

Research Article

An Explicit Numerical Method for the Fractional Cable Equation

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An explicit numerical method to solve a fractional cable equation which involves two temporal Riemann-Liouville derivatives is studied. The numerical difference scheme is obtained by approximating the first-order derivative by a forward difference formula, the Riemann-Liouville derivatives by the Grünwald-Letnikov formula, and the spatial derivative by a three-point centered formula. The accuracy, stability, and convergence of the method are considered. The stability analysis is carried out by means of a kind of von Neumann method adapted to fractional equations. The convergence analysis is accomplished with a similar procedure. The von-Neumann stability analysis predicted very accurately the conditions under which the present explicit method is stable. This was thoroughly checked by means of extensive numerical integrations.

1. Introduction

Fractional calculus is a key tool for solving some relevant scientific problems in physics, engineering, biology, chemistry, hydrology, and so on [1–6]. A field of research in which the fractional formalism has been particularly useful is that related to anomalous diffusion processes [1, 7–13]. This kind of process is singularly abundant and important in biological media [14–16]. In this context, the electrodiffusion of ions in neurons is an anomalous diffusion problem to which the fractional calculus has recently been applied. The precise origin of the anomalous character of this diffusion process is not clear (see [17] and references therein), but in any case the consideration of anomalous diffusion in the modeling of electrodiffusion of ions in neurons seems pertinent. This problem has been addressed recently by Langlands et al. [17, 18]. An equation that plays a key role in their analysis is the following fractional cable (or telegrapher's or Cattaneo) equation (model II):

$$\frac{\partial u}{\partial t} = \frac{\partial^{1-\gamma_1}}{\partial t^{1-\gamma_1}} \left(K \frac{\partial^2 u}{\partial x^2} \right) - \mu^2 \frac{\partial^{1-\gamma_2} u}{\partial t^{1-\gamma_2}}, \quad (1.1)$$

where

$$\frac{\partial^\gamma}{\partial t^\gamma} f(t) \equiv \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{d\tau^n} \int_0^t d\tau \frac{f(\tau)}{(t-\tau)^{1+\gamma-n}}, \quad (1.2)$$

with $n-1 < \gamma < n$ and $n = \text{integer}$, is the Riemann-Liouville fractional derivative. Here u is the difference between the membrane potential and the resting membrane potential, γ_1 is the exponent characterizing the anomalous flux of ions along the nerve cell, and γ_2 is the exponent characterizing the anomalous flux across the membrane [17, 18]. Some earlier fractional cable equations were discussed in [19, 20].

A variety of analytical and numerical methods to solve many classes of fractional equations have been proposed and studied over the last few years [10, 21–30]. Of the numerical methods, finite difference methods have been particularly fruitful [31–38]. These methods can be broadly classified as explicit or implicit [39]. An implicit method for dealing with (1.1) has recently been considered by Liu et al. [38]. Although implicit methods are more cumbersome than explicit methods, they usually remain stable over a larger range of parameters, especially for large timesteps, which makes them particularly suitable for fractional diffusion problems. Nevertheless, explicit methods have some features that make them widely appreciated [32, 39]: flexibility, simplicity, small computational demand, and easy generalization to spatial dimensions higher than one. Unfortunately, they can become unstable in some cases, so that it is of great importance to determine the conditions under which these methods are stable. In this paper we will discuss an explicit finite difference scheme for solving the fractional cable equation, which is close to the methods studied in [32, 33]. We shall address two main questions: (i) whether this kind of method can cope with fractional equations involving different fractional derivatives, such as the fractional cable equation; (ii) whether the von Neumann stability analysis put forward in [32, 34] is suitable for this kind of equation.

2. The Numerical Method

Henceforth, we will use the notation $x_j = j\Delta x$, $t_m = m\Delta t$, and $u(x_j, t_m) = u_j^{(m)} \simeq U_j^{(m)}$, where $U_j^{(m)}$ is the numerical estimate of the exact solution $u(x, t)$ at $x = x_j$ and $t = t_m$.

