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Research Article

Weak Subdifferential in Nonsmooth Analysis and Optimization

Şahlar F. Meherrem and Refet Polat

Department of Mathematics, Yasar University, 35100 Izmir, Turkey

Correspondence should be addressed to Şahlar F. Meherrem, sahlar.meherrem@yasar.edu.tr

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Some properties of the weak subdifferential are considered in this paper. By using the definition and properties of the weak subdifferential which are described in the papers (Azimov and Gasimov, 1999; Kasimbeyli and Mammadov, 2009; Kasimbeyli and Inceoglu, 2010), the author proves some theorems connecting weak subdifferential in nonsmooth and nonconvex analysis. It is also obtained necessary optimality condition by using the weak subdifferential in this paper.

1. Introduction

Nonsmooth analysis had its origins in the early 1970s when control theorists and nonlinear programmers attempted to deal with necessary optimality conditions for problems with nonsmooth data or with nonsmooth functions (such as the pointwise maximum of several smooth functions) that arise even in many problems with smooth data, convex functions, and max-type functions.

For this reason, it is necessary to extend the classical gradient for the smooth function to nonsmooth functions.

The first such canonical generalized gradient was the *generalized gradient* introduced by Clarke in his work [1]. He applied this generalized gradient systematically to nonsmooth problems in a variety of problems. But the nonconvex *basic or limiting normal cone* to closed sets and the corresponding subdifferential of lower semicontinuous extended-real-valued functions satisfying these requirements were introduced by Mordukhovich at the beginning of 1975. The corresponding subdifferential is called *Morduchovich subdifferential*. The initial motivation came from the intention to derive necessary optimality conditions for optimal control problems with endpoint geometric constraints by passing to the limit from free endpoint control problems, which are much easier to handle. This was published in [2]. Let us remark also that *Clarke's normal cone* is the closed convex closure of *Mordukhovich normal cone* [2].

Multifunctions (set-valued maps) naturally appear in various areas of nonlinear analysis, optimization, control theory, and mathematical economics. In Aubin and Frankowska's book [3] and in Mordukhovich's book is an excellent introduction to the theory of multifunctions. Coderivatives are convenient derivative-like objects for multifunctions and were introduced by Mordukhovich [2] motivated by applications to optimal control (see [4] for more discussions on the motivations and the relationship among coderivatives and other derivative-like objects for multifunctions). They are defined via "normal cones" to the graph of the multifunctions. *Approximate and geometric subdifferentials* are introduced by Ioffe in [5]. These subdifferentials are infinite-dimensional extensional of Mordukhovich subdifferential which may be different only in non-Asplund spaces. *Michel and Penot's derivatives* can be discussed in [6]. Rockafellar and Wets [7] provide a comprehensive overview of the field. The more information about the subdifferentials and coderivatives in nonsmooth analysis can be found also in [8]. The notion of the *weak subdifferential,* which is a generalization of the classic subdifferential, was introduced by [9].

In this paper, we investigate the relationships between the *Frechet lower subdifferential* and *weak subdifferentia and* we prove some theorems related to the weak subdifferential.

The paper is organized as follows. The definition of the weak subdifferential, strict differentiability, and the Frechet lower subdifferential are provided in the following section. In Section 2, the principal necessary theorems related to the properties of the weak subdifferential are also proved. In the third section, the necessary optimality conditions are proved. The final section presents some conclusions.

2. Main Results

To start, we provide some definitions which will be useful for some parts of the current paper.

Let $(X, \|\cdot\|_X)$ be a real normed space, and let X^* be a topological dual of X.

Definition 2.1 (strictly differentiable functions). F is called *strictly differentiable* at $\overline{x}$ (with a strict derivative $\Delta F(\overline{x})$) if

$$\operatorname*{Lim}_{u \to x, u' \to x} \frac{F(u') - F(u) - (\Delta F(x), u' - u)}{\|u' - u\|} = 0. \tag{2.1}$$

Definition 2.2 (weak subdifferential). Let $F : X \to R$ be a single-valued function, and let $\overline{x} \in X$ be a given point where $F(\overline{x})$ is finite. A pair $(x^*, c) \in X^* \times R_+$ is called *the weak subgradient* of F at $\overline{x}$ if

$$F(x) - F(\overline{x}) \geq (x^*, x - \overline{x}) - c\|x - \overline{x}\|, \quad \forall x \in X, \tag{2.2}$$

where R_+ is defined as a set of nonnegative real numbers.

