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Research Article

Convergence and Divergence of the Solutions of a Neutral Difference Equation

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We investigate the asymptotic behavior of the solutions of a neutral type difference equation of the form $\Delta[x(n) + cx(\tau(n))] + p(n)x(\sigma(n)) = 0$, where $\tau(n)$ is a general retarded argument, $\sigma(n)$ is a general deviated argument (retarded or advanced), $c \in \mathbb{R}$, $(p(n))_{n\geq 0}$ is a sequence of positive real numbers such that $p(n) \geq p$, $p \in \mathbb{R}_{+}$, and Δ denotes the forward difference operator $\Delta x(n) = x(n+1) - x(n)$. Also, we examine the asymptotic behavior of the solutions in case they are continuous and differentiable with respect to c.

1. Introduction

Neutral type differential equations are differential equations in which the highest-order derivative of the unknown function appears in the equation both with and without delays (or delays advanced). See Driver [1], Bellman and Cooke [2], and Hale [3] for questions of existence, uniqueness, and continuous dependence.

It is to be noted that, in general, the theory of neutral differential equations presents extra complications, and basic results which are true for delay differential equations may not be true for neutral equations. For example, Snow [4] has shown that, even though the characteristic roots of a neutral differential equation may all have negative real parts, it is still possible for some solutions to be unbounded.

The discrete counterparts of neutral differential equations are called *neutral difference equations*, and it is a well-known fact that there is a similarity between the qualitative theories of neutral differential equations and neutral difference equations.

Besides its theoretical interest, strong interest in the study of the asymptotic and oscillatory behavior of solutions of neutral type equations (difference or differential) is motivated by the fact that they arise in many areas of applied mathematics, such as circuit theory [5, 6], bifurcation analysis [7], population dynamics [8, 9], stability theory [10], and dynamical behavior of delayed network systems [11]. See, also, Driver [12], Hale [3], Brayton and Willoughby [13], and the references cited therein. This is the reason that during the last few decades these equations are in the main interest of the literature.

In the present paper, we are interested in the first-order neutral type difference equation of the form

$$\Delta[x(n)+cx(\tau(n))]+p(n)x(\sigma(n))=0, \quad n\geq 0 \tag{E}$$

where $(p(n))_{n\geq 0}$ is a sequence of positive real numbers such that $p(n)\geq p$, $p\in\mathbb{R}_+$, $c\in\mathbb{R}$, $(\tau(n))_{n\geq 0}$ is an increasing sequence of integers which satisfies

$$\tau(n)\leq n-1 \quad \forall n\geq 0, \qquad \lim_{n\to\infty}\tau(n)=+\infty, \tag{1.1}$$

and $(\sigma(n))_{n\geq 0}$ is an increasing sequence of integers such that

$$\sigma(n)\leq n-1 \quad \forall n\geq 0, \qquad \lim_{n\to\infty}\sigma(n)=+\infty, \tag{1.2}$$

or

$$\sigma(n)\geq n+1 \quad \forall n\geq 0. \tag{1.3}$$

Define

$$\begin{gathered} k_1=-\min_{n\geq 0}\tau(n), \qquad k_2=-\min_{n\geq 0}\sigma(n), \\ k=\max\{k_1,k_2\}. \end{gathered} \tag{1.4}$$

(Clearly, k is a positive integer.)

By a *solution* of the neutral type difference equation (E), we mean a sequence of real numbers $(x(n))_{n\geq -k}$ which satisfies (E) for all $n\geq 0$. It is clear that, for each choice of real numbers $c_{-k}, c_{-k+1}, \dots, c_{-1}, c_0$, there exists a unique solution $(x(n))_{n\geq -k}$ of (E) which satisfies the initial conditions $x(-k)=c_{-k}$, $x(-k+1)=c_{-k+1}, \dots, x(-1)=c_{-1}$, $x(0)=c_0$.

A solution $(x(n))_{n\geq -k}$ of the neutral type difference equation (E) is called *oscillatory* if for every positive integer n there exist $n_1, n_2\geq n$ such that $x(n_1)x(n_2)\leq 0$. In other words, a solution $(x(n))_{n\geq -k}$ is oscillatory if it is neither eventually positive nor eventually negative. Otherwise, the solution is said to be *nonoscillatory*.

