit function*

$$x_\infty(t,z,\lambda,\eta) = \lim_{m\to+\infty} x_m(t,z,\lambda,\eta); \tag{4.8}$$

(2) *the limit function $x_\infty(\cdot,z,\lambda,\eta)$ possesses the properties*

$$\begin{gathered} x_\infty(a,z,\lambda,\eta) = z, \\ Ax_\infty(a,z,\lambda,\eta) + Dx_\infty(b,z,\lambda,\eta) = d - B\lambda + N_q\eta; \end{gathered} \tag{4.9}$$

(3) *function* (4.8) *is a unique absolutely continuous solution of the integro-functional equation*

$$x(t) = z + \frac{t-a}{b-a}\Big(D^{-1}(d - B\lambda + N_q\eta) - \big(D^{-1}A + \mathbb{1}_n\big)z\Big) + \int_a^t ((lx)(s) + f(s))ds - \frac{t-a}{b-a}\int_a^b ((lx)(s) + f(s))ds, \quad t \in [a,b]; \tag{4.10}$$

(4) *the error estimate*

$$\max_{t\in[a,b]} \left|x_\infty(t,z,\lambda,\eta) - x_m(t,z,\lambda,\eta)\right| \le \frac{b-a}{4} Q_l^m (\mathbb{1}_n - Q_l)^{-1}\omega(z,\lambda,\eta) \tag{4.11}$$

holds, where $\omega : \mathbb{R}^{3n-q} \to \mathbb{R}^n$ *is given by the equality*

$$\omega(z,\lambda,\eta) := \operatorname*{ess\,sup}_{s\in[a,b]}\big((l\varphi_{z,\lambda,\eta})(s) + f(s)\big) - \operatorname*{ess\,inf}_{s\in[a,b]}\big((l\varphi_{z,\lambda,\eta})(s) + f(s)\big). \tag{4.12}$$

In (3.6), (4.11) and similar relations, the signs $|\cdot|$, $\le$, $\ge$, as well as the operators "max", "ess sup", "ess inf", and so forth, applied to vectors or matrices are understood componentwise.

Proof. The validity of assertion 1 is an immediate consequence of the formula (4.1). To obtain the other required properties, we will show, that under the conditions assumed, sequence (4.1) is a Cauchy sequence in the Banach space $C([a,b],\mathbb{R}^n)$ equipped with the standard uniform norm. Let us put

$$r_m(t,z,\lambda,\eta) := x_{m+1}(t,z,\lambda,\eta) - x_m(t,z,\lambda,\eta) \tag{4.13}$$

for all $z \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n$, $\eta \in \mathbb{R}^{n-q}$, $t \in [a,b]$, and $m \ge 0$. Using Lemma 3.2 and taking equality (3.4) into account, we find that (4.1) yields

$$\begin{aligned}\left|x_1(t,z,\lambda,\eta) - x_0(t,z,\lambda,\eta)\right| &= \left|\int_a^t [(l\varphi_{z,\lambda,\eta})(s) + f(s)]ds - \frac{t-a}{b-a}\int_a^b [(l\varphi_{z,\lambda,\eta})(s) + f(s)]ds\right| \\ &\le \alpha(t)\omega(z,\lambda,\eta) \\ &\le \frac{b-a}{4}\omega(z,\lambda,\eta),\end{aligned} \tag{4.14}$$

for arbitrary fixed z, λ, and η, where α is the function given by (3.3) and $\omega(\cdot)$ is defined by formula (4.12).

According to formulae (4.1), for all $t \in [a,b]$, arbitrary fixed z, λ, and η and $m = 1,2,\ldots$ we have

$$\begin{aligned} r_m(t,z,\lambda,\eta) &= \int_a^t l(x_m(\cdot,z,\lambda,\eta) - x_{m-1}(\cdot,z,\lambda,\eta))(s)ds \\ &\quad - \frac{t-a}{b-a}\int_a^b l(x_m(\cdot,z,\lambda,\eta) - x_{m-1}(\cdot,z,\lambda,\eta))(s)ds \\ &= \left(1-\frac{t-a}{b-a}\right)\int_a^t l(x_m(\cdot,z,\lambda,\eta) - x_{m-1}(\cdot,z,\lambda,\eta))(s)ds \\ &\quad - \frac{t-a}{b-a}\int_t^b l(x_m(\cdot,z,\lambda,\eta) - x_{m-1}(\cdot,z,\lambda,\eta))(s)ds. \end{aligned} \tag{4.15}$$

