Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2010, Article ID 785428, 13 pages
doi:10.1155/2010/785428

Research Article

On Regularized Quasi-Semigroups and Evolution Equations

M. Janfada

Department of Mathematics, Sabzevar Tarbiat Moallem University, P.O. Box 397, Sabzevar, Iran

Correspondence should be addressed to M. Janfada, mjanfada@gmail.com

Received 26 November 2009; Accepted 16 April 2010

Academic Editor: Wolfgang Ruess

We introduce the notion of regularized quasi-semigroup of bounded linear operators on Banach spaces and its infinitesimal generator, as a generalization of regularized semigroups of operators. After some examples of such quasi-semigroups, the properties of this family of operators will be studied. Also some applications of regularized quasi-semigroups in the abstract evolution equations will be considered. Next some elementary perturbation results on regularized quasi-semigroups will be discussed.

1. Introduction and Preliminaries

The theory of quasi-semigroups of bounded linear operators, as a generalization of strongly continuous semigroups of operators, was introduced in 1991 [1], in a preprint of Barcenas and Leiva. This notion, its elementary properties, exponentially stability, and some of its applications in abstract evolution equations are studied in [2–5]. The dual quasi-semigroups and the controllability of evolution equations are also discussed in [6].

Given a Banach space X, we denote by $B(X)$ the space of all bounded linear operators on X. A biparametric commutative family $\{R(s,t)\}_{s,t\geq 0} \subseteq B(X)$ is called a quasi-semigroup of operators if for every $s,t,r \geq 0$ and $x \in X$, it satisfies

(1) $R(t,0) = I$, the identity operator on X,

(2) $R(r,s+t) = R(r+t,s)R(r,t)$,

(3) $\lim_{(s,t)\to(s_0,t_0)} \|R(s,t)x - R(s_0,t_0)x\| = 0, x \in X$,

(4) $\|R(s,t)\| \leq M(s+t)$, for some continuous increasing mapping $M : [0,\infty) \to [0,\infty)$.

Also regularized semigroups and their connection with abstract Cauchy problems are introduced in [7] and have been studied in [8–12] and many other papers.

We mention that if $C \in B(X)$ is an injective operator, then a one-parameter family $\{T(t)\}_{\geq 0} \subseteq B(X)$ is called a C-semigroup if for any $s, t \geq 0$ it satisfies $T(s+t)C = T(s)T(t)$ and $T(0) = C$.

In this paper we are going to introduce regularized quasi-semigroups of operators.

In Section 2, some useful examples are discussed and elementary properties of regularized quasi-semigroups are studied.

In Section 3 regularized quasi-semigroups are applied to find solutions of the abstract evolution equations. Also perturbations of the generator of regularized quasi-semigroups are also considered in this section. Our results are mainly based on the work of Barcenas and Leiva [1].

2. Regularized Quasi-Semigroups

Suppose X is a Banach space and $\{K(s,t)\}_{s,t\geq 0}$ is a two-parameter family of operators in $B(X)$. This family is called commutative if for any $r, s, t, u \geq 0$,

$$K(r,t)K(s,u) = K(s,u)K(r,t). \tag{2.1}$$

Definition 2.1. Suppose C is an injective bounded linear operator on Banach space X. A commutative two-parameter family $\{K(s,t)\}_{s,t\geq 0}$ in $B(X)$ is called a regularized quasi-semigroups (or C-quasi-semigroups) if

(1) $K(t,0) = C$, for any $t \geq 0$;

(2) $CK(r,t+s) = K(r+t,s)K(r,t)$, $r,t,s \geq 0$;

(3) $\{K(s,t)\}_{s,t\geq 0}$ is strongly continuous, that is,

$$\lim_{(s,t)\to(s_0,t_0)} \|K(s,t)x - K(s_0,t_0)x\| = 0, \quad x \in X; \tag{2.2}$$

(4) there exists a continuous and increasing function $M : [0,\infty) \to [0,\infty)$, such that for any $s,t > 0$, $\|K(s,t)\| \leq M(s+t)$.

