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Research Article

Resistance Functions for Two Spheres in Axisymmetric Flow—Part I: Stream Function Theory

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We consider low-Reynolds-number axisymmetric flow about two spheres using a novel, biharmonic stream function. This enables us to calculate analytically not only the forces, but also the dipole moments (stresslets and pressure moments) and the associated resistance functions. In this paper the basics properties of axisymmetric flow and the stream function are discussed. Explicit series expansions, obtained by separation in bispherical coordinates, will be presented in a follow-up paper.

1. Introduction

The grand resistance and mobility tensors describe the hydrodynamic interaction between rigid bodies suspended in a fluid medium and play an all-important role in colloidal science [1–7]. More specifically, they express the linear relationship between the Cartesian force multipole moments exerted by the particles on the fluid and the gradients of the ambient flow velocity taken at the particle centers. Both tensors depend, in general, on the positions and orientations of all the suspended particles. However, in the special case of just two spherical bodies, owing to O(2)-symmetry about the line connecting the particle centers, the full tensors can be reduced to a set of scalar resistance and mobility functions [8–15]. Accurate knowledge of the two-body resistance functions is essential to overcome certain contact singularities—also referred to as lubrication singularities—that dominate the many-body tensors when a pair of particles comes close to touching [16, 17].

The calculation of the 2-body resistance functions is based on the solution of the Stokes boundary value problem for stationary, low-Reynolds number flow about two spheres. In this case there exists a set of curvilinear coordinates that is adapted to the physical boundaries at hand—the bispherical coordinates—and it seems natural to try to solve the problem analytically by separation in these coordinates [18–21]. However, separation of the Stokes equations in bispherical coordinates succeeds only up to a modulation factor [22], which complicates matching of the general solution to the boundary data. In most cases, it turns out that the matching problem can be reduced to a linear recursion problem that involves an infinite tridiagonal matrix. Although this cannot, in general, be solved analytically, the numerical treatment of such a tridiagonal recursion scheme [23–28] is much more efficient than recursions based on the twin-multipole expansion [9, 12–14].

Further progress can be made if the flow is axisymmetric, which happens when the spheres translate along or rotate around the line connecting their centers and when the ambient flow is axially symmetric about that line. A subset of resistance functions, including those with the strongest contact singularities, is determined by this kind of flow problems [13, 15]. It follows from the Stokes equations that for axisymmetric flow, the azimuthal velocity component is decoupled from the remaining flow variables, namely, the pressure and the radial and axial velocity components. Accordingly, there are two types of axisymmetric flow problems, problem I involving only the azimuthal velocity and problem II involving the other flow variables. Problem I can be reduced to a harmonic equation for the azimuthal velocity with Dirichlet boundary conditions. This was solved by separation in bispherical coordinates almost hundred years ago by Jeffery [29], who calculated, in particular, the torques acting on two spheres that rotate with given angular velocities around the line of centers.

Problem II is more difficult, since it involves coupled equations among three variables. However, owing to the incompressibility, the two nonvanishing velocity components, that is, radial and axial velocities, can be expressed as derivatives of a scalar potential, the so-called stream function, whose introduction can be traced back to Stokes [30]. This stream function satisfies a fourth-order differential equation that can again be separated in bispherical coordinates. In this way, Stimson and Jeffrey calculated the forces on two spheres moving with equal velocities along the line of centers [31]. Many years later, Brenner [32] and Maude [33] solved the same problem for spheres moving with opposite velocities along the line of centers. The latter case, where two spheres approach each other, is the one with the strongest contact singularity. It is possible to extract the asymptotic behavior of the forces at close proximity from the series expansion in bispherical coordinates [34, 35], and the results agree with those obtained from a singular perturbation theory in stretched cylindrical coordinates [36, 37]. Singular perturbation theory, valid for two spheres near touching, has also been used to calculate the forces and their dipole moments in the presence of an external, axisymmetric rate-of-strain flow [14, 38, 39]. Moreover, the twin-multipole expansions, valid for large distances, are well-known for these cases [12–14]. Yet, the corresponding series expansions in bispherical coordinates, valid for all distances, have not been published so far.

In this and a follow-up paper [40], we generalize and complete previous studies of axisymmetric two-sphere-flow problems in several ways. First, we suggest a slight conceptual improvement by considering from the beginning an inhomogeneous external flow, as generated, for example, by means of a localized external volume-force density. This allows us to identify easily the contributions from each sphere without being obliged to assume artificial deformations, as in [13, 38, 41]. Second, and more important, we introduce a novel stream function that differs from the classical one [30–33, 37] but has the advantage of being a biharmonic function. Its series expansion in bispherical coordinates matches the corresponding expansions of the flow fields better than the old stream function. In particular, the biharmonic stream function is expressed in associated Legendre functions instead of Gegenbauer polynomials. Third, with the new stream function, we are able to calculate analytically not only the forces, but also their dipole moments (stresslets and pressure moments).

And we will show that this can be accomplished without full knowledge of the pressure (which would require solving a tridiagonal recursion scheme). This signifies some progress, as it was claimed earlier in the literature [38] that the stresslets cannot be calculated on the basis of the stream function alone. In this paper, we present the theory of the biharmonic stream function for axisymmetric flow, while in the follow-up paper, we derive the series expansions for the forces and dipole moments in bispherical coordinates.