In the special case where $\tau(n)=n-a$ and $\sigma(n)=n-b$, $a,b\in\mathbb{N}$, (E) takes the form

$$\Delta[x(n)+cx(n-a)]+p(n)x(n-b)=0, \quad n\geq 0. \tag{E$_1$}$$

Equation (E) represents a discrete analogue of the neutral type differential equations

$$\frac{d}{dt}[x(t)+cx(\tau(t))]+p(t)x(\sigma(t))=0, \quad t\geq 0, \tag{E$_2$}$$

which, in the special case where $\tau(t) = t - \tau_0$ and $\sigma(t) = t - \sigma_0$, $\tau_0, \sigma_0 \in \mathbb{R}_+$, takes the form (see e.g., [14, 15])

$$\frac{d}{dt}[x(t) + cx(t - \tau_0)] + p(t)x(t - \sigma_0) = 0, \quad t \geq 0. \tag{E$_3$}$$

The search for the asymptotic behavior and, especially, for oscillation criteria and stability of neutral type (difference or differential) equations has received a great attention in the last few years. Hence, a large number of related papers have been published. See [4, 9, 10, 14–52] and the references cited therein. Most of these papers are concerning the special case of the delay difference equations (E_1) and (E_3) where the algebraic characteristic equation gives useful information about oscillation and stability.

The purpose of this paper is to investigate the convergence and divergence of the solutions of (E) in the case of a general delay argument $\tau(n)$ and of a general deviated (retarded or advanced) argument $\sigma(n)$.

2. Some Preliminaries

Assume that $(x(n))_{n\geq -k}$ is a nonoscillatory solution of (E). Then it is either eventually positive or eventually negative. As $(-x(n))_{n\geq -k}$ is also a solution of (E), we may (and do) restrict ourselves only to the case where $x(n) > 0$ for all large n. Let $n_1 \geq -k$ be an integer such that $x(n) > 0$ for all $n \geq n_1$. Then, there exists $n_0 \geq n_1$ such that

$$x(\tau(n)), x(\sigma(n)) > 0 \quad \text{for every } n \geq n_0. \tag{2.1}$$

Set

$$z(n) = x(n) + cx(\tau(n)). \tag{2.2}$$

Then, in view of (E) and taking into account the fact that $p(n) \geq p > 0$, we have

$$\Delta z(n) + p(n)x(\sigma(n)) = 0 \tag{2.3}$$

or

$$\Delta z(n) = -p(n)x(\sigma(n)) \leq -px(\sigma(n)) < 0 \quad \forall n \geq n_0, \tag{2.4}$$

which means that the sequence $(z(n))_{n\geq n_0}$ is strictly decreasing, regardless of the value of the real constant c.

Throughout this paper, we are going to use the following notation:

$$\tau \circ \tau = \tau^2, \quad \tau \circ \tau \circ \tau = \tau^3, \text{ and so on.} \tag{2.5}$$

Let the domain of τ be the set $D(\tau) = \mathbb{N}_{n_*} = \{n_*, n_* + 1, n_* + 2, \dots\}$, where n_* is the smallest natural number that τ is defined with. Then for every $n > n_*$ it is clear that there exists a natural number $m(n)$ such that

$$x\left(\tau^{m(n)}(n)\right) = x(\tau(n_*)), \qquad \lim_{n\to\infty} m(n) = +\infty \tag{2.6}$$

since $(m(n))$ is increasing and unbounded function of n.

The following lemma provides some tools which are useful for the main results.