Equalities (4.13) and (4.15) imply that for all $m = 1,2,\ldots$, arbitrary fixed z,λ,η and $t \in [a,b]$,

$$\begin{aligned} |r_m(t,z,\lambda,\eta)| &\le \left(1-\frac{t-a}{b-a}\right)\int_a^t \left|l\,(r_{m-1}(\cdot,z,\lambda,\eta)(s))\right|ds \\ &\quad + \frac{t-a}{b-a}\int_t^b \left|l(r_{m-1}(\cdot,z,\lambda,\eta))(s)\right|ds. \end{aligned} \tag{4.16}$$

Applying inequality (3.7) of Lemma 3.2 and recalling formulae (4.5) and (4.6), we get

$$\begin{aligned} |r_m(t,z,\lambda,\eta)| &\le \left(1-\frac{t-a}{b-a}\right)\int_a^t K_l(s)\max_{\tau\in[a,b]}|r_{m-1}(\tau,z,\lambda,\eta)|ds \\ &\quad + \frac{t-a}{b-a}\int_t^b K_l(s)\max_{\tau\in[a,b]}|r_{m-1}(\tau,z,\lambda,\eta)|ds \\ &= \left(\left(1-\frac{t-a}{b-a}\right)\int_a^t K_l(s)ds + \frac{t-a}{b-a}\int_t^b K_l(s)ds\right)\max_{\tau\in[a,b]}|r_{m-1}(\tau,z,\lambda,\eta)| \\ &= Q_l(t)\max_{\tau\in[a,b]}|r_{m-1}(\tau,z,\lambda,\eta)| \\ &\le Q_l\max_{\tau\in[a,b]}|r_{m-1}(\tau,z,\lambda,\eta)|. \end{aligned} \tag{4.17}$$

Using (4.17) recursively and taking (4.14) into account, we obtain

$$\begin{aligned} |r_m(t,z,\lambda,\eta)| &\le Q_l^m\max_{\tau\in[a,b]}|r_0(\tau,z,\lambda,\eta)| \\ &\le \frac{b-a}{4}Q_l^m\omega(z,\lambda,\eta), \end{aligned} \tag{4.18}$$

for all $m \geq 1$, $t \in [a,b]$, $z \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n$, and $\eta \in \mathbb{R}^{n-q}$. Using (4.18) and (4.13), we easily obtain that, for an arbitrary $j \in \mathbb{N}$,

$$\begin{aligned} |x_{m+j}(t,z,\lambda,\eta) - x_m(t,z,\lambda,\eta)| &= |(x_{m+j}(t,z,\lambda,\eta) - x_{m+j-1}(t,z,\lambda,\eta)) \\ &\quad + (x_{m+j-1}(t,z,\lambda,\eta) - x_{m+j-2}(t,z,\lambda,\eta)) + \cdots \\ &\quad + (x_{m+1}(t,z,\lambda,\eta) - x_{\mathrm{m}}(t,z,\lambda,\eta))| \\ &\leq \sum_{i=0}^{j-1} |r_{m+i}(t,z,\lambda,\eta)| \\ &\leq \frac{b-a}{4} \sum_{i=0}^{j-1} Q_l^{m+i} \omega(z,\lambda,\eta). \end{aligned} \tag{4.19}$$

Therefore, by virtue of assumption (4.7), it follows that

$$\begin{aligned} |x_{m+j}(t,z,\lambda,\eta) - x_m(t,z,\lambda,\eta)| &\leq \frac{b-a}{4} Q_l^m \sum_{i=0}^{+\infty} Q_l^i \omega(z,\lambda,\eta) \\ &= \frac{b-a}{4} Q_l^m (\mathbb{1}_n - Q_l)^{-1} \omega(z,\lambda,\eta) \end{aligned} \tag{4.20}$$

for all $m \geq 1$, $j \geq 1$, $t \in [a,b]$, $z \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n$, and $\eta \in \mathbb{R}^{n-q}$. We see from (4.20) that (4.1) is a Cauchy sequence in the Banach space $C([a,b],\mathbb{R}^n)$ and, therefore, converges uniformly in $t \in [a,b]$ for all $(z,\lambda,\eta) \in \mathbb{R}^{3n-q}$:

$$\lim_{m\to\infty} x_m(t,z,\lambda,\eta) = x_\infty(t,z,\lambda,\eta), \tag{4.21}$$

that is, assertion 2 holds. Since all functions $x_m(t,z,\lambda,\eta)$ of the sequence (4.1) satisfy the boundary conditions (2.3), by passing to the limit in (2.3) as $m \to +\infty$ we show that the function $x_\infty(\cdot,z,\lambda,\eta)$ satisfies these conditions.