The reader can find more information about the strict differentiable and the weak subdifferential, respectively, in [10, page 19] and [11, 12].

The set

$$\partial^w F(\overline{x}) = \{(x^*, c) \in X^* \times R_+ : F(x) - F(\overline{x}) \geq (x^*, x - \overline{x}) - c\|x - \overline{x}\|\}, \quad \forall x \in X, \tag{2.3}$$

is called the *weak subdifferential* for the F at the point $\overline{x} \in X$.

It is noted in [11, Remark 2.3, page 844] by the authors that when F is subdifferentiable at $\overline{x}$ (in the classical sense, for the convex functions), then F is also weakly subdifferentiable at $\overline{x}$, that is, if $x^* \in \partial F(\overline{x})$, then by definition $(x^*, c) \in \partial^w F(\overline{x})$ for every $c \geq 0$. It follows from the definition of the weak subdifferential that the pair $(x^*, c) \in X^* \times R_+$ is a weak subgradient of F at $\overline{x} \in X$, if there is a continuous (superlinear) concave function $g(x)$,

$$g(x) = (x^*, x - \overline{x}) + F(\overline{x}) - c\|x - \overline{x}\|, \tag{2.4}$$

such that $g(x) \leq F(x)$ for all $x \in X$ and $g(\overline{x}) = F(\overline{x})$.

But the authors do not note the boundedness of the gradient of the functional $g(x)$ which will be useful in estimating the subgradients for the finding extremum points for the nonsmooth functions. The following proof shows that the gradient of the functional $g(x)$ is also bounded. Let us prove this.

In fact, if we evaluate the gradient of the functional $g(x) = (x^*, x-\overline{x})+F(\overline{x})-c\|x-\overline{x}\|$, we can obtain $\nabla g(x) = x^*-c((x-\overline{x})/\|x-\overline{x}\|)$. Then, if we calculate the norm of the gradient $\nabla g(x)$ of the functional $g(x)$, we get $\|\nabla g(x)\| = \|x^*-c((x-\overline{x})/\|x-\overline{x}\|)\| \leq \|x^*\|+\|c((x-\overline{x})/\|x-\overline{x}\|)\| = \|x^*\| + c(\|x - \overline{x}\|/\|x - \overline{x}\|) = \|x^*\| + c \Rightarrow \|\nabla g(x)\| \leq \|x^*\| + c$ for all $x \in X$, and $x \neq \overline{x}$. Then we can add an extra useful and interesting property to Remark 2.3 in article [11, page 844] for the gradient of functional $g(x)$, namely, that is bounded by the nonnegative real number $\|x^*\| + c$.

Definition 2.3. The set

$$\partial F(\overline{x}) = \left\{x^* \in X^* : \operatorname*{Lim}_{x \to \overline{x}} \inf \frac{F(x) - F(\overline{x}) - (x^*, x - \overline{x})}{\|x - \overline{x}\|} \geq 0\right\} \tag{2.5}$$

is called a Frechet lower subdifferential of the function F at $\overline{x}$. Any element $x^* \in \partial F(\overline{x})$ is called the *Frechet lower subgradient* of the function $F(x)$.

Remark 2.4. In different books, the definition of Frechet lower subdifferential is given different names by the authors, such as presubdifferential or the Frechet subdifferential in [10, page 90, I volume] and the Frechet lower subdifferential in [7]. More information about Frechet upper and lower subdifferential can be found in [10, volume 1].

Let us note that the Frechet subdifferential may be empty for some functions.

Example 2.5. Take $F : R \to R : F(x) = -|x|,\ x \in R$. Easy calculation shows that the Frechet subdifferential for above example at the point zero is empty, that is, $\partial F(0) = \varnothing$.

Note that there is also a symmetric counterpart of the Frechet lower subdifferential, which is the Frechet upper subdifferential, described as $\partial F^+(\overline{x}) = \{x^* \in X^* : \operatorname{Lim}_{x \to \overline{x}} \inf((F(x) - F(\overline{x}) - (x^*, x - \overline{x}))/\|x - \overline{x}\|) \leq 0\}$.

The Frechet lower subdifferential and Frechet upper subdifferential are not empty for the function F if and only if the function F is Frechet differentiable. For more information about the Frechet subdifferentials (upper and lower), the reader can consult [10, page 90] and [13] and its applications to the necessary optimality conditions in [14, Chapters 5 and 6].