In order to get the numerical difference algorithm, we discretize the continuous differential and integro-differential operators as follows. For the discretization of the fractional Riemann-Liouville derivative we use the Grünwald-Letnikov formula

$$\frac{d^{1-\gamma}}{dt^{1-\gamma}} u(x, t) = \Delta_t^{1-\gamma} u(x, t_m) + O(\Delta t) \quad (2.1)$$

with

$$\Delta_t^\alpha f(t_m) = \frac{1}{(\Delta t)^\alpha} \sum_{k=0}^m \omega_k^{(\alpha)} f(t_{m-k}), \quad (2.2)$$

$$\omega_k^{(\alpha)} = \left(1 - \frac{1+\alpha}{k}\right) \omega_{k-1}^{(\alpha)}, \quad (2.3)$$

and $\omega_0^{(\alpha)} = 1$. These coefficients come from the generating function [40]

$$(1 - z)^\alpha = \sum_{k=0}^{\infty} \omega_k^{(\alpha)} z^k. \quad (2.4)$$

To discretize the integer derivatives we use standard formulas: for the second-order spatial derivative we employ the three-point centered formula

$$\frac{\partial^2}{\partial x^2} u(x_j, t_m) = \Delta_x^2 u_j^{(m)} + O(\Delta x)^2 \quad (2.5)$$

with

$$\Delta_x^2 u_j^{(m)} = \frac{u(x_{j+1}, t_m) - 2u(x_j, t_m) + u(x_{j-1}, t_m)}{(\Delta x)^2}, \quad (2.6)$$

and for the first-order time derivative we use the forward derivative

$$\frac{\partial}{\partial t} u(x_j, t_m) = \delta_t u_j^{m+1} + O(\Delta t), \quad (2.7)$$

where

$$\delta_t u_j^{m+1} = \frac{u(x_j, t_{m+1}) - u(x_j, t_m)}{\Delta t}. \quad (2.8)$$

Inserting (2.1), (2.5), and (2.7) into (1.1), one gets

$$\delta_t u_j^{(m+1)} - K \Delta_t^{1-\gamma_1} \left(\Delta_x^2 u_j^{(m)} \right) + \mu^2 \Delta_t^{1-\gamma_2} u_j^{(m)} = T(x, t), \quad (2.9)$$

where, as can easily be proved, the truncating error $T(x, t)$ is

$$T(x, t) = O(\Delta t) + O(\Delta x)^2. \quad (2.10)$$

Neglecting the truncating error we get the finite difference scheme we are seeking:

$$\delta_t U_j^{(m+1)} - K \Delta_t^{1-\gamma_1} \Delta_x^2 U_j^{(m)} + \mu^2 \Delta_t^{1-\gamma_2} U_j^{(m)} = 0, \quad (2.11)$$

that is,

$$U_j^{(m+1)} = U_j^{(m)} + S \sum_{k=0}^m \omega_k^{1-\gamma_1} \left(U_{j+1}^{(m-k)} - 2U_j^{(m-k)} + U_{j-1}^{(m-k)} \right) - \mu^2 (\Delta t)^{\gamma_2} \sum_{k=0}^m \omega_k^{1-\gamma_2} U_j^{(m-k)}, \quad (2.12)$$

where

$$S = K \frac{(\Delta t)^{\gamma_1}}{(\Delta x)^2}. \quad (2.13)$$

To test this algorithm, we solved (1.1) in the interval $-L/2 \leq x \leq L/2$, with absorbing boundary conditions, $u(x = -L/2, t) = u(x = L/2, t) = 0$, and initial condition given by a Dirac's delta function centered at $x = 0$: $u(x, 0) = \delta(x)$. The exact solution of this problem for $L \rightarrow \infty$ is [17]

$$u(x, t) = \frac{1}{\sqrt{4t^{\gamma_1} \pi}} \sum_{k=0}^{\infty} \frac{(-\mu^2 t^{\gamma_2})^k}{k!} H_{1,2}^{2,0} \left[\frac{x^2}{4t^{\gamma_1}} \left| \begin{matrix} \left(1 - \frac{\gamma_1}{2} + \gamma_2 k, \gamma_1\right) \\ (0, 1), \left(\frac{1}{2} + k, 1\right) \end{matrix} \right. \right], \quad (2.14)$$

where H denotes the Fox H function [10, 41]. In our numerical procedure, the exact initial condition $u(x, 0) = \delta(x)$ is approximated by

$$u(x_j, 0) = \begin{cases} \frac{1}{\Delta x}, & j = 0, \\ 0, & j \neq 0. \end{cases} \quad (2.15)$$

