Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2011, Article ID 165160, 15 pages
doi:10.1155/2011/165160

Research Article

Symplectic Analytical Solutions for the Magnetoelectroelastic Solids Plane Problem in Rectangular Domain

Xiao-Chuan Li[1] and Wei-An Yao[2]

[1] *School of Architecture and Civil Engineering, Shenyang University of Technology, Shenyang 110870, China*
[2] *State Key Laboratory of Structural Analysis for Industrial Equipment, Dalian University of Technology, Dalian 116023, China*

Correspondence should be addressed to Xiao-Chuan Li, li_xiaochuan@tom.com

Received 20 September 2010; Accepted 12 January 2011

Academic Editor: Pablo Gonza'lez-Vera

The transversely isotropic magnetoelectroelastic solids plane problem in rectangular domain is derived to Hamiltonian system. In symplectic geometry space with the origin variables—displacements, electric potential, and magnetic potential, as well as their duality variables—lengthways stress, electric displacement, and magnetic induction, on the basis of the obtained eigensolutions of zero-eigenvalue, the eigensolutions of nonzero-eigenvalues are also obtained. The former are the basic solutions of Saint-Venant problem, and the latter are the solutions which have the local effect, decay drastically with respect to distance, and are covered in the Saint-Venant principle. So the complete solution of the problem is given out by the symplectic eigensolutions expansion. Finally, a few examples are selected and their analytical solutions are presented.

1. Introduction

Magnetoelectroelastic solids are a kind of the emerging functional composite material. Due to possessing mechanical, electric and magnetic field coupling capacity, these materials show better foreground in many high-tech areas (see [1]). They not only convert energy from one form to the other (among magnetic, electric, and mechanical energies), but also possess some new properties of magnetoelectric effect, which are not found in single-phase piezoelectric or piezomagnetic materials. Over the years, a large amount of studies have been done in mechanics, materials science, and physics fields (see [2–6]), and it has become a new cross subject.

Due to multifields coupling, the magnetoelectroelastic solids problem is solved more difficultly than elasticity one. Zhong et al. (see [7, 8]) introduced a symplectic dual method based on the conservative Hamiltonian system to solve the elastic problem which is different from the traditional semi-inverse solution method. The complete solutions space can be obtained and a satisfactory solution can be obtained under the boundary conditions (see [9–11]). The symplectic dual method has been developed for studying piezoelectric effects (see [12]) and magnetoelectroelastic problems (see [13]).

With the symplectic approach, the plane problem of magnetoelectroelastic solids in rectangular domain is derived into the Hamiltonian system by means of the generalized variable principle of the magnetoelectroelastic solids. In symplectic geometry space with the origin variables—displacements, electric potential, and magnetic potential, as well as their duality variables—lengthways stress, electric displacement, and magnetic induction, symplectic dual equations are employed. Yao and Li have obtained all the eigensolutions of zero-eigenvalue, which have their specific physical interpretation and are the basic solutions of plane Saint-Venant problem in [13]. This paper gets the eigensolutions of nonzero-eigenvalues, which are the solutions that have the local effect, decay drastically with respect to distance, and are covered in the Saint-Venant principle. So the complete solution of the problem is given out by the symplectic eigensolutions expansion. Finally, a few examples are selected and their analytical solutions are presented.

2. Functional Equations and Boundary Conditions

The transversely isotropic magnetoelectroelastic solids are studied here, with the z-axis being the polar direction. If geometry size, load, and so forth, in y direction satisfy the given condition, the problem can be simplified as a plane problem in the xoz plane. The functional equations of the plane problem are as follows (see [14]).

(1) Governing equations:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} + f_x = 0, \qquad \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \sigma_z}{\partial z} + f_z = 0, \qquad \frac{\partial D_x}{\partial x} + \frac{\partial D_z}{\partial z} = \rho, \qquad \frac{\partial B_x}{\partial x} + \frac{\partial B_z}{\partial z} = 0. \tag{2.1}$$

(2) Gradient equations:

$$\begin{gathered} \varepsilon_x = \frac{\partial u}{\partial x}, \qquad \varepsilon_z = \frac{\partial w}{\partial z}, \qquad \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, \\ E_x = -\frac{\partial \phi}{\partial x}, \qquad E_z = -\frac{\partial \phi}{\partial z}, \qquad H_x = -\frac{\partial \psi}{\partial x}, \qquad H_z = -\frac{\partial \psi}{\partial z}. \end{gathered} \tag{2.2}$$

(3) Constitutive equations:

$$\begin{gathered} \sigma_x = c_{11}\varepsilon_x + c_{13}\varepsilon_z - e_{31}E_z - q_{31}H_z, \qquad \sigma_z = c_{13}\varepsilon_x + c_{33}\varepsilon_z - e_{33}E_z - q_{33}H_z, \\ \tau_{xz} = c_{44}\gamma_{xz} - e_{15}E_x - q_{15}H_x, \\ D_x = e_{15}\gamma_{xz} + \kappa_{11}E_x + \alpha_{11}H_x, \qquad D_z = e_{31}\varepsilon_x + e_{33}\varepsilon_z + \kappa_{33}E_z + \alpha_{33}H_z, \\ B_x = q_{15}\gamma_{xz} + \alpha_{11}E_x + \mu_{11}H_x, \qquad B_z = q_{31}\varepsilon_x + q_{33}\varepsilon_z + \alpha_{33}E_z + \mu_{33}H_z, \end{gathered} \tag{2.3}$$

where u, w are displacement components in the x, z direction, respectively; σ_x, σ_z, and τ_{xz} are stress components, respectively; D_x, D_z, and ϕ are electric displacement components and electric potential, respectively; B_x, B_z, and ψ are magnetic induction components and magnetic potential, respectively; f_x, f_z, and ρ are body force components and density of free charges in region V, respectively. c_{ij}, κ_{ij}, and μ_{ij} are elastic modus, dielectric constant, and magnetic constants, respectively; e_{ij}, q_{ij}, and α_{ij} are piezoelectric, piezomagnetic, and electromagnetic constants, respectively. Equation (2.3) can also be rewritten as follows:

$$\begin{gathered}
\varepsilon_x = s_{11}\sigma_x + s_{13}\sigma_z + d_{31}D_z + b_{31}B_z, \qquad \varepsilon_z = s_{13}\sigma_x + s_{33}\sigma_z + d_{33}D_z + b_{33}B_z, \\
\gamma_{xz} = s_{44}\tau_{xz} + d_{15}D_x + b_{15}B_x, \\
E_x = -d_{15}\tau_{xz} + \lambda_{11}D_x + \beta_{11}B_x, \qquad E_z = -d_{31}\sigma_x - d_{33}\sigma_z + \lambda_{33}D_z + \beta_{33}B_z, \\
H_x = -b_{15}\tau_{xz} + \beta_{11}D_x + \nu_{11}B_x, \qquad H_z = -b_{31}\sigma_x - b_{33}\sigma_z + \beta_{33}D_z + \nu_{33}B_z.
\end{gathered} \tag{2.4}$$

