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Research Article

Hyers-Ulam Stability of Power Series Equations

M. Bidkham,[1, 2] H. A. Soleiman Mezerji,[1, 2] and M. Eshaghi Gordji[1, 2]

[1] *Department of Mathematics, Semnan University, P.O. Box 35195-363, Semnan, Iran*
[2] *Center of Excellence in Nonlinear Analysis and Applications (CENAA), Semnan University, Semnan, Iran*

Correspondence should be addressed to M. Eshaghi Gordji, madjid.eshaghi@gmail.com

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We prove the Hyers-Ulam stability of power series equation $\sum_{n=0}^{\infty} a_n x^n = 0$, where a_n for $n = 0, 1, 2, 3, \ldots$ can be real or complex.

1. Introduction and Preliminaries

A classical question in the theory of functional equations is that "when is it true that a function which approximately satisfies a functional equation $\mathcal{E}$ must be somehow close to an exact solution of $\mathcal{E}$." Such a problem was formulated by Ulam [1] in 1940 and solved in the next year for the Cauchy functional equation by Hyers [2]. It gave rise to the *Hyers-Ulam stability* for functional equations.

In 1978, Th. M. Rassias [3] provided a generalization of Hyers' theorem by proving the existence of unique linear mappings near approximate additive mappings. On the other hand, J. M. Rassias [4–6] considered the Cauchy difference controlled by a product of different powers of norm. This new concept is known as generalized Hyers-Ulam stability of functional equations (see also [7–10] and references therein).

Recently, Li and Hua [11] discussed and proved the Hyers-Ulam stability of a polynomial equation

$$x^n + \alpha x + \beta = 0, \tag{1.1}$$

where $x \in [-1, 1]$ and proved the following.

Theorem 1.1. *If $|\alpha| > n$, $|\beta| < |\alpha| - 1$ and $y \in [-1,1]$ satisfies the inequality*

$$\left|y^n + \alpha y + \beta\right| \leq \varepsilon, \tag{1.2}$$

then there exists a solution $v \in [-1,1]$ of (1.1) such that

$$\left|y - v\right| \leq K\varepsilon, \tag{1.3}$$

where $K > 0$ is constant.

They also asked an open problem whether the real polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0 \tag{1.4}$$

has Hyers-Ulam stability for the case that this real polynomial equation has some solution in $[a,b]$.

In this paper we establish the Hyers-Ulam-Rassias stability of power series with real or complex coefficients. So we prove the generalized Hyers-Ulam stability of equation

$$f(z) = 0, \tag{1.5}$$

where f is any analytic function. First we give the definition of the generalized Hyers-Ulam stability.

Definition 1.2. Let p be a real number. We say that (1.7) has the generalized Hyers-Ulam stability if there exists a constant $K > 0$ with the following property:

for every $\varepsilon > 0$, $y \in [-1,1]$ if

$$\left|\sum_{n=0}^{\infty} a_n y^n\right| \leq \varepsilon\left(\sum_{n=0}^{\infty} \frac{|a_n|^p}{2^n}\right), \tag{1.6}$$

then there exists some $x \in [-1,1]$ satisfying

$$\sum_{n=0}^{\infty} a_n x^n = 0 \tag{1.7}$$

such that $|y - x| \leq K\varepsilon$. For the complex coefficients, $[-1,1]$ can be replaced by closed unit disc

$$D = \{z \in \mathbb{C};\ |z| \leq 1\}. \tag{1.8}$$

2. Main Results

The aim of this work is to investigate the generalized Hyers-Ulam stability for (1.7).

Theorem 2.1. *If*

$$\sum_{n=0,n\neq 1}^{\infty} |a_n| < |a_1|, \tag{2.1}$$

$$\sum_{n=2}^{\infty} n|a_n| < |a_1|, \tag{2.2}$$

then there exists an exact solution $v \in [-1,1]$ *of* (1.7).

Proof. If we set

$$g(x) = \frac{-1}{a_1}\left(\sum_{n=0,n\neq 1}^{\infty} a_n x^n\right), \tag{2.3}$$

for $x \in [-1,1]$, then we have

$$\begin{aligned} |g(x)| &= \frac{1}{|a_1|}\left|\sum_{n=0,n\neq 1}^{\infty} a_n x^n\right| \\ &\leq \frac{1}{|a_1|}\left(\sum_{n=0,n\neq 1}^{\infty} |a_n|\right) \\ &\leq 1 \end{aligned} \tag{2.4}$$

by (2.1).