Passing to the limit as $m \to \infty$ in (4.1), we show that the limit function is a solution of the integro-functional equation (4.10). Passing to the limit as $j \to \infty$ in (4.20) we obtain the estimate

$$|x_\infty(t,z,\lambda,\eta) - x_m(t,z,\lambda,\eta)| \leq \frac{b-a}{4} Q_l^m (\mathbb{1}_n - Q_l)^{-1} \omega(z,\lambda,\eta) \tag{4.22}$$

for a.e. $t \in [a,b]$ and arbitrary fixed z, λ, η, and $m = 1,2,\ldots$. This completes the proof of Theorem 4.2.

We have the following simple statement. □

Proposition 4.3. *If, under the assumptions of Theorem 4.2, one can specify some values of z, λ, and η, such that the limit function $x_\infty(\cdot, z, \lambda, \eta)$ possesses the property*

$$D^{-1}(d - B\lambda + N_q\eta) - \left(D^{-1}A + \mathbb{1}_n\right)z = \int_a^b ((lx_\infty(\cdot, z, \lambda, \eta))(s) + f(s))ds = 0, \tag{4.23}$$

then, for these z, λ, and η, it is also a solution of the boundary value problem (1.9), (2.3).

Proof. The proof is a straightforward application of the above theorem. □

5. Some Properties of the Limit Function

Let us first establish the relation of the limit function $x_\infty(\cdot, z, \lambda, \eta)$ to the auxiliary two-point boundary value problem (1.9), (2.3). Along with system (1.9), we also consider the system with a constant forcing term in the right-hand side

$$x'(t) = (lx)(t) + f(t) + \mu, \quad t \in [a, b], \tag{5.1}$$

and the initial condition

$$x(a) = z, \tag{5.2}$$

where $\mu = \operatorname{col}(\mu_1, \ldots, \mu_n)$ is a control parameter.

We will show that for arbitrary fixed $z \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n$, and $\eta \in^{n-q}$, the parameter μ can be chosen so that the solution $x(\cdot, z, \lambda, \eta, \mu)$ of the initial value problem (5.1), (5.2) is, at the same time, a solution of the two-point parametrised boundary value problem (5.1), (2.3).

Proposition 5.1. *Let $z \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n$, and $\eta \in \mathbb{R}^{n-q}$ be arbitrary given vectors. Assume that condition* (4.7) *is satisfied. Then a solution $x(\cdot)$ of the initial value problem* (5.1), (5.2) *satisfies the boundary conditions* (2.3) *if and only if $x(\cdot)$ coincides with $x_\infty(\cdot, z, \lambda, \eta)$ and*

$$\mu = \mu_{z,\lambda,\eta}, \tag{5.3}$$

where

$$\begin{aligned} \mu_{z,\lambda,\eta} := {} & \frac{1}{b-a}\left(D^{-1}(d - B\lambda + N_q\eta) - \left(D^{-1}A + \mathbb{1}_n\right)z\right) \\ & - \frac{1}{b-a}\int_a^b [(lx_\infty(\cdot, z, \lambda, \eta)(s) + f(s))]ds \end{aligned} \tag{5.4}$$

and $x_\infty(\cdot, z, \lambda, \eta)$ is the limit function of sequence (4.1).

Proof. The assertion of Proposition 5.1 is obtained by analogy to the proof of [50, Theorem 4.2]. Indeed, let $z \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n$, and $\eta \in \mathbb{R}^{n-q}$ be arbitrary.

If μ is given by (5.3), then, due to Theorem 4.2, the function $x_\infty(\cdot, z, \lambda, \eta)$ has properties (4.9) and satisfies equation (4.10), whence, by differentiation, equation (5.1) with the above-mentioned value of μ is obtained. Thus, $x_\infty(\cdot, z, \lambda, \eta)$ is a solution of (5.1), (5.2) with μ of form (5.3) and, moreover, this function satisfies the two-point boundary conditions (2.3).