Lemma 2.1. *Assume that $(x(n))_{n\geq -k}$ is a positive solution of* (E). *Then, one has the following.*

(i) *If $c \neq 0$ and*

$$\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) = S_0 < +\infty, \tag{2.7}$$

then

$$\lim_{n\to\infty} z(n) = cL = c\lim_{n\to\infty} x(\tau(\sigma(n))). \tag{2.8}$$

(ii) *If*

$$\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) = +\infty, \tag{2.9}$$

then

$$z(n) < 0 \quad \textit{eventually}. \tag{2.10}$$

(iii) *If $c \geq -1$, then*

$$\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) = S_0 < +\infty. \tag{2.11}$$

(iv) *If $c < -1$, then*

$$\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) = +\infty. \tag{2.12}$$

Proof. Summing up (2.3) from n_0 to n, we obtain

$$z(n+1) - z(n_0) + \sum_{i=n_0}^{n} p(i)x(\sigma(i)) = 0 \tag{2.13}$$

or

$$z(n+1) = z(n_0) - \sum_{i=n_0}^{n} p(i)x(\sigma(i)). \tag{2.14}$$

For the above relation, there are only two possible cases:

$$\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) = S_0 < +\infty \tag{2.15a}$$

or

$$\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) = +\infty. \tag{2.15b}$$

Assume that (2.15a) holds. Since $p(n) \geq p > 0$, we have

$$+\infty > S_0 = \sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) \geq p \sum_{i=n_0}^{\infty} x(\sigma(i)). \tag{2.16}$$

The last inequality guarantees that

$$\sum_{i=n_0}^{\infty} x(\sigma(i)) < +\infty \tag{2.17}$$

and, consequently,

$$\lim_{n\to\infty} x(\sigma(n)) = 0. \tag{2.18}$$

Also, (2.15a) guarantess that $\lim_{n\to\infty} z(n)$ exists as a real number. Now, assume that

$$\lim_{n\to\infty} z(n) = \ell \in \mathbb{R}, \qquad c \neq 0. \tag{2.19}$$

Since $(z(\sigma(n)))$ is a subsequence of $(z(n))$, it is obvious that

$$\lim_{n\to\infty} z(\sigma(n)) = \ell \tag{2.20}$$

or

$$\lim_{n\to\infty} [x(\sigma(n)) + cx(\tau(\sigma(n)))] = \ell, \tag{2.21}$$

and, in view of (2.18), we obtain

$$\lim_{n\to\infty} x(\tau(\sigma(n))) = \frac{\ell}{c} := L. \tag{2.22}$$

Hence,

$$\lim_{n\to\infty} z(n) = \ell = cL = c\lim_{n\to\infty} x(\tau(\sigma(n))). \tag{2.23}$$

The proof of part (i) of the lemma is complete.

Assume that (2.15b) holds. Then, by taking limits on both sides of (2.14) we obtain

$$\lim_{n\to\infty} z(n) = -\infty, \tag{2.24}$$

which, in view of the fact that the sequence $(z(n))$ is strictly decreasing, means that

$$z(n) < 0 \quad \text{eventually}. \tag{2.25}$$

The proof of part (ii) of the lemma is complete.

Assume that $-1 \le c < 0$, and suppose, for the sake of contradiction, that $\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) = +\infty$. Then, in view of part (ii), we have $z(n) < 0$ eventually. Thus,

$$x(n) < -cx(\tau(n)) \tag{2.26}$$

or

$$x(n) < -c\left[-cx\left(\tau^2(n)\right)\right]. \tag{2.27}$$

Repeating the above procedure we obtain

$$x(n) < (-c)^{m(n)} x\left(\tau^{m(n)}(n)\right) = (-c)^{m(n)} x(\tau(n_*)). \tag{2.28}$$

If $-1 < c < 0$, clearly, $(-c)^{m(n)} \to 0$ since $m(n) \to \infty$ as $n \to \infty$. Since $x(n) > 0$ for all large n, (2.28) guarantees that

$$\lim_{n\to\infty} x(n) = 0. \tag{2.29}$$

This implies that

$$\lim_{n\to\infty} z(n) = 0, \tag{2.30}$$

and consequently $\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) < \infty$, which contradicts our assumption.

If $c = -1$, by (2.28) we obtain

$$x(n) < x(\tau(n_*)), \tag{2.31}$$

which means that $(x(n))$ is bounded and therefore $(z(n))$ is bounded. Hence, $\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) < \infty$, which contradicts our assumption.