The rectangle domain as showing in Figure 1 is studied in this paper

$$V : 0 \le z \le l, \quad -h \le x \le h. \tag{2.5}$$

And the boundary conditions are expressed as (see [15])

$$\begin{gathered}
\sigma_x = \overline{F}_{x1}(z), \quad \tau_{xz} = \overline{F}_{z1}(z), \quad D_x = \overline{D}_{x1}(z), \quad B_x = \overline{B}_{x1}(z) \quad \text{on } x = -h, \\
\sigma_x = \overline{F}_{x2}(z), \quad \tau_{xz} = \overline{F}_{z2}(z), \quad D_x = \overline{D}_{x2}(z), \quad B_x = \overline{B}_{x2}(z) \quad \text{on } x = h.
\end{gathered} \tag{2.6}$$

In addition, there are body force and density of free charges in region V.

On $z = 0, l$, the boundary conditions are (see [15])

$$w = \overline{w}, \quad u = \overline{u}, \quad \phi = \overline{\phi}, \quad \psi = \overline{\psi} \quad \text{on } z = 0 \text{ or } l. \tag{2.7}$$

Or

$$\sigma_z = \overline{\sigma}_z, \quad \tau_{xz} = \overline{\tau}_{xz}, \quad D_z = \overline{D}_z, \quad B_z = \overline{B}_z \quad \text{on } z = 0 \text{ or } l. \tag{2.8}$$

3. Duality Equation in Symplectic Geometry Space and Separation of Variables (See [13])

Consider rectangle domain as showing in Figure 1. At first, the coordinate z here is employed to simulate the time variable in the Hamiltonian system, and a symbol "$\cdot$" in the following derivation will be used denoting the differential with respect to z, that is, $(\dot{\ })=\partial/\partial z$. For the sake of simplicity, the notations σ, τ, D, and B are introduced to represent σ_z, τ_{xz}, D_z, and B_z, respectively. If we omit body forces and density of free charges, a dual equation with the full state function vector is given as

$$\dot{\mathbf{v}} = \mathbf{H}\mathbf{v}, \tag{3.1}$$

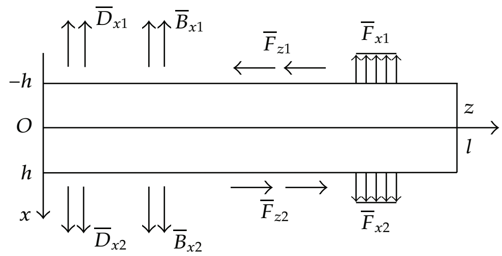

Figure 1: The rectangular domain problems on magnetoelectroelastic solids.

and $\mathbf{H}$ is the operator matrix in [13]

$$\mathbf{H}=\begin{bmatrix} 0 & C_{10}\dfrac{\partial}{\partial x} & 0 & 0 & C_{11} & 0 & C_{14} & C_{15} \\ -\dfrac{\partial}{\partial x} & 0 & -C_{21}\dfrac{\partial}{\partial x} & -C_{22}\dfrac{\partial}{\partial x} & 0 & C_{17} & 0 & 0 \\ 0 & C_{18}\dfrac{\partial}{\partial x} & 0 & 0 & C_{14} & 0 & -C_{12} & -C_{16} \\ 0 & C_{19}\dfrac{\partial}{\partial x} & 0 & 0 & C_{15} & 0 & -C_{16} & -C_{13} \\ 0 & 0 & 0 & 0 & 0 & -\dfrac{\partial}{\partial x} & 0 & 0 \\ 0 & -C_{20}\dfrac{\partial^2}{\partial x^2} & 0 & 0 & C_{10}\dfrac{\partial}{\partial x} & 0 & C_{18}\dfrac{\partial}{\partial x} & C_{19}\dfrac{\partial}{\partial x} \\ 0 & 0 & C_{23}\dfrac{\partial^2}{\partial x^2} & -C_{24}\dfrac{\partial^2}{\partial x^2} & 0 & -C_{21}\dfrac{\partial}{\partial x} & 0 & 0 \\ 0 & 0 & -C_{24}\dfrac{\partial^2}{\partial x^2} & C_{25}\dfrac{\partial^2}{\partial x^2} & 0 & -C_{22}\dfrac{\partial}{\partial x} & 0 & 0 \end{bmatrix}, \tag{3.2}$$

where the constants $C_{10} \sim C_{25}$ are listed in the appendix.

Consider the following boundary conditions on two sides in [13]:

$$\begin{gathered} \frac{1}{s_{11}}\left(\frac{\partial u}{\partial x}-s_{13}\sigma-d_{31}D-b_{31}B\right)=0, \qquad \tau=0, \\ \frac{1}{\Delta}\left(-\nu_{11}\frac{\partial\phi}{\partial x}+\beta_{11}\frac{\partial\psi}{\partial x}+C_1\tau\right)=0, \qquad \frac{1}{\Delta}\left(\beta_{11}\frac{\partial\phi}{\partial x}-\lambda_{11}\frac{\partial\psi}{\partial x}+C_2\tau\right)=0, \quad \text{on } x=\pm h, \end{gathered} \tag{3.3}$$

where constant Δ, C_1, and C_2 are listed in the appendix. Equation (3.1) can usually be solved by using the method of partition of variables. Let

$$\mathbf{v}(x,z)=\xi(z)\mathbf{Y}(x), \tag{3.4}$$

and substitute (3.4) into (3.1); the following equations can be obtained:

$$\mathbf{H}\mathbf{Y}(x) = \mu\mathbf{Y}(x), \qquad \xi(z) = e^{\mu z}, \tag{3.5}$$

where μ is the eigenvalue and $\mathbf{Y}(x)$ is the eigenfunction vector, which satisfies the boundary condition (3.3) on $x = \pm h$.