2. Stokes Equations

We consider two spheres immersed in an unbounded, simple fluid with shear viscosity η. The spheres—in the following labeled by Greek indices $\nu, \mu \in \{1, 2\}$—have radii a_ν and are centered on the 3-axis of a Cartesian frame at positions $\mathbf{X}_\nu = Z_\nu\, \mathbf{e}_3$. To exclude overlap, we assume that $|Z_1 - Z_2| > a_1 + a_2$. The dynamics of the fluid is described in terms of the local deviation from the thermal pressure, $p(\mathbf{x})$, and the flow velocity $\mathbf{u}(\mathbf{x})$ which obey the linear Stokes equations for stationary, low-Reynolds number flow with stick boundary conditions [8, 42]

$$
\begin{gathered}
\left.\begin{aligned}
\nabla \cdot \mathbf{u} &= 0 \\
\nabla p - \eta \Delta \mathbf{u} &= \mathbf{f}^{\text{ext}}
\end{aligned}\right\} \quad (\mathbf{x} \in \mathrm{B}_0), \\
(p, \mathbf{u}) \longrightarrow 0 \qquad (\|\mathbf{x}\| \longrightarrow \infty), \\
\mathbf{u} = \mathbf{U}_\nu + \boldsymbol{\Omega}_\nu \times (\mathbf{x} - \mathbf{X}_\nu) \qquad (\mathbf{x} \in \partial \mathrm{B}_\nu, \nu = 1, 2).
\end{gathered}
\tag{2.1}
$$

Here, B_0 denotes the fluid region and $\partial \mathrm{B}_\nu$ the surface of particle ν. Moreover, $\mathbf{f}^{\text{ext}}$ is an external force density acting on the fluid, while $\mathbf{U}_\nu$ and $\boldsymbol{\Omega}_\nu$ are the translational and rotational velocities of particle ν, respectively. We assume $\mathbf{f}^{\text{ext}}$ to have a compact support that is not overlapping with the particles. Equations (2.1) then pose a Dirichlet boundary value problem for $(p, \mathbf{u})$ whose solution—for fixed geometry and given sources $(\mathbf{f}^{\text{ext}}, \mathbf{U}, \boldsymbol{\Omega})$—is uniquely determined [8, 43].

We are interested here in the forces and dipole moments exerted by the particles on the fluid. These are defined by (we use Latin indices $i, j, \ldots \in \{1, 2, 3\}$ to label Cartesian components and the summation convention over repeated upper and lower Latin indices)

$$
\left.\begin{aligned}
\mathcal{F}_i^\mu &= \int_{\partial \mathrm{B}_\mu} dS F_i^\mu(\mathbf{x}), \\
\mathcal{F}_{ij}^\mu &= a_\mu \int_{\partial \mathrm{B}_\mu} dS F_i^\mu(\mathbf{x}) N_j(\mathbf{x}),
\end{aligned}\right\} \quad (\mu = 1, 2),
\tag{2.2}
$$

where the surface-force density [8, 42]

$$
\mathbf{F}^\mu = \mathbf{N} \cdot \left[p \mathbf{1} - 2\eta (\nabla \mathbf{u}) \right]_{\partial \mathrm{B}_\mu^+} \quad (\mathbf{x} \in \partial \mathrm{B}_\mu)
\tag{2.3}
$$

is determined by the solution of (2.1). Here, $\mathbf{N}$ is the normal field on $\partial \mathrm{B}_\mu$ (directed outwards), $(\nabla \mathbf{u})$ is the symmetrized velocity gradient, and the subscript $\partial \mathrm{B}_\mu^+$ indicates analytic continuation from the fluid regime onto the particle surface. It is useful to split the dipole moments into a trace, a skew-symmetric, and a symmetric-traceless part, according to

$$\mathcal{F}^{\mu}_{ij} = Q^{\mu}\delta_{ij} - \frac{1}{2}\varepsilon_{ij}{}^{m}\mathcal{T}^{\mu}_{m} + \mathcal{S}^{\mu}_{ij}, \tag{2.4}$$

where (for an arbitrary second-rank tensor with components B_{ij}, we use the notations $B_{(ij)} = (1/2)(B_{ij} + B_{ji})$, $B_{[ij]} = (1/2)(B_{ij} - B_{ji})$, $B_{\langle ij\rangle} = B_{(ij)} - (1/3)\delta^{kl}B_{kl}\,\delta_{ij}$)

$$Q^{\mu} = \frac{1}{3}\delta^{mn}\mathcal{F}^{\mu}_{mn} \qquad \mathcal{T}^{\mu}_{i} = -\varepsilon_{i}{}^{mn}\mathcal{F}^{\mu}_{[mn]} \qquad \mathcal{S}^{\mu}_{ij} = \mathcal{F}^{\mu}_{\langle ij\rangle} \tag{2.5}$$

denote the pressure moment [14], the torque, and the (deviatoric) stresslet [11, 44], respectively.

We decompose the flow $(p, \mathbf{u})$ in the form

$$\begin{aligned} p(\mathbf{x}) &= p_{\text{ext}}(\mathbf{x}) + q(\mathbf{x}), \\ \mathbf{u}(\mathbf{x}) &= \mathbf{u}_{\text{ext}}(\mathbf{x}) + \mathbf{v}(\mathbf{x}), \end{aligned} \quad (\mathbf{x} \in \mathrm{B}_0), \tag{2.6}$$

where $(p_{\text{ext}}, \mathbf{u}_{\text{ext}})$ is the flow caused by $\mathbf{f}^{\text{ext}}$ in the absence of the particles and called the external flow. The remainder $(q, \mathbf{v})$ will be referred to as the excess flow. It satisfies the Dirichlet problem

$$\begin{gathered} \left.\begin{aligned} \nabla\cdot\mathbf{v} &= 0, \\ \nabla q - \eta\Delta\mathbf{v} &= 0, \end{aligned}\right\} \quad (\mathbf{x} \in \mathrm{B}_0), \\ (q, \mathbf{v}) \longrightarrow 0, \quad (\|\mathbf{x}\| \longrightarrow \infty), \\ \mathbf{v} = \mathbf{u}_{\nu}, \quad (\mathbf{x} \in \partial\mathrm{B}_{\nu}, \nu = 1, 2), \end{gathered} \tag{2.7}$$

with applied velocities

$$\mathbf{u}_{\nu}(\mathbf{x}) = \mathbf{U}_{\nu} + \mathbf{\Omega}_{\nu} \times (\mathbf{x} - \mathbf{X}_{\nu}) - \mathbf{u}_{\text{ext}}(\mathbf{x}), \quad (\mathbf{x} \in \partial\mathrm{B}_{\nu}, \nu = 1, 2). \tag{2.8}$$