Assume that $c \geq 0$ and that $\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) = \infty$. In view of part (ii), (2.10) holds, that is, $z(n) < 0$ eventually. This contradicts $z(n) = x(n) + cx(\tau(n)) > 0$. Therefore $\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) = S_0 < +\infty$. The proof of part (iii) of the lemma is complete.

In the remainder of this proof, it will be assumed that $c < -1$.

If (2.15a) holds, that is, $\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) = S_0 < +\infty$, then, in view of part (i), (2.8) is satisfied, that is,

$$\lim_{n\to\infty} z(n) = cL, \quad \text{where } L = \lim_{n\to\infty} x(\tau(\sigma(n))). \tag{2.32}$$

If $L > 0$, then

$$\lim_{n\to\infty} z(n) = \lim_{n\to\infty} (x(n) + cx(\tau(n))) = cL < 0. \tag{2.33}$$

Since $(z(n))$ is strictly decreasing, we have

$$z(n) > cL, \tag{2.34}$$

$$x(n) > -cx(\tau(n)) + cL, \tag{2.35}$$

or

$$x(n) > -c\left[-cx\left(\tau^2(n)\right) + cL\right] + cL. \tag{2.36}$$

Repeating this procedure $m(n_\ell)$ times we have

$$\begin{aligned}
x(n) &> -c\left[-cx\left(\tau^2(n)\right) + cL\right] + cL \\
&= (-c)^2 x\left(\tau^2(n)\right) - c^2L + cL \\
&> \cdots > (-c)^{m(n_\ell)} x\left(\tau^{m(n_\ell)}(n)\right) + cL - c^2L + \cdots \pm c^{m(n_\ell)}L \\
&= (-c)^{m(n_\ell)} x(\tau(n_\lambda)) + cL\frac{(-c)^{m(n_\ell)} - 1}{-1-c} \\
&= (-c)^{m(n_\ell)} x(\tau(n_\lambda)) - \frac{cL}{1+c}\left[(-c)^{m(n_\ell)} - 1\right] \\
&= (-c)^{m(n_\ell)}\left[x(\tau(n_\lambda)) - \frac{cL}{1+c}\right] + \frac{cL}{1+c}.
\end{aligned} \tag{2.37}$$

Now, if for this index λ we have $x(\tau(n_\lambda)) - (cL/(1+c)) > 0$, then, as $n \to \infty$, $x(n) \to \infty$, which contradicts (2.18). Therefore, there exists an index s such that

$$x(\tau(n_s)) = x\left(\tau^{m(n_\mu)}(n)\right) \tag{2.38}$$

with

$$x(\tau(n_s)) - \frac{cL}{1+c} < 0. \tag{2.39}$$

Then since $\lim_{n\to\infty} z(n) = cL$, for every $\varepsilon > 0$ there exists $n_2(\varepsilon)$ such that

$$z(n) < cL + \varepsilon \quad \text{for every } n \geq \max\{\tau(n_s), n_2\} = n_3, \tag{2.40}$$

where $\tau(n_s)$ is satisfying the previous inequality.

Hence,

$$x(n) < -cx(\tau(n)) + cL + \varepsilon \quad \text{for every } n \geq n_3, \tag{2.41}$$

or

$$x(n) < -c\left[-cx\left(\tau^2(n)\right) + cL + \varepsilon\right] + cL + \varepsilon \quad \text{for every } n \geq n_3. \tag{2.42}$$

Repeating this procedure $m(n_\mu)$ times we have

$$\begin{aligned} x(n) &< -c\left[-cx\left(\tau^2(n)\right) + cL + \varepsilon\right] + cL + \varepsilon \\ &= (-c)^2 x\left(\tau^2(n)\right) - c^2L - c\varepsilon + cL + \varepsilon \\ &< \cdots < (-c)^{m(n_\mu)} x(\tau(n_s)) - \frac{cL}{1+c}\left[(-c)^{m(n_\mu)} - 1\right] + \frac{(-c)^{m(n_\mu)} - 1}{-1-c}\varepsilon \\ &= (-c)^{m(n_\mu)}\left[x(\tau(n_s)) - \frac{cL}{1+c} - \frac{\varepsilon}{1+c}\right] + \frac{cL}{1+c} + \frac{\varepsilon}{1+c} \end{aligned} \tag{2.43}$$

or

$$x(n) < (-c)^{m(n_\mu)}\left[x(\tau(n_s)) - \frac{cL}{1+c} - \frac{\varepsilon}{1+c}\right] + \frac{cL}{1+c} + \frac{\varepsilon}{1+c}. \tag{2.44}$$

Then, for sufficiently large n the above inequality gives

$$x(n) < 0, \tag{2.45}$$

which condradicts $x(n) > 0$.