The full state vectors $\mathbf{v}$ form a symplectic space according to the following definition of symplectic inner product:

$$\left\langle \mathbf{v}^{(1)}, \mathbf{v}^{(2)} \right\rangle \overset{\text{def}}{=} \int_{-h}^{h} \mathbf{v}^{(1)T}\mathbf{J}\mathbf{v}^{(2)}dx, \qquad \mathbf{J} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_4 \\ -\mathbf{I}_4 & \mathbf{0} \end{bmatrix}. \tag{3.6}$$

If only $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ meet the requirement of (3.3), the following invariant is obtained as

$$\left\langle \mathbf{v}^{(1)}, \mathbf{H}\mathbf{v}^{(2)} \right\rangle \equiv \left\langle \mathbf{v}^{(2)}, \mathbf{H}\mathbf{v}^{(1)} \right\rangle. \tag{3.7}$$

Therefore, the operator matrix $\mathbf{H}$ is the Hamiltonian operator matrix in the symplectic geometry space. So its eigenvalues have some characteristics, that is, if μ is an eigenvalue, then $-\mu$ must also be one, and the eigenfunction vectors satisfy the symplectic adjoint orthogonal relationship. After eigenvalues and eigenfunction vectors of $\mathbf{H}$ are given, the origin problem can be solved by the method of eigenfunction expansion (see [7, 8]).

4. Eigenfunction Vectors of Eigenvalue Zero (See [13])

With the free boundary condition at both sides($x = \pm h$), the magnetoelectroelastic solids plane problem in rectangular domain exists eigenvalue zero. The eigenvalue zero is a special eigenvalue of the Hamiltonian operator matrix, which eigenfunction vectors are not only the fundamental solution with the special physical significance but also a nondecaying solution. Solve the following eigenequation with conditions (3.3):

$$\mathbf{H}\mathbf{Y}(x) = 0. \tag{4.1}$$

After obtaining the fundamental eigensolutions and eigensolutions in Jordan form, the solutions of (3.1) can be given as

$$\begin{gathered}
\mathbf{v}_{0f}^{(0)} = \begin{pmatrix}1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}^T, \qquad \mathbf{v}_{0s}^{(0)} = \begin{pmatrix}0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}^T, \\
\mathbf{v}_{0t}^{(0)} = \begin{pmatrix}0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\end{pmatrix}^T, \qquad \mathbf{v}_{0r}^{(0)} = \begin{pmatrix}0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\end{pmatrix}^T, \\
\mathbf{v}_{0f}^{(1)} = \begin{pmatrix}z & b_1x & 0 & 0 & a_1 & 0 & a_2 & a_3\end{pmatrix}^T, \qquad \mathbf{v}_{0s}^{(1)} = \begin{pmatrix}-x & z & 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}^T, \\
\mathbf{v}_{0t}^{(1)} = \begin{pmatrix}0 & b_2x & z & 0 & a_2 & 0 & a_4 & a_5\end{pmatrix}^T, \qquad \mathbf{v}_{0r}^{(1)} = \begin{pmatrix}0 & b_3x & 0 & z & a_3 & 0 & a_5 & a_6\end{pmatrix}^T, \\
\mathbf{v}_{0s}^{(2)} = \begin{pmatrix}-xz & 0.5\left(z^2 - b_1x^2\right) & 0 & 0 & -a_1x & 0 & -a_2x & -a_3x\end{pmatrix}^T.
\end{gathered} \tag{4.2}$$

The component forms of $\mathbf{v}_{0s}^{(3)}$ are

$$w = a_8 x^3 + b_5 x - \frac{1}{2} x z^2, \qquad u = -\frac{1}{2} b_1 x^2 z + \frac{1}{6} z^3, \qquad \phi = a_9 \left(x^3 - 3h^2 x \right),$$
$$\psi = a_{10} \left(x^3 - 3h^2 x \right), \qquad \sigma = -a_1 x z, \qquad \tau = \frac{1}{2} a_1 \left(x^2 - h^2 \right), \qquad D = -a_2 x z, \qquad B = -a_3 x z. \tag{4.3}$$

The constants in (4.2)-(4.3) are listed in [13]. The eigensolutions of eigenvalue zero just are the fundamental solution of Saint Venant problem. But the solutions having the local effect are constituted by the eigensolutions of eigenvalue nonzero of (3.1), which are obtained in the next section.

5. Eigenfunction Vectors of Eigenvalue Nonzero

The eigenvalue equation (3.5) is a system of ordinary differential equations with respect to x which can be solved by first determining the eigenvalue $\tilde{\lambda}$ with respect to the x direction. The corresponding equation is

$$\det \begin{bmatrix} -\mu & C_{10}\tilde{\lambda} & 0 & 0 & C_{11} & 0 & C_{14} & C_{15} \\ -\tilde{\lambda} & -\mu & -C_{21}\tilde{\lambda} & -C_{22}\tilde{\lambda} & 0 & C_{17} & 0 & 0 \\ 0 & C_{18}\tilde{\lambda} & -\mu & 0 & C_{14} & 0 & -C_{12} & -C_{16} \\ 0 & C_{19}\tilde{\lambda} & 0 & -\mu & C_{15} & 0 & -C_{16} & -C_{13} \\ 0 & 0 & 0 & 0 & -\mu & -\tilde{\lambda} & 0 & 0 \\ 0 & -C_{20}\tilde{\lambda}^2 & 0 & 0 & C_{10}\tilde{\lambda} & -\mu & C_{18}\tilde{\lambda} & C_{19}\tilde{\lambda} \\ 0 & 0 & C_{23}\tilde{\lambda}^2 & -C_{24}\tilde{\lambda}^2 & 0 & -C_{21}\tilde{\lambda} & -\mu & 0 \\ 0 & 0 & -C_{24}\tilde{\lambda}^2 & C_{25}\tilde{\lambda}^2 & 0 & -C_{22}\tilde{\lambda} & 0 & -\mu \end{bmatrix} = 0, \tag{5.1}$$

where the constants $C_{10} \sim C_{25}$ are listed in the appendix. Expanding the determinant yields the eigenvalue equation

$$c_1 \tilde{\lambda}^8 + c_2 \tilde{\lambda}^6 \mu^2 + c_3 \tilde{\lambda}^4 \mu^4 + c_4 \tilde{\lambda}^2 \mu^6 + \mu^8 = 0. \tag{5.2}$$