Theorem 2.6. *If x^* is a Frechet lower subgradient (Definition 2.3) for the functional $F : X \to R$ at the point $\overline{x}$, then the couple (x^*, c) is a weak subdifferential for the functional $F(x)$ at $\overline{x}$ for any nonnegative $c \in R_+$.*

Proof. Let x^* be a Frechet subgradient for the functional $F : X \to R$ at the point $\overline{x}$, that is, $x^* \in \partial F(\overline{x})$. Then by using the definition (Definition 2.3) of the Frechet lower subdifferential provided above, we can write

$$\frac{F(x) - F(\overline{x}) - (x^*, x - \overline{x})}{\|x - \overline{x}\|} \geq 0, \tag{2.6}$$

(due to, Definition 2.3, $\operatorname{Lim}_{x \to \overline{x}} \inf(\cdot) \geq 0$). Then it reduces easily to the inequality

$$F(x) - F(\overline{x}) - (x^*, x - \overline{x}) \geq o\|x - \overline{x}\|. \tag{2.7}$$

It is easy to show that the right side $o\|x - \overline{x}\|$ of the last inequality is not less than $-c\|x - \overline{x}\|$ for any nonnegative c. Then it follows that

$$F(x) - F(\overline{x}) - (x^*, x - \overline{x}) \geq o\|x - \overline{x}\| \geq -c\|x - \overline{x}\|. \tag{2.8}$$

By using the definition of the weak subdifferential (Definition 2.2), we can say that (x^*, c) is a weak subdifferential for the functional $F(x)$ at the point $\overline{x}$. □

Theorem 2.7. *Let $F(x)$ be a finite at $\overline{x}, h(x) \in C^1$ (continuously differentiable function) in a neighborhood of $\overline{x}$. Then if $(x^*, c) \in \partial^w (F + h)(\overline{x})$, then $(x^* - h'(\overline{x}), -2c) \in \partial^w F(\overline{x})$, that is, $(x^* - h'(\overline{x}), 2c)$ is the weak subdifferential of the function $F(x)$ at the point $\overline{x}$.*

Proof. The inequality (2.2) applied to the function $-h(x)$ implies the existence of the constant c such that

$$-h(x) + h(\overline{x}) + c\|x - \overline{x}\| \geq \left(-h'(\overline{x}), x - \overline{x}\right), \quad \forall x \in X. \tag{2.9}$$

(It is easy to check that, for the differentiable functions, the weak subgradient and its derivative coincide, i.e., $x^* = h'$.)

Since $(x^*, c) \in \partial^w (F + h)(\overline{x})$, if we imply the inequality (2.2) for the function $F + h$, we can obtain

$$(F(x) + h(x)) - (F(\overline{x}) + h(\overline{x})) + c\|x - \overline{x}\| \geq (x^*, x - \overline{x}), \tag{2.10}$$

for all $x \in X$ near $\overline{x}$.

Upon adding the inequalities (2.9) and (2.10) side-by-side, we arrive with

$$F(x) - F(\overline{x}) + 2c\|x - \overline{x}\| \geq (x^* - h'(\overline{x}), x - \overline{x}) \Longrightarrow F(x) - F(\overline{x}) \geq (x^* - h'(\overline{x}), x - \overline{x}) - 2c\|x - \overline{x}\|. \tag{2.11}$$

The last inequality means that $(x^* - h'(\overline{x}), 2c) \in \partial^w F(\overline{x})$ that is, the couple $(x^* - h'(\overline{x}), 2c)$ is the weak subdifferential of the function $F(x)$ at the point $\overline{x}$. □

Theorem 2.8. *Let $F(x)$ be finite at $\overline{x}$, and (x^*, c) the weak subdifferential for $F(x)$ at $\overline{x}$ provided that $\|x^*\| \geq c$, and let one take any real number l which satisfying $c \leq l \leq \|x^*\|$. Then, for any x, where $x = (x^*/l) + \overline{x}$, the inequality $F(x) \geq F(\overline{x})$ holds.*

Proof. By using the definition of the weak subdifferential (Definition 2.2), it is easy check that if the couple (x^*, c) is the weak subdifferential for the function F, then, for any real $l \geq c$, the pair (l, c) is also the weak subdifferential for the function F. Then we can write

$$F(x) - F(\overline{x}) \geq (x^*, x - \overline{x}) - c\|x - \overline{x}\| \geq (x^*, x - \overline{x}) - l\|x - \overline{x}\|. \tag{2.12}$$