Inserting (2.6) in (2.3), one obtains a decomposition of the surface-force densities of the form

$$\begin{aligned} \mathbf{F}^{\mu} &= \mathbf{F}^{\mu}_{\text{ext}} + \mathbf{G}^{\mu}, \\ \mathbf{G}^{\mu} &= \mathbf{N}\cdot\left[q\mathbf{1} - 2\eta(\nabla\mathbf{v})\right]_{\partial\mathrm{B}^{+}_{\mu}}, \end{aligned} \tag{2.9}$$

where $\mathbf{F}^{\mu}_{\text{ext}}$ is defined as in (2.3), with $(p, \mathbf{u})$ replaced by $(p_{\text{ext}}, \mathbf{u}_{\text{ext}})$. Notice that the external flow is well defined inside the particles, where it obeys the homogeneous Stokes equations (since the support of $\mathbf{f}^{\text{ext}}$ lies outside the particles). The contribution from $\mathbf{F}^{\mu}_{\text{ext}}$ to the force moments (2.2) can therefore be evaluated by converting the surface integrals into volume integrals over the interiors $\mathtt{B}_\mu$ of the particles. Expanding then $(p_{\text{ext}}, \mathbf{u}_{\text{ext}})$ about the particle centers, $\mathbf{X}_\mu$, and using isotropy, it is easy to calculate these integrals exactly.

Assuming that the external flow varies slowly on the length scale of the particle diameters, we expand the applied velocities (2.8) up to first order about the particle centers. This yields

$$u_\nu^k(\mathbf{x}) = \mathcal{U}_\nu^k + a_\nu\left[-\varepsilon^{kl}{}_n\mathcal{W}_\nu^n + \mathcal{E}^{kl}_\nu\right]N_l(\mathbf{x}) + \cdots \quad (\mathbf{x} \in \partial\mathtt{B}_\nu, \nu = 1, 2), \tag{2.10}$$

where the expansion coefficients

$$\mathcal{U}^k_{\ \nu} = U_\nu^k - u_{\text{ext}}^k\Big|_{\mathbf{X}_\nu} \qquad \mathcal{W}^k_{\ \nu} = \Omega_\nu^k - \frac{1}{2}\varepsilon^k{}_{mn}\partial^{[m}u_{\text{ext}}^{n]}\Big|_{\mathbf{X}_\nu} \qquad \mathcal{E}_\nu^{kl} = -\partial^{(l}u_{\text{ext}}^{k)}\Big|_{\mathbf{X}_\nu} \tag{2.11}$$

are referred to as the (effective) translational and rotational velocities and the local rate of strain, respectively. We also put

$$\mathcal{P}_\nu = -p_{\text{ext}}\big|_{\mathbf{X}_\nu}. \tag{2.12}$$

It follows from incompressibility (cf. the first equation in (2.1)) that the local rates of strain are traceless, $\delta_{mn}\ \mathcal{E}_\nu^{mn} = 0$. Notice that by admitting inhomogeneous external flows, we can avoid conceptual problems of earlier approaches which were based on a uniform shear flow and which, therefore, either could not always resolve the individual contributions of the spheres [11], or, in order to be able to do so, had to assume artificial deformations [13, 38, 41].

Upon inserting (2.9) in (2.2) and (2.5), and evaluating the contributions from $\mathbf{F}^\mu_{\text{ext}}$ in the manner outlined above, one obtains

$$\begin{aligned} \mathcal{F}^\mu_{\ i} &= \mathcal{G}^\mu_{\ i}, \qquad & Q^\mu &= -\frac{4}{3}\pi a_\mu^3\mathcal{P}_\mu + \frac{1}{3}\delta^{mn}\mathcal{G}^\mu_{\ mn}, \\ \mathcal{T}^\mu_{\ i} &= -\varepsilon_i{}^{mn}\mathcal{G}^\mu_{\ [mn]}, \qquad & \mathcal{S}^\mu_{\ ij} &= \frac{8}{3}\pi\eta a_\mu^3\mathcal{E}^{ij}_{\ \mu} + \mathcal{G}^\mu_{\ \langle ij\rangle}, \end{aligned} \tag{2.13}$$

where $\mathcal{G}^\mu_{\ i}$ and $\mathcal{G}^\mu_{\ ij}$ are defined as in (2.2), with $F^\mu_{\ i}(\mathbf{x})$ replaced by $G^\mu_{\ i}(\mathbf{x})$. On the right-hand side of $\mathcal{S}^\mu_{\ ij}$, we have neglected—consistent with the expansion (2.10)—a second-order term proportional to $a_\mu^2\Delta\mathcal{E}^{ij}_{\ \mu} \ll \mathcal{E}^{ij}_{\ \mu}$.