For a C-quasi-semigroups $\{K(s,t)\}_{s,t\geq 0}$ on Banach space X, let D be the set of all $x \in X$ for which the following limits exist in the range of C:

$$\begin{gathered}\lim_{t\to 0^+} \frac{K(s,t)x - Cx}{t} = \lim_{t\to 0^+} \frac{K(s-t,t)x - Cx}{t}, \quad s > 0 \\ \lim_{t\to 0^+} \frac{K(0,t)x - Cx}{t}.\end{gathered} \tag{2.3}$$

Now for $x \in D$ and $s \geq 0$, define

$$A(s)x = C^{-1} \lim_{t\to 0^+} \frac{K(s,t)x - Cx}{t}. \tag{2.4}$$

$\{A(s)\}_{s\geq 0}$ is called the infinitesimal generator of the regularized quasi-semigroup $\{K(s,t)\}_{s,t\geq 0}$. Somewhere we briefly apply generator instead of infinitesimal generator.

Here are some useful examples of regularized quasi-semigroups.

Example 2.2. Let $\{T_t\}_{t\geq 0}$ be an exponentially bounded strongly continuous C-semigroup on Banach space X, with the generator A. Then

$$K(s,t) := T_t, \quad s,t \geq 0, \tag{2.5}$$

defines a C-quasi-semigroup with the generator $A(s) = A$, $s \geq 0$, and so $D = D(A)$.

Example 2.3. Let $\mathrm{X} = \mathrm{BUC}(\mathbb{R})$, the space of all bounded uniformly continuous functions on $\mathbb{R}$ with the supremum-norm. Define $C, K(s,t) \in B(X)$, by

$$Cf(x) = e^{-x^2} f(x), \quad K(s,t)f(x) = e^{-x^2} f\left(t^2 + 2st + x\right), \quad s,t \geq 0. \tag{2.6}$$

One can see that $\{K(s,t)\}_{s,t\geq 0}$ is a regularized C-quasi-semigroup of operators on X, with the infinitesimal generator $A(s)f = 2s\dot{f}$ on D, where $D = \{f \in \mathrm{X} : \dot{f} \in \mathrm{X}\}$.

Example 2.4. Let $\{T_t\}_{t\geq 0}$ be a strongly continuous semigroup of operators on Banach space X, with the generator A. If $C \in B(X)$ is injective and commutes with T_t, $t \geq 0$, then

$$K(s,t) := Ce^{T_{s+t} - T_s}, \quad s,t \geq 0, \tag{2.7}$$

is a C-quasi-semigroup with the generator $A(s) = AT_s$. Thus $D = D(A)$. In fact, for $x \in D$, boundedness of C implies that

$$CA(s)x = \lim_{t\to 0^+} \frac{Ce^{T_{s+t} - T_s}x - Cx}{t} = C\lim_{t\to 0^+} \frac{e^{T_{s+t} - T_s}x - x}{t} = C\frac{d}{ds}|_{t=0}(T_{s+t} - T_s)x = CAT_s x. \tag{2.8}$$

Now injectivity of C implies that $A(s)x = AT_s x$, and so $D = D(A)$.

Example 2.5. Let $\{T_t\}_{t\geq 0}$ be a strongly continuous exponentially bounded C-semigroup of operators on Banach space X, with the generator A. For $s,t \geq 0$, define

$$K(s,t) = T\left(g(s+t) - g(s)\right), \quad s,t \geq 0, \tag{2.9}$$

where $g(t) = \int_0^t a(s)ds$, and $a \in C[0,\infty)$, with $a(t) > 0$. We have $K(s,0) = T(0) = C$ and the C-semigroup properties of $\{T(t)\}_{t\geq 0}$ imply that

$$\begin{aligned} CK(r, s+t) &= CT\left(g(r+t+s) - g(r)\right) \\ &= CT\left(g(r+t+s) - g(t+r) + g(t+r) - g(r)\right) \\ &= T\left(g(r+t+s) - g(t+r)\right)T\left(g(t+r) - g(r)\right) \\ &= K(r+t, s)K(r,t). \end{aligned} \tag{2.10}$$

So $\{K(s,t)\}_{s,t\geq 0}$ is a C-quasi-semigroup (the other properties can be also verified easily). Also $D = D(A)$ and for $x \in D$, $A(s)x = a(s)Ax$.

Some elementary properties of regularized quasi-semigroups can be seen in the following theorem.