Let us fix an arbitrary $\mu \in \mathbb{R}^n$ and assume that the initial value problem (5.1), (5.2) has a solution y satisfies the two-point boundary conditions (2.3). Then

$$y(t) = z + \int_a^t \left[(ly)(s) + f(s)\right] ds + \mu(t-a), \tag{5.5}$$

for all $t \in [a, b]$. By assumption, y satisfies the two-point conditions (2.3) and, therefore, (5.5) yields

$$\begin{aligned} Ay(a) + Dy(b) &= Az + D\left(z + \int_a^b \left((ly)(s) + f(s)\right)(s) ds + \mu(b-a)\right) \\ &= d - B\lambda + N_q\eta, \end{aligned} \tag{5.6}$$

whence we find that μ can be represented in the form

$$\mu = \frac{1}{b-a} D^{-1}\left(d - B\lambda + N_q\eta - (A+D)z - \int_a^b \left((ly)(s) + f(s)\right)(s) ds\right). \tag{5.7}$$

On the other hand, we already know that the function $x_\infty(\cdot, z, \lambda, \eta)$, satisfies the two-point conditions (2.3) and is a solution of the initial value problem (5.1), (5.2) with $\mu = \mu_{z,\lambda,\eta}$, where the value $\mu_{z,\lambda,\eta}$ is defined by formula (5.4). Consequently,

$$x_\infty(t, z, \lambda, \eta) = z + \int_a^t \left[\left(lx_\infty(\cdot, z, \lambda, \eta)(s) + f(s)\right)\right] ds + \mu_{z,\lambda,\eta}(t-a), \quad t \in [a, b]. \tag{5.8}$$

Putting

$$h(t) := y(t) - x_\infty(t, z, \lambda, \eta), \quad t \in [a, b], \tag{5.9}$$

and taking (5.5), (5.8) into account, we obtain

$$h(t) = \int_a^t (lh)(s) ds + \left(\mu - \mu_{z,\lambda,\eta}\right)(t-a), \quad t \in [a, b]. \tag{5.10}$$

Recalling the definition (5.4) of $\mu_{z,\lambda,\eta}$ and using formula (5.7), we obtain

$$\begin{aligned}\mu-\mu_{z,\lambda,\eta}&=\frac{1}{b-a}\int_a^b l\big(x_\infty(\cdot,z,\lambda,\eta)-y\big)(s)ds\\&=-\frac{1}{b-a}\int_a^b(lh)(s)ds,\end{aligned}\tag{5.11}$$

and, therefore, equality (5.10) can be rewritten as

$$\begin{aligned}h(t)&=\int_a^t(lh)(s)ds-\frac{t-a}{b-a}\int_a^b(lh)(s)ds\\&=\left(1-\frac{t-a}{b-a}\right)\int_a^t(lh)(s)ds-\frac{t-a}{b-a}\int_t^b(lh)(s)ds,\quad t\in[a,b].\end{aligned}\tag{5.12}$$

Applying Lemma 3.2 and recalling notation (4.6), we get

$$\begin{aligned}|h(t)|&\le\left(\left(1-\frac{t-a}{b-a}\right)\int_a^t K_l(s)ds+\frac{t-a}{b-a}\int_t^b K_l(s)ds\right)\max_{\tau\in[a,b]}|h(\tau)|\\&\le Q_l\max_{\tau\in[a,b]}|h(\tau)|\end{aligned}\tag{5.13}$$

for an arbitrary $t\in[a,b]$. By virtue of condition (4.7), inequality (5.13) implies that

$$\max_{\tau\in[a,b]}|h(\tau)|\le Q_l^m\max_{\tau\in[a,b]}|h(\tau)|\longrightarrow 0\tag{5.14}$$

as $m\to+\infty$. According to (5.9), this means that y coincides with $x_\infty(\cdot,z,\lambda,\eta)$, and, therefore, by (5.11), $\mu=\mu_{z,\lambda,\eta}$, which brings us to the desired conclusion. □