Apparently, where $c_1 \sim c_4$ constituted by $C_{10} \sim C_{25}$, (5.2) has eight roots

$$\tilde{\lambda}_i = \lambda_i \mu, \quad \tilde{\lambda}_{4+i} = -\lambda_i \mu \ (i = 1, 2, 3, 4). \tag{5.3}$$

Only discuss the general case that there are eight different roots $\tilde{\lambda}_i$. Obviously, $-\tilde{\lambda}_i$ must be an eigenvalue if $\tilde{\lambda}_i$ is an eigenvalue, so the general solution of (5.2) can be expressed as

$$\begin{aligned}
\hat{w} &= \sum_{i=1}^{4} A_{1i}\cosh(\lambda_i\mu x) + \sum_{i=1}^{4} D_{1i}\sinh(\lambda_i\mu x), &\quad \hat{u} &= \sum_{i=1}^{4} A_{2i}\sinh(\lambda_i\mu x) + \sum_{i=1}^{4} D_{2i}\cosh(\lambda_i\mu x),\\
\hat{\phi} &= \sum_{i=1}^{4} A_{3i}\cosh(\lambda_i\mu x) + \sum_{i=1}^{4} D_{3i}\sinh(\lambda_i\mu x), &\quad \hat{\varphi} &= \sum_{i=1}^{4} A_{4i}\cosh(\lambda_i\mu x) + \sum_{i=1}^{4} D_{4i}\sinh(\lambda_i\mu x),\\
\hat{\sigma} &= \sum_{i=1}^{4} A_{5i}\cosh(\lambda_i\mu x) + \sum_{i=1}^{4} D_{5i}\sinh(\lambda_i\mu x), &\quad \hat{\tau} &= \sum_{i=1}^{4} A_{6i}\sinh(\lambda_i\mu x) + \sum_{i=1}^{4} D_{6i}\cosh(\lambda_i\mu x),\\
\hat{D} &= \sum_{i=1}^{4} A_{7i}\cosh(\lambda_i\mu x) + \sum_{i=1}^{4} D_{7i}\sinh(\lambda_i\mu x), &\quad \hat{B} &= \sum_{i=1}^{4} A_{8i}\cosh(\lambda_i\mu x) + \sum_{i=1}^{4} D_{8i}\sinh(\lambda_i\mu x).
\end{aligned} \tag{5.4}$$

It shows that the partial solutions relevant to A are the solutions of symmetric deformation with the z-axis while the partial solutions relevant to D are the solutions of antisymmetric deformation with the z-axis.

Firstly discuss the general solution of symmetric deformation

$$\begin{gathered}
\hat{w} = \sum_{i=1}^{4} A_{1i}\cosh(\lambda_i\mu x), \qquad \hat{u} = \sum_{i=1}^{4} A_{2i}\sinh(\lambda_i\mu x), \qquad \hat{\phi} = \sum_{i=1}^{4} A_{3i}\cosh(\lambda_i\mu x),\\
\hat{\varphi} = \sum_{i=1}^{4} A_{4i}\cosh(\lambda_i\mu x), \qquad \hat{\sigma} = \sum_{i=1}^{4} A_{5i}\cosh(\lambda_i\mu x), \qquad \hat{\tau} = \sum_{i=1}^{4} A_{6i}\sinh(\lambda_i\mu x),\\
\hat{D} = \sum_{i=1}^{4} A_{7i}\cosh(\lambda_i\mu x), \qquad \hat{B} = \sum_{i=1}^{4} A_{8i}\cosh(\lambda_i\mu x),
\end{gathered} \tag{5.5}$$

where constants A_{ji} $(j = 1, 2, \ldots, 8;\ i = 1, 2, 3, 4)$ are not all independent. Substitute (5.5) into (3.5); A_{ji} can be expressed by the independent constants A_{6i} $(i = 1, 2, 3, 4)$

$$A_{ji} = \frac{f_{Nj}(\lambda_i)}{f_{Dj}(\lambda_i)\mu} A_{6i}(j = 1, 2, 3, 4), \quad A_{5i} = -\lambda_i A_{6i}, \quad A_{ji} = \frac{f_{Nj}(\lambda_i)}{f_{Dj}(\lambda_i)} A_{6i}(j = 7, 8), \tag{5.6}$$

where $f_{Nj}(\lambda_i)$ and $f_{Dj}(\lambda_i)$ are expressions of λ_i.

So far, only A_{6i} $(i = 1,2,3,4)$ are unknown constants in the general solution. Substituting (5.5) and (5.6) into the boundary conditions (3.3) yields

$$\begin{gathered}
\sum_{i=1}^{4}\left(\lambda_i \frac{f_{N2}(\lambda_i)}{f_{D2}(\lambda_i)} + s_{13}\lambda_i - d_{31}\frac{f_{N7}(\lambda_i)}{f_{D7}(\lambda_i)} - b_{31}\frac{f_{N8}(\lambda_i)}{f_{D8}(\lambda_i)}\right)\cosh(\lambda_i \mu h)A_{6i} = 0, \\
\sum_{i=1}^{4}\sinh(\lambda_i \mu h)A_{6i} = 0, \\
\sum_{i=1}^{4}\left(C_1 - \nu_{11}\lambda_i\frac{f_{N3}(\lambda_i)}{f_{D3}(\lambda_i)} + \beta_{11}\lambda_i\frac{f_{N4}(\lambda_i)}{f_{D4}(\lambda_i)}\right)\sinh(\lambda_i \mu h)A_{6i} = 0, \\
\sum_{i=1}^{4}\left(C_2 + \beta_{11}\lambda_i\frac{f_{N3}(\lambda_i)}{f_{D3}(\lambda_i)} - \lambda_{11}\lambda_i\frac{f_{N4}(\lambda_i)}{f_{D4}(\lambda_i)}\right)\sinh(\lambda_i \mu h)A_{6i} = 0.
\end{gathered} \tag{5.7}$$

For the sake of simplicity, (5.7) is denoted as

$$\{B_{ji}\}\{A_{6i}\} = \{0\} \quad (i, j = 1,2,3,4). \tag{5.8}$$

For nontrivial solution to exist, the determinant of coefficient matrix vanishes.