The explicit difference scheme (2.12) is tested by comparing the analytical solution with the numerical solution for several cases of the problem described following (2.13) with different values of γ_1 and γ_2 . We have computed the analytical solution by means of (2.14) truncating the series at $k = 20$. The corresponding Fox H function was evaluated by means of the series expansion described in [10, 41] truncating the infinite series after the first 50 terms. In Figures 1 and 2 we show the analytical and numerical solution for two values of γ_1 and γ_2 at $x = 0$ and $x = 0.5$. The differences between the exact and the numerical solution are shown in Figures 3 and 4. One sees that, except for very short times, the agreement is quite good. The large value of the error for small times is due in part to the approximation of the Dirac's delta function at $x = 0$ by (2.15). This is clearly appreciated when noticing the quite different scales of Figures 3 and 4: the error is much smaller for $x = 0.5$ than for $x = 0$. For the cases with $\gamma_1 = 1/2$ we used a smaller value of Δt and, simultaneously, a larger value of Δx than for the cases with $\gamma_1 = 1$ in order to keep the numerical scheme stable. This issue will be discussed in Section 3.

3. Stability

As usual for explicit methods, the present explicit difference scheme (2.12) is not unconditionally stable, that is, for any given set of values of γ_1 , γ_2 , μ , and K there are choices of Δx and Δt for which the method is unstable. Therefore, it is important to determine the conditions under which the method is stable. To this end, here we shall employ the fractional von Neumann stability analysis (or Fourier analysis) put forward in [32] (see also [33–35]). The question we address is to what extent this procedure is valid for fractional diffusion equations that involve fractional derivatives of different order.

Proceeding as [32], we start by recognizing that the solution of our problem can be written as the linear combination of subdiffusive modes, $u_j^{(m)} = \sum_q \zeta_q^{(m)} e^{iqj\Delta x}$, where the

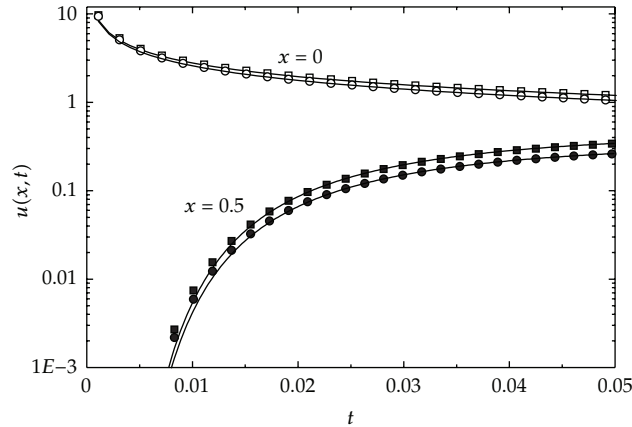


Figure 1: Numerical solution at the mid-point $x = 0$ (hollow symbols) and $x = 0.5$ (filled symbols) of the fractional cable equation for $\gamma_1 = 1$ and $\gamma_2 = 1$ (squares) and $\gamma_2 = 1/2$ (circles) with $\Delta x = 1/20$, $\Delta t = 10^{-4}$, $K = \mu = 1$, and $L = 5$. Lines are the exact solutions given by (2.14).

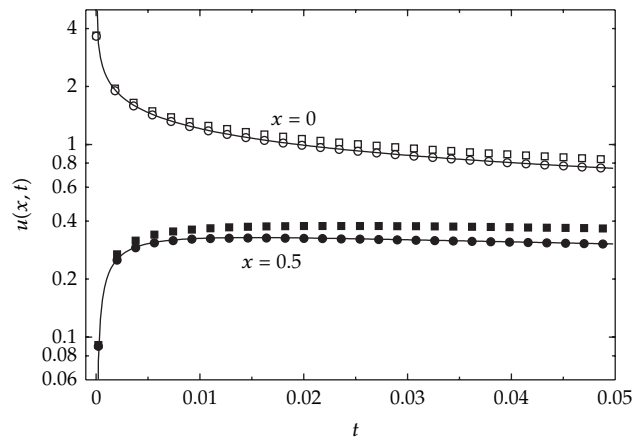


Figure 2: Numerical solution at the mid-point $x = 0$ (hollow symbols) and $x = 0.5$ (filled symbols) of the fractional cable equation for $\gamma_1 = 1/2$ and $\gamma_2 = 1$ (squares) and $\gamma_2 = 1/2$ (circles) with $\Delta x = 1/10$, $\Delta t = 10^{-5}$, $K = \mu = 1$, and $L = 5$. Lines are the exact solutions given by (2.14).