3. Axisymmetric Flow

Axisymmetric scalar- and vector-valued fields are of the generic form

$$\begin{aligned} q(\mathbf{x}) &= q(s,z), \\ \mathbf{v}(\mathbf{x}) &= v^s(s,z)\mathbf{e}_s(\varphi) + v^\varphi(s,z)\mathbf{e}_\varphi(\varphi) + v^z(s,z)\mathbf{e}_z, \end{aligned} \tag{3.1}$$

where (s, φ, z) denote cylindrical coordinates about the 3-axis and $(\mathbf{e}_s(\varphi), \mathbf{e}_\varphi(\varphi), \mathbf{e}_z)$ the associated unit vectors along the coordinate curves. Explicitly,

$$\mathbf{x}(s, \varphi, z) = \begin{pmatrix} s\cos\varphi \\ s\sin\varphi \\ z \end{pmatrix} \quad \left(s \geq 0, \varphi \in \frac{\mathbb{R}}{2\pi}, z \in \mathbb{R}\right),$$
$$\mathbf{e}_s(\varphi) = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}, \qquad \mathbf{e}_\varphi(\varphi) = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}, \qquad \mathbf{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{3.2}$$

Notice in (3.1) that the scalar and the vector components are independent of the azimuthal angle φ. Since the Stokes operator $(\mathbf{u}_1, \mathbf{u}_2) \mapsto (q, \mathbf{v})$, defined by the Dirchlet problem (2.7), is invariant under rotations about the 3-axis, it follows [15] that the flow $(q, \mathbf{v})$ is axisymmetric if and only if the applied velocities (2.8) are axisymmetric. Expressing (2.10) in cylindrical coordinates one finds that this is the case if and only if the local velocity gradients (2.11) assume the special form

$$\left(\mathcal{U}_\nu^j\right) = \mathcal{U}_\nu \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \left(\mathcal{W}_\nu^j\right) = \mathcal{W}_\nu \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \left(\mathcal{E}_\nu^{jl}\right) = \mathcal{E}_\nu \frac{1}{\sqrt{6}} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \tag{3.3}$$

where we have used that $\operatorname{tr} \boldsymbol{\mathcal{E}}_\nu = 0$. Inserting (3.3) in (2.10), one obtains

$$\mathbf{u}_\nu(\mathbf{x}) = -\sqrt{\frac{1}{6}} s\mathcal{E}_\nu \mathbf{e}_s(\varphi) + s\mathcal{W}_\nu \mathbf{e}_\varphi(\varphi) + \left[\mathcal{U}_\nu + \sqrt{\frac{2}{3}}(z - Z_\nu)\mathcal{E}_\nu\right]\mathbf{e}_z \quad \left((s, z) \in \partial\widetilde{\mathrm{B}}_\nu\right), \tag{3.4}$$

where $(s, z) \in \partial\widetilde{\mathrm{B}}_\nu$ means $\mathbf{x}(s, \varphi, z) \in \partial\mathrm{B}_\nu$ for some (and thus all) $\varphi \in [0, 2\pi)$. Likewise, we will use the notation $(s, z) \in \widetilde{\mathrm{B}}_0$.

Upon inserting (3.1) and (3.4) into (2.7), one obtains two Dirichlet problems: one for v^φ and one for (q, v^s, v^z). For brevity, we refer to these as problem I and problem II, respectively. Introducing the Laplace operators

$$\Delta_m = \frac{1}{s}\partial_s s \partial_s - \frac{m^2}{s^2} + \partial_z\partial_z \quad (m = 0, 1), \tag{3.5}$$

problem I reads

$$\Delta_1 v^\varphi = 0, \quad \left((s, z) \in \widetilde{\mathrm{B}}_0\right),$$
$$v^\varphi \longrightarrow 0, \quad \left(\sqrt{s^2 + z^2} \longrightarrow \infty\right), \tag{3.6}$$
$$v^\varphi = s\mathcal{W}_\nu, \quad \left((s, z) \in \partial\widetilde{\mathrm{B}}_\nu, \nu = 1, 2\right),$$

while problem II reads

$$\frac{1}{s}\partial_s s v^s + \partial_z v^z = 0, \quad \partial_s q - \eta\Delta_1 v^s = 0, \quad \partial_z q - \eta\Delta_0 v^z = 0, \quad \left((s,z)\in\widetilde{\mathrm{B}}_0\right),$$
$$(q, v^s, v^z) \longrightarrow 0, \quad \left(\sqrt{s^2+z^2}\longrightarrow\infty\right), \tag{3.7}$$
$$v^s = -\sqrt{\frac{1}{6}}s\mathcal{E}_\nu, \quad v^z = \mathcal{U}_\nu + \sqrt{\frac{2}{3}}(z - Z_\nu)\mathcal{E}_\nu, \quad \left((s,z)\in\partial\widetilde{\mathrm{B}}_\nu, \nu = 1,2\right).$$

These two problems are completely decoupled, since (3.6) is determined by the $\mathcal{W}_\nu$, while (3.7) is determined by the $\mathcal{U}_\nu$ and $\mathcal{E}_\nu$. Also, from the general uniqueness theorem [8, 43], it follows that both problems have at most one solution.

In the following, we parameterize the surfaces $\partial\mathrm{B}_\nu$ in terms of the azimuthal angle φ and a polar angle ϑ according to

$$\partial\mathrm{B}_\nu = \left\{ \mathbf{x} = \begin{pmatrix} s_\nu(\vartheta)\cos\varphi \\ s_\nu(\vartheta)\sin\varphi \\ z_\nu(\vartheta) \end{pmatrix} \mid \vartheta\in[0,\pi], \varphi\in[0,2\pi) \right\} \quad (\nu = 1,2), \tag{3.8}$$

where the mapping $\vartheta \mapsto (s_\nu(\vartheta), z_\nu(\vartheta)) \in \partial\widetilde{\mathrm{B}}_\nu$ parameterizes the solutions of the surface constraint

$$s^2 + (z - Z_\nu)^2 = a_\nu^2. \tag{3.9}$$

This mapping is assumed to be smooth and nonsingular, with $\dot{z}_\nu(\vartheta) \neq 0$ for all $\vartheta \in (0,\pi)$. Then, $\operatorname{sgn}\dot{z}_\nu = \pm 1$ denotes the orientation of the parameterization, and $dS = a\,|\dot{z}_\nu(\vartheta)|\,d\vartheta\,d\varphi$ is the surface element. The parameterization will be made explicit in the following paper [40], where we introduce bispherical coordinates; for the present purpose, it is sufficient stay general.