Theorem 2.6. *Suppose $\{K(s,t)\}_{s,t\geq 0}$ is a C-quasi-semigroup with the generator $\{A(s)\}_{s\geq 0}$ on Banach space X. Then*

(i) *for any $x \in D$ and s_0, $t_0 \geq 0$, $K(s_0, t_0)x \in D$ and*

$$K(s_0,t_0)A(s)x = A(s)K(s_0,t_0)x; \tag{2.11}$$

(ii) *for each $x_0 \in D$,*

$$\frac{\partial}{\partial t}K(r,t)Cx_0 = A(r+t)K(r,t)Cx_0 = K(r,t)A(r+t)Cx_0; \tag{2.12}$$

(iii) *if $A(s)$ is locally integrable, then for each $x_0 \in D$ and $r \geq 0$,*

$$K(r,t)x_0 = Cx_0 + \int_0^t A(r+s)K(r,s)x_0 ds, \quad t \geq 0; \tag{2.13}$$

(iv) *let $f : [0,\infty) \to X$ be a continuous function; then for every $t \in [0,\infty)$,*

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} K(s,u)f(u)du = K(s,t)f(t); \tag{2.14}$$

(v) *Let $C' \in B(X)$ be injective and for any $s,t \geq 0$, $C'K(s,t) = K(s,t)C'$. Then $R(s,t) := C'K(s,t)$ is a CC'-quasi-semigroup with the generator $\{A(s)\}_{s\geq 0}$,*

(vi) *Suppose $\{R(s,t)\}_{s,t\geq 0}$ is a quasi-semigroup of operators on Banach space X with the generator $\{A(s)\}_{s\geq 0}$, and $C \in B(X)$ commutes with every $R(s,t)$, $s,t \geq 0$. Then $K(s,t) := CR(s,t)$ is a C-quasi-semigroup of operators on X with the generator $\{A(s)\}_{s\geq 0}$.*

Proof. First we note that from the commutativity of $\{K(s,t)\}_{s,t\geq 0}$;

$$CK(s,t) = K(s,t)C \quad s,t \geq 0. \tag{2.15}$$

Also $x \in D$ implies that

$$\lim_{t\to 0^+}\frac{K(s,t)x - Cx}{t} = CA(s)x \quad s \geq 0. \tag{2.16}$$

Thus from continuity of $K(s_0,t_0)$, we have

$$\begin{aligned}\lim_{t\to 0^+}\frac{K(s,t)K(s_0,t_0)x-CK(s_0,t_0)x}{t}&=K(s_0,t_0)\lim_{t\to 0^+}\frac{K(s,t)x-Cx}{t}\\&=K(s_0,t_0)CA(s)x\\&=CK(s_0,t_0)A(s)x.\end{aligned}\tag{2.17}$$

Thus $K(s_0,t_0)x\in D$ and $A(s)K(s_0,t_0)=K(s_0,t_0)A(s)x$.

To prove (ii), consider the quotient

$$\begin{aligned}\frac{K(r,t+s)Cx_0-K(r,t)Cx_0}{s}&=\frac{K(r+t,s)K(r,t)x_0-K(r,t)Cx_0}{s}\\&=K(r,t)\frac{K(r+t,s)x_0-Cx_0}{s},\end{aligned}\tag{2.18}$$

which tends to $K(r,t)CA(r+t)x_0$ as $s\to 0^+$.

Also for $s<0$,

$$\begin{aligned}\frac{K(r,t+s)Cx_0-K(r,t)Cx_0}{s}&=\frac{K(r,t)Cx_0-K(r,t+s)Cx_0}{-s}\\&=\frac{K(r+t+s,-s)K(r,t+s)x_0-K(r,t+s)Cx_0}{-s}\\&=K(r,t+s)\frac{K(r+t+s,-s)x_0-Cx_0}{-s}\\&=K(r,t+s)\frac{1}{-s}(K(r+t+s,-s)x_0\\&\quad-K(r+t,-s)x_0+K(r+t,-s)x_0-Cx_0).\end{aligned}\tag{2.19}$$

Now the strongly continuity of $\{K(s,t)\}_{s,t\geq 0}$ implies that

$$\lim_{s\to 0^-}K(r+t+s,-s)x_0-K(r+t,-s)x_0=0.\tag{2.20}$$

Thus

$$\lim_{s\to 0^-}\frac{K(r+t+s,-s)x_0-Cx_0}{-s}=CA(r+t)x_0.\tag{2.21}$$

Hence by the strongly continuity of $K(s,t)$,

$$\lim_{s\to 0^-}\frac{K(r,t+s)Cx_0-K(r,t)Cx_0}{s}=K(r,t)CA(r+t)x_0.\tag{2.22}$$

Thus $(\partial/\partial t)K(r,t)Cx_0=K(r,t)CA(r+t)x_0$. The second equality holds by (i).

Now integrating of this equation, we have

$$K(r,t)Cx_0 - Cx_0 = C\int_0^t K(r,s)A(r+s)x_0 ds. \tag{2.23}$$

Hence injectivity of C implies (iii).