If $L = 0$, then

$$\lim_{n\to\infty} z(n) = 0 \tag{2.46}$$

or

$$\lim_{n\to\infty} (x(n) + cx(\tau(n))) = 0, \tag{2.47}$$

and since $(z(n))$ is stricly decreasing, it is obvious that $z(n) > 0$ eventually or

$$x(n) > -cx(\tau(n)). \tag{2.48}$$

In view of (2.6), the last inequality becomes

$$x(n) > (-c)^{m(n)} x(\tau(n_*)), \tag{2.49}$$

and consequently

$$\lim_{n\to\infty} x(n) \geq \lim_{n\to\infty} \left[(-c)^{m(n)} x(\tau(n_*))\right] = +\infty, \tag{2.50}$$

which contradicts (2.18). Hence, it is clear that $\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) = +\infty$. The proof of part (iv) of the lemma is complete.

The proof of the lemma is complete. □

3. Main Results

The asymptotic behavior of the solutions of the neutral type difference equation (E) is described by the following theorem.

Theorem 3.1. *For* (E) *one has the following.*

(I) *Every nonoscillatory solution is unbounded if $c < -1$.*

(II) *Every solution oscillates if $c = -1$.*

(III) *Every nonoscillatory solution tends to zero if $-1 < c < 1$.*

(IV) *Every nonoscillatory solution is bounded if $c \geq 1$.*

Furthermore, if any solution of (E) *is continuous with respect to c, one has the following.*

(V) *Every solution is zero if $c \leq -1$.*

(VI) *Every solution tends to zero if $-1 < c \leq 1$.*

(VII) *If, additionally, any solution of* (E) *has continuous derivatives of any order and convergent Taylor series for every $c \in \mathbb{R}$, then the solution is zero.*

Proof. Assume that $(x(n))_{n\geq -k}$ is a nonoscillatory solution of (E). Then it is either eventually positive or eventually negative. As $(-x(n))_{n\geq -k}$ is also a solution of (E), we may (and do) restrict ourselves only to the case where $x(n) > 0$ for all large n. Let $n_1 \geq -k$ be an integer such that $x(n) > 0$ for all $n \geq n_1$. Then, there exists $n_0 \geq n_1$ such that

$$x(\tau(n)), x(\sigma(n)) > 0 \quad \forall n \geq n_0, \tag{3.1}$$

which, in view of the previous section, means that the sequence $(z(n))_{n\geq n_0}$ is strictly decreasing, regardless of the value of the real constant c.

Assume that $c < -1$. Then, in view of part (iv) of Lemma 2.1, we have $\sum_{i=n_0}^{\infty} p(i) x(\sigma(i)) = +\infty$, and, consequently, in view of part (ii) of Lemma 2.1, $z(n) < 0$ eventually. Therefore,

$$\lim_{n\to\infty} z(n) = -\infty \tag{3.2}$$

or

$$\lim_{n\to\infty} (x(n) + cx(\tau(n))) = -\infty. \tag{3.3}$$

Since $c < -1$, the last relation guarantees that

$$\lim_{n\to\infty} x(\tau(n)) = +\infty, \tag{3.4}$$

which means that $(x(n))$ is unbounded. The proof of part (I) of the theorem is complete.