Since the mapping $(q,\mathbf{v}) \mapsto (\mathbf{G}^1, \mathbf{G}^2)$, defined by the constitutive equation (2.9), is invariant under rotations about the 3-axis, it follows [15] that the surface-force densities $\mathbf{G}^\mu(\mathbf{x})$ caused by axisymmetric flow $(q,\mathbf{v})$ are again axisymmetric

$$\mathbf{G}^\mu(\mathbf{x}) = G_s^\mu(\vartheta)\mathbf{e}_s(\varphi) + G_\varphi^\mu(\vartheta)\mathbf{e}_\varphi(\varphi) + G_z^\mu(\vartheta)\mathbf{e}_z. \tag{3.10}$$

Calculating the associated moments, defined as in (2.2), one can easily perform the azimuthal integrations to obtain the generic form

$$\left(\mathcal{G}_i^\mu\right) = \begin{pmatrix} 0 \\ 0 \\ \mathcal{G}_3^\mu \end{pmatrix} \qquad \left(\mathcal{G}_{ij}^\mu\right) = \begin{pmatrix} \mathcal{G}_{11}^\mu & -\mathcal{G}_{21}^\mu & 0 \\ \mathcal{G}_{21}^\mu & \mathcal{G}_{11}^\mu & 0 \\ 0 & 0 & \mathcal{G}_{33}^\mu \end{pmatrix}, \tag{3.11}$$

where the nonvanishing elements involve polar integrations over the components of (3.10). Inserting these in (2.13), one finds

$$\left(\mathcal{F}^{\mu}_{i}\right) = \mathcal{F}^{\mu}\begin{pmatrix}0\\0\\1\end{pmatrix}, \qquad \mathcal{Q}^{\mu} = -\frac{4}{3}\pi a_{\mu}^{3}\mathcal{P}_{\mu} + \Delta\mathcal{Q}^{\mu},$$

$$\left(\mathcal{S}^{\mu}_{ij}\right) = \mathcal{S}^{\mu}\frac{1}{\sqrt{6}}\begin{pmatrix}-1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 2\end{pmatrix}, \tag{3.12}$$

$$\left(\mathcal{T}^{\mu}_{i}\right) = \mathcal{T}^{\mu}\begin{pmatrix}0\\0\\1\end{pmatrix}, \qquad \mathcal{S}^{\mu} = \frac{8}{3}\pi\eta a_{\mu}^{3}\mathcal{E}_{\mu} + \Delta\mathcal{S}^{\mu},$$

with

$$\begin{aligned}\mathcal{F}^{\mu} &= \mathcal{G}^{\mu}_{3} = 2\pi a_{\mu}(\operatorname{sgn}\dot{z}_{\mu})\int_{0}^{\pi} d\vartheta\dot{z}_{\mu}G^{\mu}_{z},\\
\mathcal{T}^{\mu} &= 2\mathcal{G}^{\mu}_{21} = 2\pi a_{\mu}(\operatorname{sgn}\dot{z}_{\mu})\int_{0}^{\pi} d\vartheta\dot{z}_{\mu}s_{\mu}G^{\mu}_{\varphi},\\
\Delta\mathcal{Q}^{\mu} &= \frac{1}{3}\left(2\mathcal{G}^{\mu}_{11} + \mathcal{G}^{\mu}_{33}\right) = \frac{2}{3}\pi a_{\mu}(\operatorname{sgn}\dot{z}_{\mu})\int_{0}^{\pi} d\vartheta\dot{z}_{\mu}\left[(z_{\mu} - Z_{\mu})G^{\mu}_{z} + s_{\mu}G^{\mu}_{s}\right],\\
\Delta\mathcal{S}^{\mu} &= \sqrt{\frac{2}{3}}\left(\mathcal{G}^{\mu}_{33} - \mathcal{G}^{\mu}_{11}\right) = \sqrt{\frac{8}{3}}\pi a_{\mu}(\operatorname{sgn}\dot{z}_{\mu})\int_{0}^{\pi} d\vartheta\dot{z}_{\mu}\left[(z_{\mu} - Z_{\mu})G^{\mu}_{z} - \frac{1}{2}s_{\mu}G^{\mu}_{s}\right].\end{aligned} \tag{3.13}$$

To determine the components of (3.10), we insert (3.1) in (2.9) and use the incompressibility condition. This yields

$$\begin{aligned}G^{\mu}_{s}(\vartheta) &= \frac{1}{a_{\mu}}sq(s,z) - \frac{\eta}{a_{\mu}}\left[(\partial_{\mathrm{N}} - 1)v^{s}(s,z) + \partial_{\mathrm{T}}v^{z}(s,z)\right],\\
G^{\mu}_{\varphi}(\vartheta) &= -\frac{\eta}{a_{\mu}}(\partial_{\mathrm{N}} - 1)v^{\varphi}(s,z),\\
G^{\mu}_{z}(\vartheta) &= \frac{1}{a_{\mu}}(z - Z_{\mu})q(s,z) + \frac{\eta}{a_{\mu}}\left[\left(\partial_{\mathrm{T}} + \frac{z - Z_{\mu}}{s}\right)v^{s}(s,z) - \partial_{\mathrm{N}}v^{z}(s,z)\right],\end{aligned} \tag{3.14}$$

where

$$\begin{aligned}\partial_{\mathrm{N}} &= s\partial_{s} + (z - Z_{\mu})\partial_{z},\\
\partial_{\mathrm{T}} &= (z - Z_{\mu})\partial_{s} - s\partial_{z}\end{aligned} \tag{3.15}$$

denote the derivatives in the normal and tangential directions, respectively, and (s, z) is to be put equal to $(s_\mu(\vartheta), z_\mu(\vartheta))$ after the differentiations have been carried out. The expressions (3.14) must now be inserted in (3.13). To simplify the integrals we use the boundary conditions on ∂B_μ, as specified in (3.6) and (3.7), and the relations