We show that one can choose certain values of parameters $z=z_*,\ \lambda=\lambda_*,\ \eta=\eta_*$ for which the function $x_\infty(\cdot,\ z_*,\ \lambda_*,\ \eta_*)$ is the solution of the original three-point boundary value problem (1.9), (1.10). Let us consider the function $\Delta:\mathbb{R}^{3n-q}\to\mathbb{R}^n$ given by formula

$$\Delta(z,\lambda,\eta):=g(z,\lambda,\eta)-\int_a^b\big((lx_\infty(\cdot,z,\lambda,\eta))(s)+f(s)\big)ds\tag{5.15}$$

with

$$g(z,\lambda,\eta):=D^{-1}(d-B\lambda+N_q\eta)-\left(D^{-1}A+\mathbb{1}_n\right)z\tag{5.16}$$

for all z, λ, and η, where x_∞ is the limit function (4.8).

The following statement shows the relation of the limit function (4.8) to the solution of the original three-point boundary value problem (1.9), (1.10).

Theorem 5.2. *Assume condition* (4.7). *Then the function* $x_\infty(\cdot, z, \lambda, \eta)$ *is a solution of the three-point boundary value problem* (1.9), (1.10) *if and only if the triplet* z, λ, η *satisfies the system of* $3n - q$ *algebraic equations*

$$\Delta(z, \lambda, \eta) = 0, \tag{5.17}$$

$$e_1 x_\infty(\xi, z, \lambda, \eta) = \lambda_1, \; e_2 x_\infty(\xi, z, \lambda, \eta) = \lambda_2, \; \ldots, \; e_n x_\infty(\xi, z, \lambda, \eta) = \lambda_n, \tag{5.18}$$

$$e_{q+1} x_\infty(b, z, \lambda, \eta) = \eta_1, \; e_{q+2} x_\infty(b, z, \lambda, \eta) = \eta_2, \ldots, \; e_{q+\infty} x_\infty(b, z, \lambda, \eta) = \eta_{n-q}. \tag{5.19}$$

Proof. It is sufficient to apply Proposition 5.1 and notice that the differential equation in (5.1) coincides with (1.9) if and only if the triplet (z, λ, η) satisfies (5.17). On the other hand, (5.18) and (5.19) bring us from the auxiliary two-point parametrised conditions to the three-point conditions (1.10). □

Proposition 5.3. *Assume condition* (4.7). *Then, for any* (z^j, λ^j, η^j), $j = 0, 1$, *the estimate*

$$\max_{t \in [a,b]} \left| x_\infty\left(t, z^0, \lambda^0, \eta^0\right) - x_\infty\left(t, z^1, \lambda^1, \eta^1\right) \right| \le (\mathbb{1}_n - Q_l)^{-1} v\left(z^0, \lambda^0, \eta^0, z^1, \lambda^1, \eta^1\right) \tag{5.20}$$

holds, where

$$v\left(z^0, \lambda^0, \eta^0, z^1, \lambda^1, \eta^1\right) := \max_{t \in [a,b]} \left| \varphi_{z^0, \lambda^0, \eta^0}(t) - \varphi_{z^1, \lambda^1, \eta^1}(t) \right|. \tag{5.21}$$

Proof. Let us fix two arbitrary triplets (z^j, λ^j, η^j), $j = 0, 1$, and put

$$u_m(t) := x_m\left(t, z^0, \lambda^0, \eta^0\right) - x_m\left(t, z^1, \lambda^1, \eta^1\right), \quad t \in [a, b]. \tag{5.22}$$

Consider the sequence of vectors c_m, $m = 0, 1, \ldots$, determined by the recurrence relation

$$c_m := c_0 + Q_l c_{m-1}, \quad m \ge 1, \tag{5.23}$$

with

$$c_0 := v\left(z^0, \lambda^0, \eta^0, z^1, \lambda^1, \eta^1\right). \tag{5.24}$$