From the relation $x = (x^*/l) + \overline{x}$, it is easy to define $x^* = l(x - \overline{x})$. If, in the right side of the last inequality, we substitute x^* with $l(x - \overline{x})$, then we get

$$F(x) - F(\overline{x}) \geq l\|x - \overline{x}\|^2 - l\|x - \overline{x}\| = l\|x - \overline{x}\|(\|x - \overline{x}\| - 1). \tag{2.13}$$

Since $c \leq l \leq \|x^*\|$ and $x = (x^*/l) + \overline{x}$, then $\|x - \overline{x}\| = \|x^*/l\| \geq 1$. If we consider the estimate $\|x - \overline{x}\| \geq 1$ in the inequality

$$F(x) - F(\overline{x}) \geq l\|x - \overline{x}\|^2 - l\|x - \overline{x}\| = l\|x - \overline{x}\|(\|x - \overline{x}\| - 1), \tag{2.14}$$

we can obtain $F(x) \geq F(\overline{x})$. □

Theorem 2.9. *If F is strictly differentiable at $\overline{x}$ with a derivative $\Delta F(\overline{x})$, then, for any $(x^*, c) \in \partial^w F(u)$, there exists $\delta > 0$ such that $x^* \in \Delta F(\overline{x}) + 2cB^*$, where $u \in B_\delta(\overline{x})$-sphere with radius δ and B^*-unit sphere, provided that $\partial^w F(u) \neq 0$ for any $u \in B_\delta(\overline{x})$.*

Proof. It follows from the definition of the strict differentiable (Definition 2.1) that for any $\varepsilon > 0$ there exist $\delta > 0$ such that

$$|F(\overline{u}) - F(u) - (\Delta F(\overline{x}), \overline{u} - u)| \leq \varepsilon\|\overline{u} - u\|, \quad \forall \overline{u}, u \in B_\delta(\overline{x}). \tag{2.15}$$

Let us take $u \in B_\delta(\overline{x})$ and assume that $(x^*, c) \in \partial^w F(u)$. Then it follows from the definition of the weak subdifferential that

$$F(\overline{u}) - F(u) - (x^*, \overline{u} - u) \geq -c\|\overline{u} - u\|. \tag{2.16}$$

Let us substitute ε with the c in the inequality (2.15), that is, put $\varepsilon = c$ in (2.15). Then (2.15) can be formulated as follows.

For $c > 0$, there exists $\delta > 0$ with the condition that

$$|F(\overline{u}) - F(u) - (\Delta F(\overline{x}), \overline{u} - u)| \leq c\|\overline{u} - u\|, \quad \forall \overline{u}, u \in B_\delta(\overline{x}). \tag{2.17}$$

If we estimate the absolute value in (2.17) using the above, we can get for such c there exists $\delta > 0$ which satisfies the following inequality

$$F(\overline{u}) - F(u) - (\Delta F(\overline{x}), \overline{u} - u) \leq c\|\overline{u} - u\|, \quad \forall \overline{u}, u \in B_\delta(\overline{x}). \tag{2.18}$$

Let us multiply both sides of the inequality (2.16) by "minus". Then we obtain

$$-F(\overline{u}) + F(u) + (x^*, \overline{u} - u) \leq c\|\overline{u} - u\|, \quad \forall \overline{u}, u \in B_\delta(\overline{x}). \tag{2.19}$$

Adding up the inequalities (2.18) and (2.19) side-by-sides, we get the following estimate:

$$(\Delta F(\overline{x}) - x^*, \overline{u} - u) \leq c\|\overline{u} - u\| + c\|\overline{u} - u\| = 2c\|\overline{u} - u\|. \tag{2.20}$$

Dividing both sides of the relations (2.2) by $\|\overline{u} - u\|$, we get

$$\left(\Delta F(\overline{x}) - x^*, \frac{\overline{u} - u}{\|\overline{u} - u\|}\right) \leq 2c, \quad \text{for } \forall \overline{u}, u \in B_\delta(\overline{x}). \tag{2.21}$$

If we take the supremum with the respect to the variables $\overline{u}$ and u in the last inequality, then (2.21) reduces to

$$\sup\left(\Delta F(\overline{x}) - x^*, \frac{\overline{u} - u}{\|\overline{u} - u\|}\right) \leq 2c, \quad \text{for } \forall \overline{u}, u \in B_\delta(\overline{x}). \tag{2.22}$$

If we consider the norm of the functional or operator [7], then we get

$$\|\Delta F(\overline{x}) - x^*\| \leq 2c, \tag{2.23}$$

which can be reduced to the following form:

$$x^* \in \Delta F(\overline{x}) + 2c. \tag{2.24}$$

This is the end of the proof. □

3. Necessary Optimality Conditions via the Weak Subdifferential

In this section, we present the necessary optimality condition for the weakly differentiable function.