$$\frac{d}{d\vartheta} = \dot{s}\partial_s + \dot{z}\partial_z \qquad s\dot{s} + (z - Z_\mu)\dot{z} = 0 \qquad \dot{z}\partial_{\mathrm{T}} = -s\frac{d}{d\vartheta}, \tag{3.16}$$

which follow from (3.9) and (3.15). Employing the notation

$$q_\mu(\vartheta) = q(s,z)\big|_{(s,z)=(s_\mu(\vartheta), z_\mu(\vartheta))} \qquad (\partial_{\mathrm{N}} v^z)_\mu(\vartheta) = \partial_{\mathrm{N}} v^z(s,z)\big|_{(s,z)=(s_\mu(\vartheta), z_\mu(\vartheta))}, \tag{3.17}$$

and so forth, and using that

$$(s_\mu(0), z_\mu(0)) = (0, Z_\mu - a\,\mathrm{sgn}\,\dot{z}_\mu) \qquad (s_\mu(\pi), z_\mu(\pi)) = (0, Z_\mu + a\,\mathrm{sgn}\,\dot{z}_\mu), \tag{3.18}$$

we finally obtain

$$\begin{aligned}
\mathcal{F}^\mu &= 2\pi\eta(\mathrm{sgn}\,\dot{z}_\mu)\int_0^\pi d\vartheta\left[\frac{1}{2\eta}s_\mu^2\dot{q}_\mu - \dot{z}_\mu(\partial_{\mathrm{N}}v^z)_\mu\right],\\
\mathcal{T}^\mu &= \frac{8}{3}\pi\eta a_\mu^3\mathcal{W}_\mu - 2\pi\eta(\mathrm{sgn}\,\dot{z}_\mu)\int_0^\pi d\vartheta\dot{z}_\mu s_\mu(\partial_{\mathrm{N}}v^\varphi)_\mu,\\
\Delta Q^\mu &= \frac{2}{3}\pi a_\mu^3\left[q_\mu(0) + q_\mu(\pi)\right]\\
&\quad - \frac{2}{3}\pi\eta(\mathrm{sgn}\,\dot{z}_\mu)\int_0^\pi d\vartheta\left\{\frac{a_\mu^2}{\eta}(z_\mu - Z_\mu)\dot{q}_\mu + \dot{z}_\mu\left[(z_\mu - Z_\mu)(\partial_{\mathrm{N}}v^z)_\mu + s_\mu(\partial_{\mathrm{N}}v^s)_\mu\right]\right\},\\
\Delta\mathcal{S}^\mu &= -\frac{4}{3}\pi\eta a_\mu^3\mathcal{E}_\mu + \sqrt{\frac{8}{3}}\pi\eta(\mathrm{sgn}\,\dot{z}_\mu)\\
&\quad\times\int_0^\pi d\vartheta\left\{\frac{1}{2\eta}(z_\mu - Z_\mu)s_\mu^2\dot{q}_\mu - \dot{z}_\mu\left[(z_\mu - Z_\mu)(\partial_{\mathrm{N}}v^z)_\mu - \frac{1}{2}s_\mu(\partial_{\mathrm{N}}v^s)_\mu\right]\right\}.
\end{aligned}\tag{3.19}$$

Notice from (3.7) and (3.16) that

$$\frac{1}{\eta}\dot{q}_\mu = \dot{s}_\mu(\Delta_1 v^s)_\mu + \dot{z}_\mu(\Delta_0 v^z)_\mu. \tag{3.20}$$

From (3.19), it follows immediately that for axisymmetric flow, the $\mathcal{T}^\mu$ are determined via v^φ by the $\mathcal{W}_\nu$ (problem I, (3.6)), while the $(\mathcal{F}^\mu, \mathcal{S}^\mu, \Delta Q^\mu)$ are determined via (q, v^s, v^z) by the $(\mathcal{U}_\nu, \mathcal{E}_\nu)$ (problem II, (3.7)). It is convenient to quantify the resulting linear relationships between the force moments and the applied velocity gradients in the scaled form [15]

$$\widetilde{\mathcal{T}}^\mu = \sum_{\nu=1,2} \sigma_{TW}^{\mu\nu} \widetilde{\mathcal{W}}_\nu,$$
$$\begin{pmatrix} \widetilde{\mathcal{F}}^\mu \\ \widetilde{\mathcal{S}}^\mu \\ \Delta\widetilde{Q}^\mu \end{pmatrix} = \sum_{\nu=1,2} \begin{pmatrix} \sigma_{FU}^{\mu\nu} & \sigma_{FE}^{\mu\nu} \\ \sigma_{SU}^{\mu\nu} & \sigma_{SE}^{\mu\nu} \\ \sigma_{QU}^{\mu\nu} & \sigma_{QE}^{\mu\nu} \end{pmatrix} \begin{pmatrix} \widetilde{\mathcal{U}}_\nu \\ \widetilde{\mathcal{E}}_\nu \end{pmatrix} \quad (\mu = 1, 2), \tag{3.21}$$

where the $\sigma_{..}^{\mu\nu}$ are dimensionless resistance functions and

$$\begin{aligned} \widetilde{\mathcal{F}}^\mu &= (6\pi\eta a_\mu)^{-1/2} \mathcal{F}^\mu \qquad & \widetilde{\mathcal{U}}_\nu &= (6\pi\eta a_\nu)^{1/2} \mathcal{U}_\nu, \\ \widetilde{\mathcal{T}}^\mu &= \left(8\pi\eta a_\mu^3\right)^{-1/2} \mathcal{T}^\mu \qquad & \widetilde{\mathcal{W}}_\nu &= \left(8\pi\eta a_\nu^3\right)^{1/2} \mathcal{W}_\nu, \\ \widetilde{\mathcal{S}}^\mu &= \left(\frac{20}{3}\pi\eta a_\mu^3\right)^{-1/2} \mathcal{S}^\mu \qquad & \widetilde{\mathcal{E}}_\nu &= \left(\frac{20}{3}\pi\eta a_\nu^3\right)^{1/2} \mathcal{E}_\nu, \\ \Delta\widetilde{Q}^\mu &= \left(\frac{20}{3}\pi\eta a_\mu^3\right)^{-1/2} \Delta Q^\mu. \end{aligned} \tag{3.22}$$