Let us show that

$$\max_{t \in [a,b]} |u_m(t)| \le c_m \tag{5.25}$$

for all $m \geq 0$. Indeed, estimate (5.25) is obvious for $m = 0$. Assume that

$$\max_{t\in[a,b]}|u_{m-1}(t)| \leq c_{m-1}. \tag{5.26}$$

It follows immediately from (4.1) that

$$\begin{aligned} u_m(t) &= \varphi_{z^0,\lambda^0,\eta^0}(t) - \varphi_{z^1,\lambda^1,\eta^1}(t) + \int_a^t (lu_{m-1})(s)ds - \frac{t-a}{b-a}\int_a^b (lu_{m-1})(s)ds \\ &= \varphi_{z^0,\lambda^0,\eta^0}(t) - \varphi_{z^1,\lambda^1,\eta^1}(t) \\ &\quad + \left(1 - \frac{t-a}{b-a}\right)\int_a^t (lu_{m-1})(s)ds - \frac{t-a}{b-a}\int_t^b (lu_{m-1})(s)ds, \end{aligned} \tag{5.27}$$

whence, by virtue of (5.21), estimate (3.7) to Lemma 3.2, and assumption (5.26),

$$\begin{aligned} |u_m(t)| &\leq \left|\varphi_{z^0,\lambda^0,\eta^0}(t) - \varphi_{z^1,\lambda^1,\eta^1}(t)\right| \\ &\quad + \left(1 - \frac{t-a}{b-a}\right)\int_a^t |(lu_{m-1})(s)|ds + \frac{t-a}{b-a}\int_t^b |(lu_{m-1})(s)|ds \\ &\leq v\left(z^0,\lambda^0,\eta^0,z^1,\lambda^1,\eta^1\right) \\ &\quad + \left(1 - \frac{t-a}{b-a}\right)\int_a^t K_l(s)ds \max_{t\in[a,b]}|u_{m-1}(t)| + \frac{t-a}{b-a}\int_t^b K_l(s)ds \max_{t\in[a,b]}|u_{m-1}(t)| \\ &\leq v\left(z^0,\lambda^0,\eta^0,z^1,\lambda^1,\eta^1\right) + \left(\left(1 - \frac{t-a}{b-a}\right)\int_a^t K_l(s)ds + \frac{t-a}{b-a}\int_t^b K_l(s)ds\right)c_{m-1} \\ &\leq v\left(z^0,\lambda^0,\eta^0,z^1,\lambda^1,\eta^1\right) + Q_l c_{m-1}, \end{aligned} \tag{5.28}$$

which estimate, in view of (5.23) and (5.24), coincides with the required inequality (5.25). Thus, (5.25) is true for any m. Using (5.23) and (5.25), we obtain

$$\begin{aligned} \max_{t\in[a,b]}|u_m(t)| &\leq c_0 + Q_l c_{m-1} = c_0 + Q_l c_0 + Q_l^2 c_{m-2} = \cdots \\ &= \sum_{k=0}^{m-1} Q_l^k c_0 + Q_l^m c_0. \end{aligned} \tag{5.29}$$

Due to assumption (4.7), $\lim_{m\to+\infty} Q_l^m = 0$. Therefore, passing to the limit in (5.29) as $m \to +\infty$ and recalling notation (5.22), we obtain the estimate

$$\max_{t\in[a,b]}\left|x_*\left(t,z^0,\lambda^0,\eta^0\right) - x_*\left(t,z^1,\lambda^1,\eta^1\right)\right| \leq \sum_{k=0}^{+\infty} Q_l^k c_0 = (\mathbb{1}_n - Q_l)^{-1}c_0, \tag{5.30}$$

which, in view of (5.24), coincides with (5.20). □

Now we establish some properties of the "determining function" $\Delta : \mathbb{R}^{3n-q} \to \mathbb{R}^n$ given by equality (5.15).

Proposition 5.4. *Under condition* (3.10), *formula* (5.15) *determines a well-defined function* $\Delta : \mathbb{R}^{3n-q} \to \mathbb{R}^n$, *which, moreover, satisfies the estimate*

$$\begin{aligned}\left|\Delta\left(z^0,\lambda^0,\eta^0\right)-\Delta\left(z^1,\lambda^1,\eta^1\right)\right| &\leq \left|G[z^0-z^1,\lambda^0-\lambda^1,\eta^0-\eta^1]^*\right| \\ &\quad + R_l \max_{t\in[a,b]}\left|z^0-z^1+\frac{t-a}{b-a}G[z^0-z^1,\lambda^0-\lambda^1,\eta^0-\eta^1]^*\right|,\end{aligned} \tag{5.31}$$

for all (z^j,λ^j,η^j), $j=0,1$, *where the* $(n\times n)$*-matrices* G *and* R_l *are defined by the equalities*

$$\begin{aligned}G &:= D^{-1}\left[A+D,B,N_q\right], \\ R_l &:= \int_a^b K_l(s)ds\ (\mathbb{1}_n-Q_l)^{-1}.\end{aligned} \tag{5.32}$$