Assume that $c = -1$. Then, in view of part (iii) of Lemma 2.1, we have $\sum_{i=n_0}^{\infty} p(i) x(\sigma(i)) < \infty$, and, therefore, in view of part (i) of Lemma 2.1, we obtain

$$\lim_{n\to\infty} [x(n) - x(\tau(n))] = -L. \tag{3.5}$$

Assume that $L > 0$. Then there exists a natural number n_λ such that $z(n) < 0$ for every $n \geq n_\lambda$, and therefore

$$x(n) < x(\tau(n)) < \cdots < x\left(\tau^{m(n_\ell)}(n_\lambda)\right), \tag{3.6}$$

which means that $(x(n))$ is bounded. Since $(x(n))$ is bounded, let

$$M = \limsup x(n), \quad \text{where } M \geq L. \tag{3.7}$$

Then there exists a subsequence $(x(\theta(n)))$ of $(x(n))$ such that

$$\lim_{n\to\infty} x(\theta(n)) = M. \tag{3.8}$$

Thus,

$$\lim_{n\to\infty} z(\theta(n)) = -L, \tag{3.9}$$

$$\lim_{n\to\infty} [x(\theta(n)) - x(\tau(\theta(n)))] = -L, \tag{3.10}$$

or

$$\lim_{n\to\infty} x(\tau(\theta(n))) = M + L > M, \tag{3.11}$$

which contradicts (3.7). Therefore, $L > 0$ is not valid. Hence, $L = 0$ and consequently $\lim_{n\to\infty} z(n) = 0$. Furthermore, taking into account the fact that the sequence $(z(n))$ is strictly decreasing we conclude that $z(n) > 0$ or equivalently

$$x(n) > x(\tau(n)) > \cdots > x(\tau(n_*)). \tag{3.12}$$

Since $(x(n))$ has a lower bound greater than zero, it cannot have any subsequence that tends to zero. Thus, $\lim_{n\to\infty} x(\sigma(n)) = 0$ is not valid, and therefore $\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) = \infty$ which contradicts our previous conclusions. Hence, if $c = -1$, $(x(n))$ oscillates. The proof of part (II) of the theorem is complete.

Assume that $-1 < c < 0$.

Assume that $L > 0$. Then there exists a natural number n_λ such that $z(n) < 0$ for every $n \geq n_\lambda$, and therefore

$$x(n) < (-c)x(\tau(n)) < \cdots < (-c)^{m(n_\ell)} x\left(\tau^{m(n_\ell)}(n_\lambda)\right), \tag{3.13}$$

which means that $(x(n))$ tends to zero as $n \to \infty$. Therefore, $(z(n))$ tends to zero as $n \to \infty$, that is, $L = 0$, which contradicts $L > 0$. Hence $L = 0$. Taking into account the fact that the sequence $(z(n))$ is strictly decreasing, it is obvious that $z(n) > 0$. Hence, for every $\epsilon > 0$ there exists a natural number n_4 such that for every $n \geq n_4$

$$x(n) + cx(\tau(n)) < \epsilon \tag{3.14}$$

or

$$x(n) < -cx(\tau(n)) + \epsilon < -c\left[-cx\left(\tau^2(n)\right) + \epsilon\right] + \epsilon = c^2 x\left(\tau^2(n)\right) - c\epsilon + \epsilon. \tag{3.15}$$

For sufficiently large n, after m-steps we obtain

$$x(n) < c^{m+1} x\left(\tau^{m+1}(n)\right) + \epsilon - c\epsilon + \cdots + (-c)^m \epsilon. \tag{3.16}$$

As $n \to \infty$, clearly $m \to \infty$, and therefore

$$\lim_{n\to\infty} x(n) \leq \lim_{m\to\infty} \left[\epsilon - c\epsilon + \cdots + (-c)^m \epsilon\right] = \frac{\epsilon}{1+c}. \tag{3.17}$$

Since ϵ is an arbitrary real positive number and, taking into account the fact that $(x(n)) > 0$, it is clear that

$$\lim_{n\to\infty} x(n) = 0. \tag{3.18}$$

Let $c = 0$. In view of part (iii) of Lemma 2.1, we have

$$\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) < +\infty. \tag{3.19}$$