The scaling (3.22) is based on the one-sphere results [13]

$$\begin{aligned} \mathcal{F}_i &= 6\pi\eta a \mathcal{U}_i, \\ \mathcal{T}_i &= 8\pi\eta a^3 \mathcal{W}_i, \\ \mathcal{F}_{ij} &= \frac{20}{3}\pi\eta a^3 \mathcal{E}_{ij}, \end{aligned} \tag{3.23}$$

and differs from earlier scalings [11–13]. An advantage of the present choice is that our resistance functions satisfy $\sigma_{FU}^{\mu\nu}, \sigma_{TW}^{\mu\nu}, \sigma_{SE}^{\mu\nu} \to \delta^{\mu\nu}$ as the distance between the particles goes to infinity. From Lorentz-reciprocity [8, 9], it follows that $\sigma_{SU}^{\mu\nu} = \sigma_{FE}^{\nu\mu}$.

4. Stream Function

Since problem I is already a scalar one and needs no further treatment at this point, we now focus on problem II and the calculation of $(\mathcal{F}^\mu, \mathcal{S}^\mu, \Delta Q^\mu)$. To solve (3.7), we make the ansatz

$$\begin{aligned} v^s &= \partial_z \Psi, \\ v^z &= -\frac{1}{s}\partial_s s \Psi, \end{aligned} \tag{4.1}$$

where $\Psi(s, z)$ is a scalar stream function. Obviously, (4.1) satisfies the condition of incompressibility, $(1/s)\partial_s s\, v^s + \partial_z v^z = 0$. Our stream function differs by a factor s from the classical stream function introduced by Stokes [30] and used hitherto [31–33, 36, 37] but has the advantage of being a biharmonic function. In fact, using the identity

$$\partial_s \frac{1}{s} \partial_s s = \frac{1}{s} \partial_s s \partial_s - \frac{1}{s^2}, \tag{4.2}$$

one obtains from (3.5), (3.7), and(4.1)

$$\begin{aligned} \partial_s q &= \eta \partial_z \Delta_1 \Psi, \\ \partial_z q &= -\eta \Delta_0 \frac{1}{s} \partial_s s \Psi = -\eta \frac{1}{s} \partial_s s \Delta_1 \Psi, \end{aligned} \tag{4.3}$$

which leads to

$$0 = \partial_z \partial_s q - \partial_s \partial_z q = \eta \left[\partial_z \partial_z + \partial_s \frac{1}{s} \partial_s s \right] \Delta_1 \Psi = \eta \Delta_1 \Delta_1 \Psi. \tag{4.4}$$

To satisfy the boundary condition $(q, v^s, v^z) \to 0$ as $\sqrt{s^2 + z^2} \to \infty$, it is sufficient to require that Ψ stay bounded at infinity.

Next, we turn to the boundary conditions on the particle surfaces. Since the applied velocities $\mathbf{u}_\nu$ satisfy the homogenous Stokes equations inside particles, they can be expressed in terms of stream functions, too. We denote these applied stream functions by Φ_ν, $(\nu = 1, 2)$, and determine them from the conditions

$$\begin{gathered} u_\nu^s = \partial_z \Phi_\nu = -\sqrt{\frac{1}{6}} s \boldsymbol{\mathcal{E}}_\nu, \\ u_\nu^z = -\frac{1}{s} \partial_s s \Phi_\nu = \mathcal{U}_\nu + \sqrt{\frac{2}{3}} (z - Z_\nu) \boldsymbol{\mathcal{E}}_\nu \quad \left((s, z) \in \widetilde{\mathrm{B}}_\nu,\ \nu = 1, 2 \right), \\ \Phi_\nu \ \text{regular} (s \longrightarrow 0), \end{gathered} \tag{4.5}$$

which follow from (3.4) and (4.1). The solutions are

$$\Phi_\nu(s, z) = -\frac{1}{2} s \mathcal{U}_\nu - \sqrt{\frac{1}{6}} s (z - Z_\nu) \boldsymbol{\mathcal{E}}_\nu \quad \left((s, z) \in \widetilde{\mathrm{B}}_\nu, \nu = 1, 2 \right). \tag{4.6}$$

It is easy to show that the boundary condition $\mathbf{v} = \mathbf{u}_\nu$ on $\partial \mathrm{B}_\nu$ is satisfied, if

$$\left.\begin{aligned} \Psi &= \Phi_\nu \\ \partial_{\mathrm{N}} \Psi &= \partial_{\mathrm{N}} \Phi_\nu \end{aligned}\right\} \quad (\mathbf{x} \in \partial \mathrm{B}_\nu). \tag{4.7}$$

The proof is based on the identities

$$\begin{aligned} &\frac{d}{d\vartheta}(\Psi - \Phi_\nu) = -\dot{s}(v^z - u_\nu^z) + \dot{z}(v^s - u_\nu^s) - \frac{\dot{s}}{s}(\Psi - \Phi_\nu) \\ &\partial_{\mathrm{N}}(\Psi - \Phi_\nu) = (z - Z_\nu)(v^s - u_\nu^s) - s(v^z - u_\nu^z) - (\Psi - \Phi_\nu) \end{aligned} \qquad (\vartheta \in (0,\pi)), \tag{4.8}$$

which follow from (3.15), (3.16), (4.1), and (4.5). Now, if (4.7) holds, these identities reduce to the homogeneous 2×2-system

$$\begin{pmatrix} \dot{z} & -\dot{s} \\ z - Z_\nu & -s \end{pmatrix}\begin{pmatrix} v^s - u_\nu^s \\ v^z - u_\nu^z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{4.9}$$

which has a nonzero determinant. Hence, (4.9) admits only the trivial solution, $\mathbf{v} - \mathbf{u}_\nu = 0$.