$$\left|\{B_{ji}\}\right| = 0. \tag{5.9}$$

Equation (5.9) can be solved by numerical methods. If μ_n $(n = 1,2,\ldots)$ is roots of (5.9), each μ_n in reality has its symplectic adjoint eigenvalue $-\mu_n$ and their complex conjugate eigenvalues. After obtaining A_{6i} $(i, j = 1,2,3,4)$ by substituting μ_n into (5.8), the corresponding eigenvector function is

$$\mathbf{Y}_n = \begin{pmatrix}\widehat{w}_n & \widehat{u}_n & \widehat{\phi}_n & \widehat{\varphi}_n & \widehat{\sigma}_n & \widehat{\tau}_n & \widehat{D}_n & \widehat{B}_n\end{pmatrix}^T, \tag{5.10}$$

where

$$\begin{gathered}
\widehat{w}_n = \sum_{i=1}^{4}\frac{f_{N1}(\lambda_i)}{f_{D1}(\lambda_i)\mu}A_{6i}\cosh(\lambda_i\mu_n x), \qquad \widehat{u}_n = \sum_{i=1}^{4}\frac{f_{N2}(\lambda_i)}{f_{D2}(\lambda_i)\mu}A_{6i}\sinh(\lambda_i\mu_n x), \\
\widehat{\phi}_n = \sum_{i=1}^{4}\frac{f_{N3}(\lambda_i)}{f_{D3}(\lambda_i)\mu}A_{6i}\cosh(\lambda_i\mu_n x), \qquad \widehat{\varphi}_n = \sum_{i=1}^{4}\frac{f_{N4}(\lambda_i)}{f_{D4}(\lambda_i)\mu}A_{6i}\cosh(\lambda_i\mu_n x), \\
\widehat{\sigma}_n = \sum_{i=1}^{4}-\lambda_i A_{6i}\cosh(\lambda_i\mu_n x), \qquad \widehat{\tau}_n = \sum_{i=1}^{4}A_{6i}\sinh(\lambda_i\mu_n x), \\
\widehat{D}_n = \sum_{i=1}^{4}\frac{f_{N7}(\lambda_i)}{f_{D7}(\lambda_i)}A_{6i}\cosh(\lambda_i\mu_n x), \qquad \widehat{B}_n = \sum_{i=1}^{4}\frac{f_{N8}(\lambda_i)}{f_{D8}(\lambda_i)}A_{6i}\cosh(\lambda_i\mu_n x).
\end{gathered} \tag{5.11}$$

The corresponding solution of (3.1) in symmetric deformation is

$$\mathbf{v}_n = e^{\mu_n z}\mathbf{Y}_n. \tag{5.12}$$

Likewise, the corresponding eigenvector function of eigenvalues μ'_n $(n = 1, 2, \ldots)$ in antisymmetric deformation is

$$\mathbf{Y}'_n = \begin{pmatrix} \widehat{w}'_n & \widehat{u}'_n & \widehat{\phi}'_n & \widehat{\varphi}'_n & \widehat{\sigma}'_n & \widehat{\tau}'_n & \widehat{D}'_n & \widehat{B}'_n \end{pmatrix}^T, \tag{5.13}$$

where

$$\begin{gathered}
\widehat{w}'_n = \sum_{i=1}^{4} D_{1i}\sinh(\lambda_i\mu'_n x), \qquad \widehat{u}'_n = \sum_{i=1}^{4} D_{2i}\cosh(\lambda_i\mu'_n x), \\
\widehat{\phi}'_n = \sum_{i=1}^{4} D_{3i}\sinh(\lambda_i\mu'_n x), \qquad \widehat{\varphi}'_n = \sum_{i=1}^{4} D_{4i}\sinh(\lambda_i\mu'_n x), \\
\widehat{\sigma}'_n = \sum_{i=1}^{4} D_{5i}\sinh(\lambda_i\mu'_n x), \qquad \widehat{\tau}'_n = \sum_{i=1}^{4} D_{6i}\cosh(\lambda_i\mu'_n x), \\
\widehat{D}'_n = \sum_{i=1}^{4} D_{7i}\sinh(\lambda_i\mu'_n x), \qquad \widehat{B}'_n = \sum_{i=1}^{4} D_{8i}\sinh(\lambda_i\mu'_n x).
\end{gathered} \tag{5.14}$$

The corresponding solution of (3.1) in antisymmetric deformation is

$$\mathbf{v}'_n = e^{\mu'_n z}\mathbf{Y}'_n. \tag{5.15}$$

Thus, all eigensolutions of nonzero eigenvalues are obtained. These solutions are covered in the Saint-Venant principle and decay with distance depending on the characteristics of eigenvalues (see [7, 8]). Together with the eigensolutions of zero eigenvalue, they constitute a complete adjoint symplectic orthonormal basis and the expansion theorem is then applicable. Further discussion concerns solutions of magnetoelectroelastic solids plane problems in rectangular domain.

6. Solutions of Generalized Plane Problems in Rectangular Domain

The solution of homogeneous (3.1) by the method of separation variables has been discussed in the previous several sections. The analytical expressions of eigensolutions of zero eigenvalue and of nonzero eigenvalues have also been presented. Based on expansion theorems, the general solution of homogeneous equation (3.1) for magnetoelectroelastic solids plane problems in rectangular domain is

$$\mathbf{v} = \sum_{i=0}^{1}\left(a_{0f}^{(i)}\mathbf{v}_{0f}^{(i)} + a_{0t}^{(i)}\mathbf{v}_{0t}^{(i)} + a_{0r}^{(i)}\mathbf{v}_{0r}^{(i)}\right) + \sum_{i=0}^{3} a_{0s}^{(i)}\mathbf{v}_{0s}^{(i)} + \sum_{i=1}^{\infty}\widetilde{a}_i\mathbf{v}_i + \sum_{i=1}^{\infty}\widetilde{b}_i\mathbf{v}'_i, \tag{6.1}$$

where $a_{0f}^{(i)}$, $a_{0s}^{(i)}$, $a_{0t}^{(i)}$, $a_{0r}^{(i)}$, $\tilde{a}_i$, and $\tilde{b}_i$ are undetermined constants. Determining these constants needs the variational equations corresponding to the two-point boundary conditions obtained by applying the variational principle.