This guarantees that $(z(n))$ is bounded, and therefore, since $z(n) = x(n)$, $(x(n))$ is bounded. Also, since $(z(n))$ is stricly decreasing, $\lim_{n\to\infty} z(n)$ exists, that is, $\lim_{n\to\infty} x(n)$ exists. In view of (2.18), we conclude that

$$\lim_{n\to\infty} x(n) = 0. \tag{3.20}$$

Assume that $0 < c < 1$, then

$$\lim_{n\to\infty} z(n) = cL, \quad \text{where } \lim_{n\to\infty} x(\tau(\sigma(n))) = L \tag{3.21}$$

or

$$\lim_{n\to\infty} [x(n) + cx(\tau(n))] = cL. \tag{3.22}$$

Therefore,

$$\lim_{n\to\infty} z(\tau(\sigma(n))) = cL, \tag{3.23}$$

$$\lim_{n\to\infty} \left[x(\tau(\sigma(n))) + cx\left(\tau^2(\sigma(n))\right)\right] = cL, \tag{3.24}$$

or

$$\lim_{n\to\infty} x\left(\tau^2(\sigma(n))\right) = L - \frac{L}{c} \leq 0, \tag{3.25}$$

which means that $L = 0$. Hence,

$$\lim_{n\to\infty} z(n) = 0, \tag{3.26}$$

and since

$$z(n) \geq x(n) > 0, \tag{3.27}$$

we have

$$\lim_{n\to\infty} x(n) = 0. \tag{3.28}$$

The proof of part (III) of the theorem is complete.

Assume that $c \geq 1$. In view of Part (iii) of Lemma 2.1, it is clear that $\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) = S_0 < +\infty$ which combined with (2.14) implies that $(z(n))$ is bounded and, therefore $(x(n))$ is bounded. The proof of part (IV) of the theorem is complete.

In the remainder of this proof, it will be assumed that $x(n)$ is a continuous function with respect to c, and, therefore, instead of $x(n)$ we will write $x(c,n)$.

Let $c = -1$. Then, in view of part (II), $(x(-1,n))$ oscillates. On the other hand, since $x(c,n)$ is continuous, we have

$$\lim_{c\to -1^-} x(c,n) = x(-1,n). \tag{3.29}$$

But $x(c,n) > 0$ for all large n, and therefore its limit is always nonnegative. Thus, $x(-1,n) > 0$ for all large n which contradicts that $(x(-1,n))$ oscillates. Therefore, $x(-1,n) = 0$ eventually.

Let $c < -1$. In view of part (I) we have $(x(c,n))$ is unbounded, and therefore $x(c,\tau(n)) \to \infty$. Let $M > 0$, then there exists an index n_5 such that, for every $n \geq n_5$, $x(c,\tau(n)) > M$. Since the function $x(c,n)$ is continuous, $x(c,\tau(n))$ is continuous, and therefore

$$\lim_{c\to -1^-} x(\mathrm{c},\tau(n)) = x(-1,\tau(n)) = 0 \quad \text{for every } n \geq n_5. \tag{3.30}$$

Hence, there exists $h > 0$ so that, if $c > -1-h$ then $x(c,\tau(n)) < M$ for every $n \geq n_5$, which contradicts $x(c,\tau(n)) > M$. This implies that, there exists an interval $(a,-1)$ such that

$$x(c,n) = 0 \quad \text{for every } c \in (a,-1) \text{ eventually}. \tag{3.31}$$

Let

$$A = \inf\{c \mid x(c,n) = 0\}. \tag{3.32}$$

Then, with the same argument as above, we conclude that

$$x(A,n) = 0 \text{ eventually}. \tag{3.33}$$

So, by a similar procedure we obtain an interval $(B,A]$ such that

$$x(c,n) = 0 \quad \text{for every } c \in (B,A] \text{ eventually}. \tag{3.34}$$

This contradicts our assumption for A. Thus, if $c < -1$, we have $x(c,n) = 0$ eventually. The proof of part (V) of the theorem is complete.