To summarize *the flow velocity* (v^s, v^z) *can be expressed by* (4.1) *in terms of a scalar stream function* $\Psi(s,z)$ *that satisfies the biharmonic two-body Dirichlet problem*

$$\begin{aligned} &\Delta_1 \Delta_1 \Psi = 0 \quad \Big((s,z) \in \widetilde{\mathrm{B}}_0\Big), \\ &\Psi \text{ bounded} \quad \Big(\sqrt{s^2 + z^2} \longrightarrow \infty\Big), \\ &\Psi = \Phi_\nu \quad \partial_{\mathrm{N}}\Psi = \partial_{\mathrm{N}}\Phi_\nu \quad \Big((s,z) \in \partial\widetilde{\mathrm{B}}_\nu, \nu = 1,2\Big), \end{aligned} \tag{4.10}$$

where the applied stream functions Φ_ν *are given by* (4.6).

The pressure at a given point $(s,z) \in \widetilde{\mathrm{B}}_0$ can, in principle, be calculated from the stream function by integration of (4.3) along any curve in $\widetilde{\mathrm{B}}_0$ that starts at infinity and ends in the point (s,z). However, for all but a few special points, these integrals cannot be performed analytically [40].

To establish the existence of a stream function as specified by (4.10), we consider two harmonic functions, φ_1 and φ_2, that vanish at infinity,

$$\begin{aligned} &\Delta_1 \varphi_i = 0 \\ &\varphi_i \to \mathcal{O}\left(\frac{1}{\sqrt{s^2 + z^2}}\right) \quad \sqrt{s^2 + z^2} \longrightarrow \infty \end{aligned} \qquad (i = 1,2). \tag{4.11}$$

It is known from potential theory that such functions exist in bispherical geometry, and this is, of course, explicitly confirmed by the solution of problem I. Following [31], we put

$$\Psi(s,z) = \varphi_1(s,z) + z\varphi_2(s,z), \tag{4.12}$$

and obtain

$$\Delta_1\Psi = 2\ \partial_z\psi_2 \qquad \Delta_1\Delta_1\Psi = 0,$$
$$\Psi \longrightarrow \mathcal{O}\left(\frac{z}{\sqrt{s^2+z^2}}\right) \quad \left(\sqrt{s^2+z^2} \longrightarrow \infty\right). \tag{4.13}$$

Thus Ψ, as given by (4.12), is biharmonic and stays bounded at infinity. Moreover, the ansatz (4.12), with two independent harmonic functions, is general enough to allow specification of Ψ and $\partial_N\Psi$ on the boundaries, as will be confirmed explicitly in [40].

Finally, we express the force moments $(\mathcal{F}^\mu, \mathcal{S}^\mu, \Delta Q^\mu)$ in the stream function by inserting (3.20) and (4.1) into (3.19). We quote the auxiliary formulae, valid on $\partial\widetilde{\mathrm{B}}_\mu$,

$$\frac{1}{\eta}\frac{dq}{d\vartheta} = -\dot{z}\frac{1}{s}(\partial_N + 1)\Delta_1\Psi \qquad \dot{z}\partial_N v^z = -\dot{z}s\Delta_1\Psi + \frac{d}{d\vartheta}sv^s,$$
$$\dot{z}(z - Z_\mu)\partial_N v^z = -\dot{z}(z - Z_\mu)s\Delta_1\Psi - \dot{z}sv^s + \frac{d}{d\vartheta}(\cdots), \tag{4.14}$$
$$\dot{z}s\partial_N v^s = \dot{z}(z - Z_\mu)s\Delta_1\Psi + \dot{z}sv^s + \frac{d}{d\vartheta}(\cdots),$$

where $(s, z) = (s_\mu(\vartheta), z_\mu(\vartheta))$ and the unquoted expressions $(\cdots)$ tend to zero for $s \to 0$ and, therefore, do not contribute to the integrals in (3.19). The final results are

$$\mathcal{F}^\mu = -\pi\eta(\operatorname{sgn}\dot{z}_\mu)\int_0^\pi d\vartheta\dot{z}_\mu s_\mu((\partial_N - 1)\Delta_1\Psi)_\mu,$$
$$\mathcal{S}^\mu = \sqrt{\frac{2}{3}}\pi\eta(\operatorname{sgn}\dot{z}_\mu)\int_0^\pi d\vartheta\dot{s}_\mu s_\mu^2((\partial_N - 2)\Delta_1\Psi)_\mu, \tag{4.15}$$
$$\Delta Q^\mu = \frac{2}{3}\pi a_\mu^3[q_\mu(0) + q_\mu(\pi)] - \frac{2\pi\eta}{3}a_\mu^2(\operatorname{sgn}\dot{z}_\mu)\int_0^\pi d\vartheta\dot{s}_\mu((\partial_N + 1)\Delta_1\Psi)_\mu.$$

With (4.15), we have succeeded to express the forces and stresslets in the stream function alone. The pressure moments require, in addition, the quantities $q_\mu(0)$ and $q_\mu(\pi)$, that is, the pressure at the poles of the spheres. As will be shown [40], the poles belong to the few points, where the pressure can be determined analytically in terms of the stream function.

The further evaluations in bispherical coordinates and the associated series expansions for the resistance functions will be presented in [40].

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