Let $X = [-1,1]$, $d(x,y) = |x-y|$. Then (X,d) is a complete metric space and g map X to X. Now, we will show that g is a contraction mapping from X to X. For any $x, y \in X$, we have

$$\begin{aligned} d(g(x), g(y)) &= \left|\frac{1}{a_1}\left(-a_0 - a_2x^2 - \cdots\right) - \frac{1}{a_1}\left(-a_0 - a_1y^2 - \cdots\right)\right| \\ &\leq \frac{1}{|a_1|}|x-y|\left\{\sum_{n=2}^{\infty} n|a_n|\right\}. \end{aligned} \tag{2.5}$$

For $x, y \in [-1,1]$, $x \neq y$, from (2.2), we obtain

$$d(g(x), g(y)) \leq \lambda d(x,y), \tag{2.6}$$

where

$$\lambda = \frac{\sum_{n=2}^{\infty} n|a_n|}{|a_1|} < 1. \tag{2.7}$$

Thus g is a contraction mapping from X to X. By the Banach contraction mapping theorem, there exists a unique $v \in X$, such that

$$g(v) = v. \tag{2.8}$$

Hence, (1.7) has a solution on $[-1, 1]$. □

Theorem 2.2. *Under the conditions of Theorem 2.1,* (1.7) *has the generalized Hyers-Ulam stability.*

Proof. Let $\varepsilon > 0$ and $y \in [-1, 1]$ be such that

$$\left|\sum_{n=0}^{\infty} a_n y^n\right| \leq \varepsilon\left(\sum_{n=0}^{\infty} \frac{|a_n|^p}{2^n}\right). \tag{2.9}$$

We will show that there exists a constant K independent of ε, v, and y such that

$$|y - v| \leq K\varepsilon \tag{2.10}$$

for some $v \in [-1, 1]$ satisfying (1.7).

Let us introduce the abbreviation $K = 2/(|a_1|^{1-p}(1 - \lambda))$. Then

$$\begin{aligned} |y - v| &= |y - g(y) + g(y) - g(v)| \leq |y - g(y)| + |g(y) - g(v)| \\ &\leq \left|y - \left(\frac{-1}{a_1} \sum_{n=0, n\neq 1}^{\infty} a_n y^n\right)\right| + \lambda|y - v| \\ &= \frac{1}{|a_1|}\left|\sum_{n=0}^{\infty} a_n y^n\right| + \lambda|y - v|. \end{aligned} \tag{2.11}$$

Thus, we have

$$\begin{aligned} |y - v| &\leq \frac{1}{|a_1|(1 - \lambda)}\left|\sum_{n=0}^{\infty} a_n y^n\right| \leq \frac{1}{|a_1|(1 - \lambda)}\left(\sum_{n=0}^{\infty} \frac{|a_n|^p}{2^n}\right)\epsilon \\ &\leq \frac{1}{|a_1|(1 - \lambda)}\left(\sum_{n=0}^{\infty} \frac{|a_1|^p}{2^n}\right)\epsilon \\ &\leq K\varepsilon \end{aligned} \tag{2.12}$$

by (2.9) and so the result follows. □

Next, for equation of complex power series

$$\sum_{n=0}^{\infty} a_n z^n = 0, \tag{2.13}$$

as an application of Rouche's theorem, we prove the following theorem which is much better than above result. In fact, we prove the following.

Theorem 2.3. *If*

$$\sum_{n=0, n\neq 1}^{\infty} |a_n| < |a_1|. \tag{2.14}$$

Then there exists an exact solution in open unit disc for (2.13).

Proof. If we set

$$g(z) = \frac{-1}{a_1}\left(\sum_{n=0, n\neq 1}^{\infty} a_n z^n\right), \tag{2.15}$$

for $|z| \leq 1$. Such as above we have

$$\begin{aligned} |g(z)| &= \frac{1}{|a_1|}\left|\sum_{n=0, n\neq 1}^{\infty} a_n z^n\right| \\ &\leq \frac{1}{|a_1|}\left(\sum_{n=0, n\neq 1}^{\infty} |a_n|\right), \quad \text{for } |z| \leq 1 \\ &< 1 \end{aligned} \tag{2.16}$$

by (2.14).

Since $|g(z)| < 1$ for $|z| = 1$, hence for $|g(z)| < |-z| = 1$ and by Rouche's theorem, we observe that $g(z) - z$ has exactly one zero in $|z| < 1$ which implies that g has a unique fixed point in $|z| < 1$. □

Corollary 2.4. *Under the conditions of Theorem 2.1,* (2.13) *has the generalized Hyers-Ulam stability.*

For $R \geqslant 1$, we have the following corollary.

Corollary 2.5. *If*

$$\sum_{n=0, n\neq 1}^{\infty} |a_n| R^n < |a_1| R, \tag{2.17}$$

then there exists an exact solution in $\{z \in \mathbb{C};\ |z| < R\}$ *for* (2.13).

The proof is similar to previous and details are omitted.

Remark 2.6. By the similar way, one can easily prove the generalized Hyers-Ulam stability of (1.7) on any finite interval $[a, b]$.

Remark 2.7. By replacing $a_n = f^{(n)}(0)$ in (2.14), we can prove the generalized Hyers-Ulam stability for (1.5).

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