sum is over all the wave numbers q supported by the lattice. Therefore, following the von Neumann ideas, we reduce the problem of analyzing the stability of the solution to the problem of analyzing the stability of a single generic subdiffusion mode, $\zeta^{(m)} e^{iqj\Delta x}$. Inserting this expression into (2.12) one gets

$$\zeta^{(m+1)} = \zeta^{(m)} + S \sum_{k=0}^m \omega_k^{(1-\gamma_1)} \left(e^{iq\Delta x} - 2 + e^{-iq\Delta x} \right) \zeta^{(m-k)} - \mu^2 (\Delta t) \gamma_2 \sum_{k=0}^m \omega_k^{(1-\gamma_2)} \zeta^{(m-k)}. \quad (3.1)$$

The stability of the mode is determined by the behavior of $\zeta^{(m)}$. Writing

$$\zeta^{(m+1)} = \xi \zeta^{(m)} \quad (3.2)$$

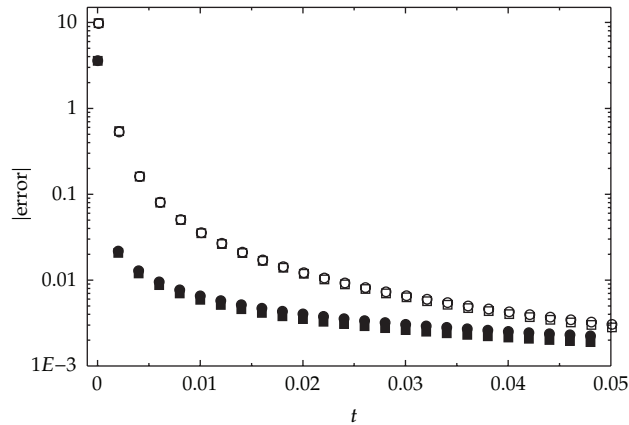


Figure 3: Absolute error $|U_j^{(m)} - u_j^{(m)}|$ of the numerical method for the problems considered in Figures 1 and 2 at $x = 0$. Squares: $\gamma_2 = 1$; circles: $\gamma_2 = 1/2$; hollow symbols: $\gamma_1 = 1$; filled symbols: $\gamma_1 = 1/2$.

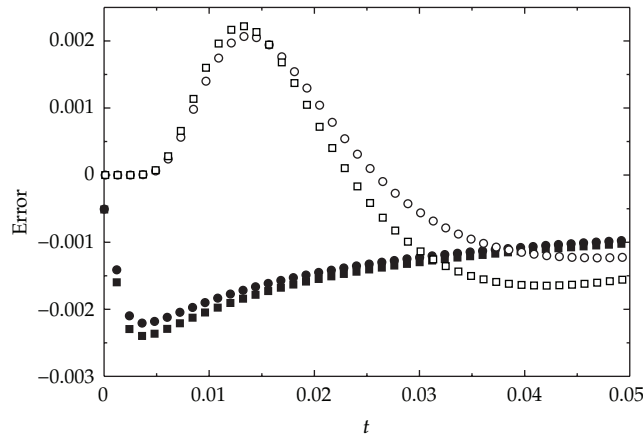


Figure 4: Error $U_j^{(m)} - u_j^{(m)}$ of the numerical method for the problem considered in Figures 1 and 2 at $x = 0.5$. Squares: $\gamma_2 = 1$; circles: $\gamma_2 = 1/2$; hollow symbols: $\gamma_1 = 1$; filled symbols: $\gamma_1 = 1/2$.

and assuming that the amplification factor ξ of the subdiffusive mode is independent of time, we get

$$\xi = 1 + S \sum_{k=0}^m \omega_k^{(1-\gamma_1)} \left(e^{iq\Delta x} - 2 + e^{-iq\Delta x} \right) \xi^{-k} - \mu^2 (\Delta t)^{\gamma_2} \sum_{k=0}^m \omega_k^{(1-\gamma_2)} \xi^{-k}. \quad (3.3)$$