Given a function $F : X \to R$ finite at the reference point and nonempty subset S of the normed space X, we consider the following minimization problem:

$$\text{minimize } F(x) \quad \text{subject to } x \in S \subset X. \tag{3.1}$$

The following is a well-known optimality condition in nonsmooth convex analysis (see [15, Proposition 1.8.1, page 168]) which states that if $F : R^n \to R$ is a convex function, then vector $\overline{x}$ minimizes F over a convex set $S \in R^n$ if and only if there exists a subgradient $x^* \in \partial F(\overline{x})$ such that

$$(x^*, x - \overline{x}) \geq 0, \quad \forall x \in S, \tag{3.2}$$

where

$$0 \in \partial F(\overline{x}) = \{x^* \in R^n : F(x) - F(\overline{x}) \geq (x^*, x - \overline{x})\}, \quad \forall x \in R^n. \tag{3.3}$$

But the optimality conditions in [16, Proposition 1.8.1, page 168] are proved for convex functions.

Let us formulate the necessary optimality conditions for problem (3.1) by using weak subdifferential in the case where the minimizing functional is nonconvex. In fact, the weak subdifferential is given for nonconvex functions, while the classical subgradient does not enable us to find the minimum point in cases where the minimizing function is nonconvex [11, 12].

The interested reader can find out more about the convex function and the subdifferential for convex functions in [1, 7].

Theorem 3.1. *Let the function $F(x)$ have a minimum at the point $\overline{x} \in S$ in problem* (3.1). *If the function $F(x)$ is weakly subdifferentiable at $\overline{x}$, that is, $\partial^w F(\overline{x}) \neq 0$, then the couple $(0, c)$ belongs to the $\partial^w F(\overline{x})$, for any nonnegative real number c.*

Proof. Let the function $F(x)$ take minimum value at the point $\overline{x}$. If $F(x)$ is weakly subdifferentiable at the point $\overline{x}$, then, by using Definition 2.2, we can write that

$$0 \neq \partial^w F(\overline{x}) = \{(x^*, c) \in X^* \times R_+ : F(x) - F(\overline{x}) \geq (x^*, x - \overline{x}) - c\|x - \overline{x}\|\}, \quad \forall x \in S. \tag{3.4}$$

Since $F(x)$ takes its minimum at the point $\overline{x}$ over the set $S \in X$, then we can write that

$$F(x) \geq F(\overline{x}), \quad \forall x \in S. \tag{3.5}$$

We can reduce last inequality to the following form:

$$F(x) - F(\overline{x}) \geq 0 = (0, x - \overline{x}) \geq (0, x - \overline{x}) - c\|x - \overline{x}\|, \tag{3.6}$$

for all $x \in S$ and nonnegative real number $c \geq 0$.

Comparing the definition of the weak subdifferential (Definition 2.2) with the inequality (3.6), we can say that

$$(0, c) \in \partial F^{w}(\overline{x}). \tag{3.7}$$

□

4. Conclusion

Comparison of the current status of smooth subdifferential theory and the corresponding smooth theory reveals a glaring lack of a second order theory. In finite dimensional space a beautiful sum rule for a second-order derivative-like object close to the fuzzy sum rule was derived in [17]. There are many other approaches and results in nonsmooth optimizations and variational analysis in infnite-dimensional spaces. In infinite dimensions the field is little developed. Applications in optimal control, mathematical programming, and other related problems are critical for the healthy development of further nonsmooth analysis theory.

A further research topic is also the development of methods for obtaining the optimality condition for the nonsmooth optimal control problem by using the weak subdifferential. Open problems, including the existence of the solution, the exploration of the necessary conditions in the nonsmooth case, the solution of the HJB (Hamilton-Jacobi-Belmann) equation, and the use of numerical methods, still present considerable challenges.

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