If the end boundary conditions for specified generalized displacements are (2.7), it can be expressed as

$$\begin{aligned} \mathbf{q}_0 &= \overline{\mathbf{q}}_0(x) = \begin{bmatrix} \overline{w}_0(x) & \overline{u}_0(x) & \overline{\phi}_0(x) & \overline{\varphi}_0(x) \end{bmatrix}^T \quad \text{at } z = 0, \\ \mathbf{q}_l &= \overline{\mathbf{q}}_l(x) = \begin{bmatrix} \overline{w}_l(x) & \overline{u}_l(x) & \overline{\phi}_l(x) & \overline{\varphi}_l(x) \end{bmatrix}^T \quad \text{at } z = l, \end{aligned} \tag{6.2}$$

where $\mathbf{q}_0$ and $\mathbf{q}_l$ denote the values of variable $\mathbf{q}$ at $z = 0$ and $z = l$, respectively.

If the end boundary conditions for specified generalized forces are (2.8), it can be expressed as

$$\begin{aligned} \mathbf{p}_0 &= \overline{\mathbf{p}}_0(x) = \begin{bmatrix} \overline{\sigma}_0(x) & \overline{\tau}_0(x) & \overline{D}_0(x) & \overline{B}_0(x) \end{bmatrix}^T \quad z = 0, \\ \mathbf{p}_l &= \overline{\mathbf{p}}_l(x) = \begin{bmatrix} \overline{\sigma}_l(x) & \overline{\tau}_l(x) & \overline{D}_l(x) & \overline{B}_l(x) \end{bmatrix}^T \quad z = l, \end{aligned} \tag{6.3}$$

where $\mathbf{p}_0$ and $\mathbf{p}_l$ denote the values of variable $\mathbf{p}$ at $z = 0$ and $z = l$, respectively.

The boundary conditions at both ends ($z = 0$ or l) can also be mixed boundary conditions.

According to approach derived by Zhong et al. in [7, 8] in the Hamiltonian system, the process of obtaining the undetermined constants of the general solution is presented here. For the sake of simplicity, the general solution is given as

$$\mathbf{v} = \sum_{i=1}^{n} A_i \mathbf{v}_i, \tag{6.4}$$

where n is the number of eigensolutions, $\mathbf{v}_i$ are all solutions that relate to zero eigenvalue and nonzero eigenvalues, and A_i are constants determined by a linear system of equations resulted from the Hamiltonian variational principle. If we specify generalized displacement $\overline{\mathbf{q}}_0$ at $z = 0$ and generalized force $\overline{\mathbf{p}}_l$ at $z = l$, the variational equation is obtained as

$$\int_{-h}^{h} \left(\sum_{i=1}^{n} \delta A_i \mathbf{p}_{0i}^T \right) \left(\sum_{j=1}^{n} A_j \mathbf{q}_{0j} - \overline{\mathbf{q}}_0 \right) \mathrm{d}x + \int_{-h}^{h} \left(\sum_{i=1}^{n} \delta A_i \mathbf{q}_{li}^T \right) \left(\sum_{j=1}^{n} A_j \mathbf{p}_{lj} - \overline{\mathbf{p}}_l \right) \mathrm{d}x = 0. \tag{6.5}$$

Equation (6.5) can be expressed as

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \left[\int_{-h}^{h} \left[A_j \left(\mathbf{p}_{0i}^T \mathbf{q}_{0j} + \mathbf{q}_{li}^T \mathbf{p}_{lj} \right) - \mathbf{p}_{0i}^T \overline{\mathbf{q}}_0 - \mathbf{q}_{li}^T \overline{\mathbf{p}}_l \right] \mathrm{d}x \right] \delta A_i = 0. \tag{6.6}$$

A_i can be determined by the following equations:

$$\begin{bmatrix} \tilde{c}_{11} & \tilde{c}_{12} & \cdots & \tilde{c}_{1n} \\ \tilde{c}_{21} & \tilde{c}_{22} & \cdots & \tilde{c}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{c}_{n1} & \tilde{c}_{n2} & \cdots & \tilde{c}_{nn} \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{Bmatrix} = \begin{Bmatrix} \tilde{d}_1 \\ \tilde{d}_2 \\ \vdots \\ \tilde{d}_n \end{Bmatrix}, \tag{6.7}$$

where

$$\tilde{c}_{ij} = \int_{-h}^{h} \left(\mathbf{p}_{0i}^T \mathbf{q}_{0j} + \mathbf{q}_{li}^T \mathbf{p}_{lj} \right) \mathrm{d}x, \quad \tilde{d}_i = \int_{-h}^{h} \left(\mathbf{p}_{0i}^T \overline{\mathbf{q}}_0 + \mathbf{q}_{li}^T \overline{\mathbf{p}}_l \right) \mathrm{d}x \quad (i, j = 1, 2, \ldots, n). \tag{6.8}$$

7. Numerical Examples

The Saint-Venant principle is applicable to the problem for $l \gg h$ in a rectangular domain. The influence of self-equilibrium forces at both ends ($z = 0$ or l) is only confined to the vicinity of the region. It is then appropriate to neglect the solutions of nonzero eigenvalues and, therefore, to apply only the solutions of zero eigenvalue in the expansion theorem.

Example 7.1. Consider a magnetoelectroelastic rectangle domain, under uniform axial tension, electrical displacement, or magnetic induction, respectively. Three load cases are considered and the boundary conditions are given by

$z = l$:

(a) $\sigma_z = P_0$, $\tau_{xz} = D_z = B_z = 0$, under the uniform tension;
(b) $D_z = D_0$, $\sigma_z = \tau_{xz} = B_z = 0$, under the uniform electric displacement;
(c) $B_z = B_0$, $\sigma_z = \tau_{xz} = D_z = 0$, under the uniform magnetic induction.

$z = 0$:

$$\tau_{xz} = w = \phi = \psi = 0; \; x = \pm h: \; \sigma_x = \tau_{xz} = D_x = B_x = 0.$$

The problem is treated as a symmetric deformation one. The solution is formed from (4.2).

For load (a),

$$\mathbf{v}_1 = \left(s_{33} \mathbf{v}_{0f}^{(1)} + d_{33} \mathbf{v}_{0t}^{(1)} + b_{33} \mathbf{v}_{0r}^{(1)} \right) P_0. \tag{7.1}$$

For load (b),

$$\mathbf{v}_2 = \left(d_{33} \mathbf{v}_{0f}^{(1)} - \lambda_{33} \mathbf{v}_{0t}^{(1)} - \beta_{33} \mathbf{v}_{0r}^{(1)} \right) D_0. \tag{7.2}$$

Table 1: The numerical results in the rectangular domain problem under three-load case.