If $|\xi| > 1$ for some q , the temporal factor of the solution grows to infinity [c.f., (3.2)], and the mode is unstable. Considering the extreme value $\xi = -1$, one gets from (3.3) that the numerical method is stable if this inequality holds:

$$\bar{S} \leq S_x^m = \frac{-2 + \mu^2 (\Delta t)^{\gamma_2} \sum_{k=0}^m \omega_k^{(1-\gamma_2)} (-1)^k}{-4 \sum_{k=0}^m \omega_k^{(1-\gamma_1)} (-1)^k}, \quad (3.4)$$

where

$$\bar{S} = S \sin^2\left(\frac{q\Delta x}{2}\right). \quad (3.5)$$

If one defines $S_x = \lim_{m \rightarrow \infty} S_x^m$, one gets

$$\bar{S} \leq S_x = \frac{-2 + \mu^2(\Delta t)^{\gamma_2} \sum_{k=0}^{\infty} \omega_k^{(1-\gamma_2)} (-1)^k}{-4 \sum_{k=0}^{\infty} \omega_k^{1-\gamma_1} (-1)^k}. \quad (3.6)$$

But from (2.4) with $z = -1$ one sees that $\sum_{k=1}^{\infty} (-1)^k \omega_k^{(1-\gamma)} = 2^{1-\gamma}$, so that

$$S_x = \frac{2^{\gamma_2} - \mu^2(\Delta t)^{\gamma_2}}{2^{2+\gamma_2-\gamma_1}}. \quad (3.7)$$

Therefore, because $S \leq \bar{S}$, we find that a sufficient condition for the present method to be stable is that $S \leq S_x$. In Figures 5 and 6 we show two representative examples of the problem considered in Figure 2 but for two values of S , respectively, larger and smaller than the stability bound provided by (3.7). One sees that the value of S that one chooses is crucial: when S is smaller than S_x one is inside the stable region and gets a sensible numerical solution (Figure 5); otherwise one gets an evidently unstable and nonsensical solution (Figure 6).

4. Numerical Check of the Stability Analysis

In this section we describe a comprehensive check of the validity of our stability analysis by using many different values of the parameters γ_1 , γ_2 , Δt , and Δx and testing whether the stability of the numerical method is as predicted by (3.7). Without loss of generality, we assume $\mu = K = 1$ in all cases. We proceed in the following way. First, we choose a set of values of γ_1 , γ_2 , Δx , and S and integrate the corresponding fractional cable equation. If

$$\left| U_j^{m-1} - U_j^m \right| > \lambda \quad (4.1)$$

for $\lambda = 10$ within the first 1000 integrations, then we say the method is unstable; otherwise, we label the method as stable. We generated Figure 7 by starting the integration for values of S well below the theoretical stability limit given by (3.7) and kept increasing its value by 0.001 until condition (4.1) was first reached. The last value for which the method was stable is recorded and plotted in Figure 7. The limit value $\lambda = 10$ is arbitrary, but choosing any other reasonable value does not significantly change these plots.

5. Convergence Analysis

In this section we show that the present numerical method is convergent, that is, that the numerical solution converges towards the exact solution when the size of the spatiotemporal

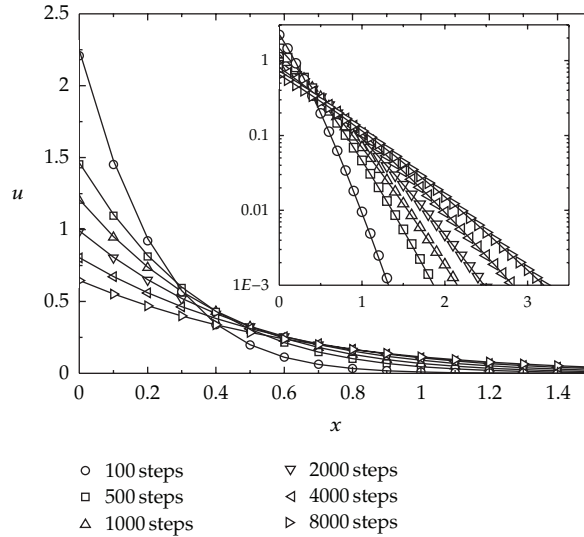


Figure 5: Exact solution (lines) and numerical solution (symbols) provided by our method for the fractional cable equation with $\gamma_1 = 0.5$ and $\gamma_2 = 0.5$ for different numbers of timesteps when $\Delta x = 1/10$, $\Delta t = 10^{-5}$, $K = \mu = 1$, $L = 5$, and $S = (\Delta t)^\gamma / (\Delta x)^2 = 0.316$. This case is inside the stability region because S is smaller than the stability limit $S_x = (2^{\gamma_2} - \mu^2(\Delta t)^{\gamma_2}) / (2^{2+\gamma_2-\gamma_1}) \simeq 0.352\dots$ provided by (3.7). The inset shows the results on logarithmic scale.