Proof. According to the definition (5.15) of Δ, we have

$$\begin{aligned}\Delta\left(z^0,\lambda^0,\eta^0\right)-\Delta\left(z^1,\lambda^1,\eta^1\right) &= g\left(z^0,\lambda^0,\eta^0\right)-g\left(z^1,\lambda^1,\eta^1\right) \\ &\quad -\int_a^b\left(l\left(x_\infty\left(\cdot,z^0,\lambda^0,\eta^0\right)-x_\infty\left(\cdot,z^1,\lambda^1,\eta^1\right)\right)(s)\right)ds,\end{aligned} \tag{5.33}$$

whence, due to Lemma 3.2,

$$\begin{aligned}\left|\Delta\left(z^0,\lambda^0,\eta^0\right)-\Delta\left(z^1,\lambda^1,\eta^1\right)\right| &\leq \left|g\left(z^0,\lambda^0,\eta^0\right)-g\left(z^1,\lambda^1,\eta^1\right)\right| \\ &\quad +\int_a^b\left|l\left(x_\infty\left(\cdot,z^0,\lambda^0,\eta^0\right)-x_\infty\left(\cdot,z^1,\lambda^1,\eta^1\right)\right)(s)\right|ds \\ &\leq \left|g\left(z^0,\lambda^0,\eta^0\right)-g\left(z^1,\lambda^1,\eta^1\right)\right| \\ &\quad +\int_a^b K_l(s)ds\max_{\tau\in[a,b]}\left|x_\infty\left(\tau,z^0,\lambda^0,\eta^0\right)-x_\infty\left(\tau,z^1,\lambda^1,\eta^1\right)(s)\right|.\end{aligned} \tag{5.34}$$

Using Proposition 5.3, we find

$$\begin{aligned}\left|\Delta\left(z^0,\lambda^0,\eta^0\right)-\Delta\left(z^1,\lambda^1,\eta^1\right)\right| &\le \left|g\left(z^0,\lambda^0,\eta^0\right)-g\left(z^1,\lambda^1,\eta^1\right)\right| \\ &\quad+\int_a^b K_l(s)ds(\mathbb{1}_n-Q_l)^{-1}v\left(z^0,\lambda^0,\eta^0,z^1,\lambda^1,\eta^1\right).\end{aligned} \tag{5.35}$$

On the other hand, recalling (4.2) and (5.21), we get

$$v\left(z^0,\lambda^0,\eta^0,z^1,\lambda^1,\eta^1\right)=\max_{t\in[a,b]}\left|z^0-z^1+\frac{t-a}{b-a}\left(g\left(z^0,\lambda^0,\eta^0\right)-g\left(z^1,\lambda^1,\eta^1\right)\right)\right|. \tag{5.36}$$

It follows immediately from (5.16) that

$$\begin{aligned}g\left(z^0,\lambda^0,\eta^0\right)-g\left(z^1,\lambda^1,\eta^1\right)&=-D^{-1}B\left(\lambda^0-\lambda^1\right)-D^{-1}N_q\left(\eta^0-\eta^1\right)-\left(D^{-1}A+\mathbb{1}_n\right)\left(z^0-z^1\right)\\ &=-D^{-1}\left[B\left(\lambda^0-\lambda^1\right)+N_q\left(\eta^0-\eta^1\right)+(A+D)\left(z^0-z^1\right)\right]\\ &=D^{-1}\left[A+D,B,N_q\right]\begin{pmatrix}z^0-z^1\\ \lambda^0-\lambda^1\\ \eta^0-\eta^1\end{pmatrix}.\end{aligned} \tag{5.37}$$

Therefore, (5.35) and (5.36) yield the estimate

$$\begin{aligned}&\left|\Delta\left(z^0,\lambda^0,\eta^0\right)-\Delta\left(z^1,\lambda^1,\eta^1\right)\right|\\ &\quad\le\left|D^{-1}\left[A+D,B,N_q\right]\begin{pmatrix}z^0-z^1\\ \lambda^0-\lambda^1\\ \eta^0-\eta^1\end{pmatrix}\right|\\ &\qquad+\int_a^b K_l(s)ds(\mathbb{1}_n-Q_l)^{-1}\max_{t\in[a,b]}\left|z^0-z^1+\frac{t-a}{b-a}D^{-1}\left[A+D,B,N_q\right]\begin{pmatrix}z^0-z^1\\ \lambda^0-\lambda^1\\ \eta^0-\eta^1\end{pmatrix}\right|,\end{aligned} \tag{5.38}$$

which, in view of (5.32), coincides with (5.31). □

Properties stated by Propositions 5.3 and 5.4 can be used when analysing conditions guaranteeing the solvability of the determining equations.