Assume that $-1 < c \leq 1$. Taking into account part (III), it is enough to show part (VI) only for $c = 1$. In view of parts (iii) and (i) of Lemma 2.1, we have

$$\lim_{n\to\infty} x(c, \tau(\sigma(n))) = L. \tag{3.35}$$

Assume, for the sake of contradiction, that $L > 0$. Taking into account the fact that $\lim_{n\to\infty} x(c, n) = 0$ when $0 < c < 1$, there exists $n_6 \in \mathbb{N}$ such that, for every $n \geq n_6$, we have

$$x(c, \tau(\sigma(n))) < \frac{L}{2}, \tag{3.36}$$

and, since $x(c, \tau(\sigma(n)))$ is continuous,

$$\lim_{c\to 1^-} x(c, \tau(\sigma(n))) = x(1, \tau(\sigma(n))). \tag{3.37}$$

Hence,

$$x(1, \tau(\sigma(n))) \leq \frac{L}{2}, \quad \forall n \geq n_6 \tag{3.38}$$

which contradicts $\lim_{n\to\infty} x(c, \tau(\sigma(n))) = L$. Therefore, $L > 0$ is impossible. Thus, $L = 0$, and, in view of (2.8), we conclude that

$$\lim_{n\to\infty} z(1, n) = 0, \tag{3.39}$$

which means that

$$\lim_{n\to\infty} x(1, n) = 0. \tag{3.40}$$

The proof of part (VI) of the theorem is complete.

Finally, since $x(c, n)$ has convergent Taylor series, we have

$$x(c, n) = \sum_{m=0}^{\infty} \frac{x^{(m)}(a, n)}{m!} c^m \quad \text{for every } a \in \mathbb{R}. \tag{3.41}$$

Choose $a < -1$. Then $x^{(m)}(a, n) = 0$ and, therefore, $x(c, n) = 0$. The proof of part (VII) of the theorem is complete.

The proof of the theorem is complete. □

By way of illustration and for purely pedagogical purposes, the asymptotic behavior of nonoscillatory solutions of (E) is presented in Figure 1.

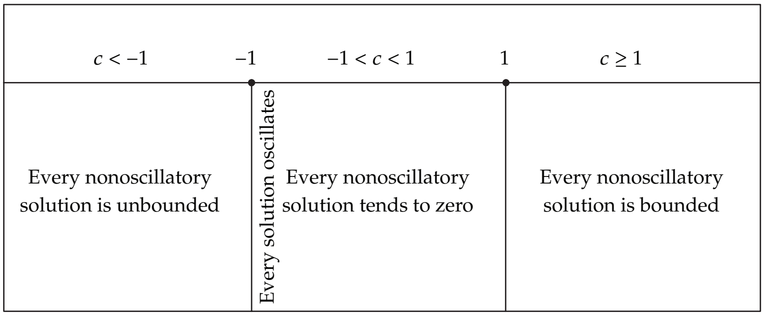

Figure 1

Corollary 3.2. *For* (E_1) *one hs the following.*

(i) *Every nonoscillatory solution tends to* $\pm\infty$ *if* $c < -1$.

(ii) *Every solution oscillates if* $c = -1$.

(iii) *Every nonoscillatory solution tends to zero if* $c > -1$.

Furthermore, if any solution of (E_1) *is continuous with respect to* c, *one has the following.*

(iv) *Every solution is zero if* $c \leq -1$.

(v) *If, additionally, any solution of* (E_1) *has continuous derivatives of any order and convergent Taylor series for every* $c \in \mathbb{R}$, *then the solution is zero.*

Proof. In part (I) of Theorem 3.1 we have proved that

$$\lim_{n\to\infty} x(\tau(n)) = +\infty \tag{3.42}$$

and consequently

$$\lim_{n\to\infty} x(n-a) = +\infty, \tag{3.43}$$

which means that

$$\lim_{n\to\infty} x(n) = +\infty. \tag{3.44}$$

The proof of part (i) of the corollary is complete.

Part (ii) is direct from part (II) of Theorem 3.1.

As we have proved in parts (III) and (IV) of Theorem 3.1, if $c > -1$, then

$$\lim_{n\to\infty} x(\sigma(n)) = 0 \tag{3.45}$$

and consequently

$$\lim_{n\to\infty} x(n-b) = 0, \tag{3.46}$$

which means that

$$\lim_{n \to \infty} x(n) = 0. \tag{3.47}$$

The proof of part (iii) of the corollary is complete.

Parts (iv) and (v) are direct from parts (V) and (VII) of Theorem 3.1.

The proof of the corollary is complete. □

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