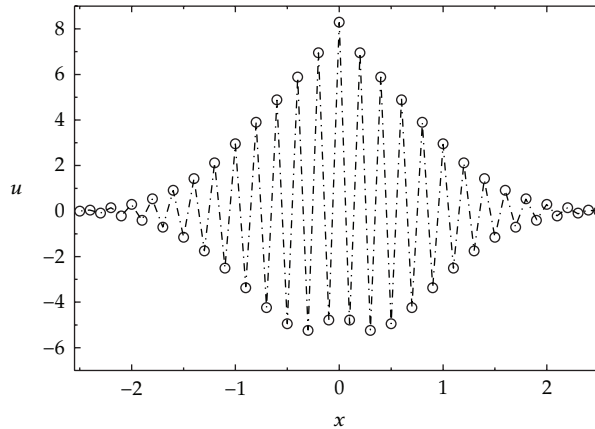


Figure 6: Numerical solution (circles) provided by our explicit method for the fractional cable equation with $\gamma_1 = 0.5$ and $\gamma_2 = 0.5$ after 100 timesteps when $\Delta x = 1/10$, $\Delta t = 1.3 \times 10^{-5}$, $K = \mu = 1$, $L = 5$, and $S = (\Delta t)^\gamma / (\Delta x)^2 = 0.36$. Note that this value is larger than the stability limit $S_x = (2^{\gamma_2} - \mu^2(\Delta t)^{\gamma_2}) / (2^{2+\gamma_2-\gamma_1}) \simeq 0.352\dots$ provided by (3.7). The broken line is to guide the eye.

discretization goes to zero. Let us define $e_j^{(k)}$ as the difference between the exact and numerical solutions at the point (x_j, t_m) : $e_j^{(k)} = u_j^{(k)} - U_j^{(k)}$. Taking into account (2.9) and (2.11), one gets the equation that describes how this difference evolves:

$$\begin{aligned}
 e_j^{(m+1)} - e_j^{(m)} - S \sum_{k=0}^m \omega_k^{(1-\gamma_1)} \left(e_{j+1}^{(m-k)} - 2e_j^{(m-k)} + e_{j-1}^{(m-k)} \right) + \mu^2(\Delta t)^{\gamma_2} \sum_{k=0}^m \omega_k^{(1-\gamma_2)} e_j^{(m-k)} \\
 = T(x_j, t_k) \equiv T_j^{(m)}.
 \end{aligned} \tag{5.1}$$

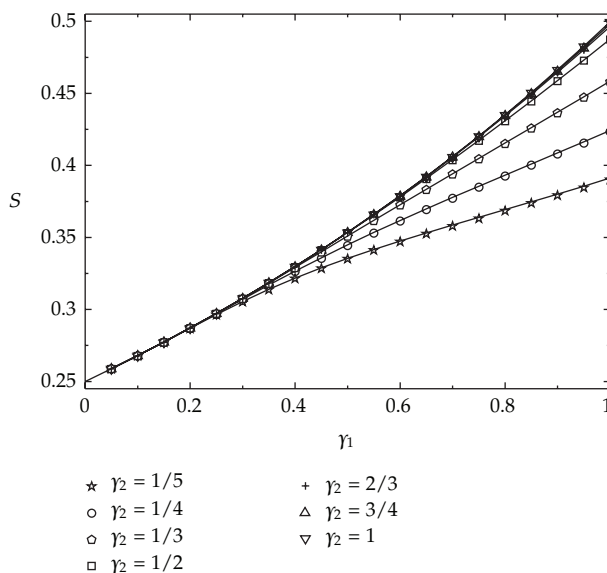


Figure 7: Stability bound S versus γ_1 for several values of γ_2 where $\Delta x = 1/20$, and $K = \mu = 1$. Symbols are numerical estimates. Lines correspond to the theoretical prediction (3.7).