6. On the Numerical-Analytic Algorithm of Solving the Problem

Theorems 4.2 and 5.2 allow one to formulate the following numerical-analytic algorithm for the construction of a solution of the three-point boundary value problem (1.9), (1.10).

(1) For any vector $z \in \mathbb{R}^n$, according to (4.1), we analytically construct the sequence of functions $x_m(\cdot, z, \lambda, \eta)$ depending on the parameters z, λ, η and satisfying the auxiliary two-point boundary condition (2.3).

(2) We find the limit $x_\infty(\cdot, z, \lambda, \eta)$ of the sequence $x_m(\cdot, z, \lambda, \eta)$ satisfying (2.3).

(3) We construct the algebraic determining system (5.17), (5.18), and (5.19) with respect $3n - q$ scalar variables.

(4) Using a suitable numerical method, we (approximately) find a root

$$z_* \in \mathbb{R}^n, \qquad \lambda_* \in \mathbb{R}^n, \qquad \eta_* \in \mathbb{R}^{n-q} \tag{6.1}$$

of the determining system (5.17), (5.18), and (5.19).

(5) Substituting values (6.1) into $x_\infty(\cdot, z, \lambda, \eta)$, we obtain a solution of the original three-point boundary value problem (1.9), (1.10) in the form

$$x(t) = x_\infty(t, z_*, \lambda_*, \eta_*), \quad t \in [a, b]. \tag{6.2}$$

This solution (6.2) can also be obtained by solving the Cauchy problem

$$x(a) = z_* \tag{6.3}$$

for (1.9).

The fundamental difficulty in the realization of this approach arises at point (2) and is related to the analytic construction of the function $x_\infty(\cdot, z, \lambda, \eta)$. This problem can often be overcome by considering certain approximations of form (4.1), which, unlike the function $x_\infty(\cdot, z, \lambda, \eta)$, are known in the analytic form. In practice, this means that we fix a suitable $m \geq 1$, construct the corresponding function $x_m(\cdot, z, \lambda, \eta)$ according to relation (4.1), and define the function $\Delta_m : \mathbb{R}^{3n-q} \to \mathbb{R}^n$ by putting

$$\Delta_m(z, \lambda, \eta) := D^{-1}(d - B\lambda + N_q\eta) - \left(D^{-1}A + \mathbb{1}_n\right)z - \int_a^b \left[(lx_m(\cdot, z, \lambda, \eta)(s) + f(s))\right] ds, \tag{6.4}$$

for arbitrary z, λ, and η. To investigate the solvability of the three-point boundary value problem (1.9), (1.10), along with the determining system (5.17), (5.18), and (5.19), one considers the mth approximate determining system

$$\begin{gathered} \Delta_m(z, \lambda, \eta) = 0, \\ e_1 x_m(\xi, z, \lambda, \eta) = \lambda_1, e_2 x_m(\xi, z, \lambda, \eta) = \lambda_2, \ldots, e_n x_m(\xi, z, \lambda, \eta) = \lambda_n, \\ e_{q+1} x_m(b, z, \lambda, \eta) = \eta_1, \ldots, e_n x_m(b, z, \lambda, \eta) = \eta_{n-q}, \end{gathered} \tag{6.5}$$

where e_i, $i = 1, 2, \ldots, n$, are the vectors given by (5.15).

It is natural to expect (and, in fact, can be proved) that, under suitable conditions, the systems (5.17), (5.18), (5.19), and (6.5) are "close enough" to one another for m sufficiently large. Based on this circumstance, existence theorems for the three-point boundary value problem (1.9), (1.10) can be obtained by studying the solvability of the approximate determining system (6.5) (in the case of periodic boundary conditions, see, e.g., [35]).

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