Case	u (10^{-13} m)	w (10^{-12} m)	ϕ (10^{-3} V)	φ (10^{-4} A)	σ_z (N/m^2)	D_z (10^{-9} C/m^2)	B_z (10^{-7} N/Am)
a	−9.4998	5.6834	9.4955	2.1391	10.0000	0.0000	0.0000
	(−9.4998)	(5.6834)	(9.4955)	(2.1391)	(10.0000)	(0.0000)	(0.0000)
b	−2.1079	0.94955	−6.2894	0.25669	0.0000	1.0000	0.0000
	(−2.1079)	(0.94955)	(−6.2894)	(0.25669)	(0.0000)	(1.0000)	(0.0000)
c	5.0767	2.1391	2.5669	−7.5213	0.0000	0.0000	1.0000
	(5.0767)	(2.1391)	(2.5669)	(−7.5213)	(0.0000)	(0.0000)	(1.0000)

Table 2: The eigenvalues with respect to the x-direction.

i	1	2	3	4
λ_i	$0.58880i$	$1.0711i$	$0.60197 + 1.1734i$	$0.60197 - 1.1734i$

For load (c),

$$\mathbf{v}_3 = \left(b_{33}\mathbf{v}_{0f}^{(1)} - \beta_{33}\mathbf{v}_{0r}^{(1)} - \nu_{33}\mathbf{v}_{0r}^{(1)} \right) B_0. \tag{7.3}$$

For numerical calculation, the composite materials $BaTiO_3$-$CoFe_2O_4$ are specified, which material constants are given in [5]. Take $\sigma_z = 10\,\mathrm{N/m^2}$, $D_z = 10^{-9}\,\mathrm{C/m^2}$, $B_z = 10^{-7}\,\mathrm{N/Am}$, $h = 0.03\,\mathrm{m}$, and $l = 0.1\,\mathrm{m}$. The numerical results at point (h, l) are given in Table 1, where the values in the parenthesis are the exact ones which are calculated with the formulas presented by Ding and Jiang in [5]. The results show that the numerical solutions by the method have higher accuracy. The eigensolutions of zero eigenvalue correspond to the solutions of Saint-Venant problems. On the other hand, the solutions of the portion covered by the Saint-Venant principle correspond to the eigensolutions with nonzero eigenvalues. Subsequently, a very simple example is presented here.

Example 7.2. Consider a simple tension problem of composite materials $BaTiO_3$-$CoFe_2O_4$ semi-infinite strip fixed at $z = 0$, and take $-h \le x \le h$, $h = 0.1\,\mathrm{m}$. There is only the tension stress σ_∞ for $z \to \infty$, and $\tau_{zx} = D_z = B_z = 0$. The stress distribution at the fixed end is determined here.

Firstly, by utilizing (5.2) and (5.3), λ_i $(i = 1, 2, \ldots, 4)$ are calculated and listed in Table 2 for the composite materials $BaTiO_3$-$CoFe_2O_4$.

Secondly, obtain nonzero eigenvalues. With reference to the problem, there is only the tension stress σ_∞ for $z \to \infty$ and the deformation is symmetric with respect to z-axis. Solve (5.9) by numerical methods. Table 3 lists the first five eigenvalues in the first quadrant. Each μ_n in reality has its symplectic adjoint eigenvalue $-\mu_n$ and their complex conjugate eigenvalue.

Thirdly, the general solution is then formed from the eigensolution of the zero eigenvalue (4.2) and (4.3) and the eigensolution of the nonzero eigenvalues of the symmetric deformation (5.12). Consider the deformation is symmetric with respect to z-axis and rigid body translation, constant electric potential or constant magnetic potential has no effect on

Table 3: Nonzero eigenvalues for symmetric deformation.

n	1	2	3	4	5
$\mu_n h$	1.8209 + 1.2193i	2.8349	3.8972	3.9331 + 2.1502i	5.8470

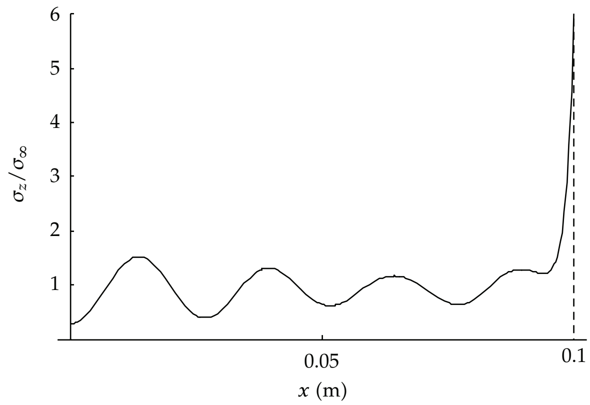

Figure 2: Normal stress analysis with fixed end.

stress, only $\mathbf{v}_{0f}^{(1)}$, $\mathbf{v}_{0t}^{(1)}$, and $\mathbf{v}_{0r}^{(1)}$ are selected in (4.2). Their coefficients are determined by the boundary conditions at $z \to \infty$. So the general solution can be expressed as

$$\mathbf{v} = \sigma_\infty \left(s_{33}\mathbf{v}_{0f}^{(1)} + d_{33}\mathbf{v}_{0t}^{(1)} + b_{33}\mathbf{v}_{0r}^{(1)} \right) + \sum_{i=1}^{n} A_i \mathbf{v}_i, \tag{7.4}$$

where n is the number of the eigensolutions of nonzero eigenvalues of the symmetric deformation. For the above general solution, only eigensolutions of α-set in (5.12) are adopted, that is, Re $\mu_i < 0$. Thus it consistent with the far-end boundary condition at $z \to \infty$.

Finally, Substituting (7.4) into the equation of variation (6.5) and adopting a total of ten eigensolutions of nonzero eigenvalues in the calculation yield σ_z/σ_∞ at the fixed end $z = 0$ as illustrated in Figure 2. The figure shows that there is stress singularity at the edge corner. Fluctuation of stress observed is common when truncated finite terms are adopted in the expansion. Such phenomenon has been observed, for instance, when finite terms in a Fourier series are assumed. Figure 3 shows that the normal stress distribution approximates uniformity at $z = 2h$.

8. Conclusions

In this paper, the transversely isotropic magnetoelectroelastic solids plane problem in rectangular domain is considered from a symplectic approach. The eigensolutions of nonzero-eigenvalues obtained decay drastically with respect to distance can express the end effects and corner stresses. There is no requirement of experience in the symplectic approach for solving the problem since it is a rational, analytical approach to satisfy the boundary conditions in a straightforward manner.