As we did in the previous section for $U_j^{(k)}$, we write $e_j^{(k)}$ and $T_j^{(m)}$ as a combination of (sub) diffusion modes, $e_j^{(k)} = \sum_q \zeta_q^{(k)} e^{iqj\Delta x}$ and $T_j^{(m)} = \sum_q \chi_q^{(m)} e^{iqj\Delta x}$, and analyze the convergence of a single but generic q -mode [36, 39, 42]. Therefore, replacing $e_j^{(k)}$ by $\zeta^{(k)} e^{iqj\Delta x}$ and $T_j^{(m)}$ by $\chi^{(m)} e^{iqj\Delta x}$ in (5.1), we get

$$\zeta^{(m+1)} = \zeta^{(m)} + \bar{S} \sum_{k=0}^m \omega_k^{(1-\gamma_1)} \zeta^{(m-k)} + \mu^2 (\Delta t)^{\gamma_2} \sum_{k=0}^m \omega_k^{(1-\gamma_2)} \zeta^{(m-k)} + \chi^{(m)}. \tag{5.2}$$

Now we will prove by induction that $|\zeta^{(m)}| = O(\Delta t) + O(\Delta x)^2$ for all m . To start, $U_j^{(0)}$ satisfies the initial condition by construction, so that $e_j^{(0)} = 0$. This means that $\zeta^{(0)} = 0$. Therefore, from (5.2) one gets $\zeta^{(1)} = \chi^{(0)}$. But from (2.10) one knows that $|T_j^{(0)}| = |\chi^{(0)}| = O(\Delta t) + O(\Delta x)^2$, so that $|\zeta^{(1)}| = O(\Delta t) + O(\Delta x)^2$. Let us now assume that $|\zeta^{(k)}| = O(\Delta t) + O(\Delta x)^2$ holds for $k = 1, \dots, m$. Then we will prove that $|\zeta^{(m+1)}| = O(\Delta t) + O(\Delta x)^2$. From (5.2) we obtain

$$|\zeta^{(m+1)}| \leq |\chi^{(m)}| + |\zeta^{(m)}| + \bar{S} |\zeta^{(m)}| \left| \sum_{k=0}^m \omega_k^{(1-\gamma_1)} \right| - \mu^2 (\Delta t)^{\gamma_2} |\zeta^{(m)}| \left| \sum_{k=0}^m \omega_k^{(1-\gamma_2)} \right|, \tag{5.3}$$

where $|\zeta^{(m)}|$ is the maximum value of $|\zeta^{(k)}|$ for $k = 0, \dots, m$. Taking into account (2.4), using the value $z = 1$, and because $\omega_0^{(\alpha)} = 1$, it is easy to see that $\sum_{k=1}^\infty \omega_k^{(\alpha)} = -1$ or, equivalently, $\sum_{k=1}^\infty |\omega_k^{(\alpha)}| = 1$ since $\omega_k^{(\alpha)} < 0$ for $k \geq 1$ (see (2.3)). Therefore $\sum_{k=0}^m |\omega_k^{(1-\gamma)}|$ is bounded (in

fact, it is smaller than 2). Using this result in (5.3), together with $|\zeta^{(k)}| \leq C(\Delta t + (\Delta x)^2)$ and $|x^{(k)}| \leq C(\Delta t + (\Delta x)^2)$, we find that

$$|\zeta^{(m+1)}| \leq C(\Delta t + (\Delta x)^2). \quad (5.4)$$

Therefore the amplitude of the subdiffusive modes goes to zero when the spatiotemporal mesh goes to zero. Employing the Parseval relation, this means that the norm of the error $\|e^{(k)}\|^2 \equiv \sum_j |e_j^{(k)}|^2 = \sum_q |\zeta_q^{(k)}|^2$ goes to zero when Δt and Δx go to zero. This is what we aimed to prove.

6. Conclusions

An explicit method for solving a kind of fractional diffusion equation that involves several fractional Riemann-Liouville derivatives, which are approximated by means of the Grünwald-Letnikov formula, has been considered. The method was used to solve a class of equations of this type (fractional cable equations) with free boundary conditions, Dirac's delta initial condition, and different fractional exponents. The error of the numerical method is compatible with the truncating error, which is of order $O(\Delta t) + O(\Delta x)^2$. It was also proved that the method is convergent. Besides, it was also found that a fractional von-Neumann stability analysis, which provides very precise stability conditions for standard fractional diffusion equations, leads also to a very accurate estimate of the stability conditions for cable equations.

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