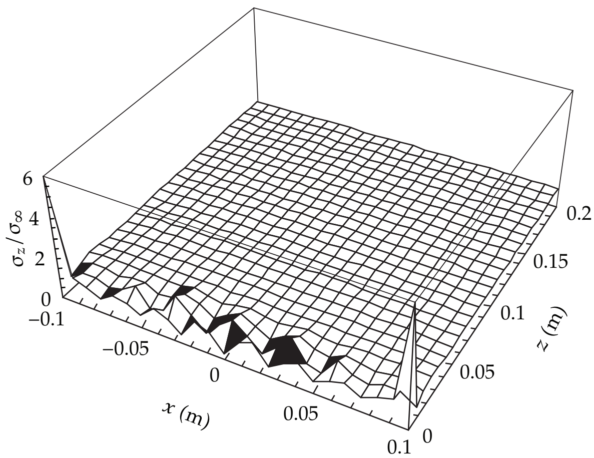

Figure 3: Normal stress distribution in the partial region of the left end.

Appendix

$$\Delta = \lambda_{11}\nu_{11} - \beta_{11}^2, \qquad C_1 = \nu_{11}d_{15} - \beta_{11}b_{15}, \qquad C_2 = \lambda_{11}b_{15} - \beta_{11}d_{15},$$
$$C_{10} = \frac{s_{13}}{s_{11}}, \qquad C_{11} = \frac{s_{11}s_{33} - s_{13}^2}{s_{11}}, \qquad C_{12} = \frac{s_{11}\lambda_{33} + d_{31}^2}{s_{11}},$$
$$C_{13} = \frac{s_{11}\nu_{33} + b_{31}^2}{s_{11}}, \qquad C_{14} = \frac{s_{11}d_{33} - s_{13}d_{31}}{s_{11}}, \qquad C_{15} = \frac{s_{11}b_{33} - s_{13}b_{31}}{s_{11}},$$
$$C_{16} = \frac{s_{11}\beta_{33} + b_{31}d_{31}}{s_{11}}, \qquad C_{17} = s_{44} + \frac{(d_{15}C_1 + b_{15}C_2)}{\Delta}, \tag{A.1}$$
$$C_{18} = \frac{d_{31}}{s_{11}}, \qquad C_{19} = \frac{b_{31}}{s_{11}}, \qquad C_{20} = \frac{1}{s_{11}}, \qquad C_{21} = \frac{C_1}{\Delta},$$
$$C_{22} = \frac{C_2}{\Delta}, \qquad C_{23} = \frac{\nu_{11}}{\Delta}, \qquad C_{24} = \frac{\beta_{11}}{\Delta}, \qquad C_{25} = \frac{\lambda_{11}}{\Delta}.$$

Acknowledgment

The work was supported by the National Natural Science Foundation of China (no. 10902072).

References

[1] J. H. Huang and W. S. Kuo, "The analysis of piezoelectric/piezomagnetic composite materials containing ellipsoidal inclusions," *Journal of Applied Physics*, vol. 81, no. 3, pp. 1378–1386, 1997.

[2] B. M. Singh, J. Rokne, and R. S. Dhaliwal, "Closed-form solutions for two anti-plane collinear cracks in a magnetoelectroelastic layer," *European Journal of Mechanics, A/Solids*, vol. 28, no. 3, pp. 599–609, 2009.

[3] S. M. Xiong and G. Z. Ni, “2D Green’s functions for semi-infinite transversely isotropic electro-magneto-thermo-elastic composite,” *Journal of Magnetism and Magnetic Materials*, vol. 321, no. 12, pp. 1867–1874, 2009.

[4] C. C. Ma and J. M. Lee, “Theoretical analysis of generalized loadings and image forces in a planar magnetoelectroelastic layered half-plane,” *Journal of the Mechanics and Physics of Solids*, vol. 57, no. 3, pp. 598–620, 2009.

[5] H. J. Ding and A. M. Jiang, “A boundary integral formulation and solution for 2D problems in magneto-electro-elastic media,” *Computers and Structures*, vol. 82, no. 20-21, pp. 1599–1607, 2004.

[6] W. Q. Chen, K. Y. Lee, and H. J. Ding, “On free vibration of non-homogeneous transversely isotropic magneto-electro-elastic plates,” *Journal of Sound and Vibration*, vol. 279, no. 1-2, pp. 237–251, 2005.

[7] W. X. Zhong, *Duality System in Applied Mechanics and Optimal Control*, Kluwer Academic Publishers, Boston, Mass, USA, 2004.

[8] W. A. Yao, W. X. Zhong, and C. W. Lim, *Symplectic Elasticity*, World Scientific, Singapore, 2009.

[9] W. A. Yao and H. T. Yang, “Hamiltonian system based Saint Venant Solutions for multi-layered compostie plane anisotropic plates,” *International Journal of Solids and Structures*, vol. 38, no. 32-33, pp. 5807–5817, 2001.

[10] A. Y. T. Leung and J. J. Zheng, “Closed form stress distribution in 2D elasticity for all boundary conditions,” *Applied Mathematics and Mechanics*, vol. 28, no. 12, pp. 1629–1642, 2007.

[11] C. W. Lim, C. F. Lü, Y. Xiang, and W. A. Yao, “On new symplectic elasticity approach for exact free vibration solutions of rectangular Kirchhoff plates,” *International Journal of Engineering Science*, vol. 47, no. 1, pp. 131–140, 2009.

[12] X. S. Xu, Q. Gu, A. Y. T. Leung, and J. J. Zheng, “Symplectic eigensolution method in transversely isotropic piezoelectric cylindrical media,” *Journal of Zhejiang University Science*, vol. 6, no. 9, pp. 922–927, 2005.

[13] W. A. Yao and X. C. Li, “Symplectic duality system on plane magnetoelectroelastic solids,” *Applied Mathematics and Mechanics*, vol. 27, no. 2, pp. 195–205, 2006.

[14] E. Pan, “Exact solution for simply supported and multilayered magneto-electro-elastic plates,” *Journal of Applied Mechanics, Transactions ASME*, vol. 68, no. 4, pp. 608–618, 2001.

[15] X. Wang and Y. P. Shen, “Inclusions of arbitrary shape in magnetoelectroelastic composite materials,” *International Journal of Engineering Science*, vol. 41, no. 1, pp. 85–102, 2003.