(iv) is trivial from continuity of f and strongly continuity of $\{K(s,t)\}_{s,t\geq 0}$. In (v), obviously $\{R(s,t)\}_{s,t\geq 0}$ is a $C'C$-quasi-semigroup. For $x \in D$, we have

$$\frac{R(s,t)x - CC'x}{t} = C'\frac{K(s,t)x - Cx}{t}, \tag{2.24}$$

which tends to $C'CA(s)$, as $t \to 0^+$. This proves (v).

(vi) can be seen easily. □

3. Evolution Equations and Regularized Quasi-Semigroups

Suppose C is an injective bounded linear operator on Banach space X and $r > 0$. In this section, we study the solutions of the following abstract evolution equation using the regularized quasi-semigroups:

$$\begin{aligned} \dot{x}(t) &= A(t+r)x(t), \quad t > 0, \\ x(0) &= C^2 x_0, \quad x_0 \in X. \end{aligned} \tag{3.1}$$

One can see [13, 14] for a comprehensive studying of abstract evolution equations.

Theorem 3.1. *Let $\{A(s)\}_{s\geq 0}$ be the infinitesimal generator of a C-quasi-semigroups $\{K(s,t)\}_{s,t\geq 0}$ on Banach space X, with domain D. Then for each $x_0 \in D$ and $r \geq 0$, the initial value problem* (3.1) *admits a unique solution.*

Proof. Let $x(t) = K(r,t)Cx_0$. By Theorem 2.6(ii), $x(t)$ is a solution of (3.1).

Now we show that this solution is unique. Suppose $y(s)$ is another solution of (3.1). Trivially $y(s) \in D$. Let $t > 0$. For $s \in [0,t]$ and $x \in X$, define

$$F(s)x = K(r+s, t-s)Cx, \qquad G(s) = F(s)Cy(s). \tag{3.2}$$

From C-quasi-semigroup properties, for small enough $h > 0$, we have

$$\begin{aligned} K(r+s,t-s)C &= K(r+s+t-s-(t-s-h), t-s-h)K(r+s, t-s-(t-s-h)) \\ &= K(r+s+h, t-s-h)K(r+s,h). \end{aligned} \tag{3.3}$$

So

$$\frac{F(s+h)x - F(s)x}{h} = \frac{K(r+s+h,t-s-h)Cx - K(r+s+h,t-s-h)K(r+s,h)x}{h}$$
$$= -K(r+s+h,t-s-h)\left[\frac{K(r+s,h)x - Cx}{h}\right] \tag{3.4}$$
$$\longrightarrow -K(r+s,t-s)CA(r+s)x, \quad \text{as } h \longrightarrow 0.$$

This means that

$$\dot{F}(s)x = -K(r+s,t-s)CA(r+s)x. \tag{3.5}$$

Therefore, from this, the fact that $y(s)$ satisfies (3.1), and $CF(s) = F(s)C$, we obtain that

$$\dot{G}(s) = \dot{F}(s)Cy(s) + F(s)C\dot{y}(s) = -K(r+s,t-s)CA(r+s)Cy(s) + K(r+s,t-s)C^2\dot{y}(s)$$
$$= -K(r+s,t-s)CA(r+s)Cy(s) + K(r+s,t-s)C^2A(r+s)y(s) = 0. \tag{3.6}$$

Hence for every $s \in (0,t)$, $\dot{G}(s) = 0$. Consequently, $G(s)$ is a constant function on $[0,t]$. In particular, $G(0) = G(t)$. So from $y(0) = Cx_0$, we have

$$G(0) = F(0)Cy(0) = K(r,t)C^2x_0 = G(t) = F(t)Cy(t) = K(r+t,0)C^2y(t) = C^3y(t). \tag{3.7}$$

Hence $C^2K(r,t)x_0 = C^3y(t)$. Now injectivity of C implies that $y(t) = K(r,t)Cx_0$, which proves the uniqueness of the solution. □

Now with the above notation, we consider the inhomogeneous evolution equation

$$\dot{x}(t) = A(r+t)x(t) + C^2f(t), \quad 0 < t \le T,$$
$$x(0) = C^2x_0, \quad x_0 \in D. \tag{3.8}$$

The following theorem guarantees the existence and uniqueness of solutions of (3.8) with some sufficient conditions on f.

Theorem 3.2. *Let $K(s,t)$ be a C-quasi-semigroup on Banach space X, with the generator $\{A(s)\}_{s\ge 0}$ whose domain is D. If $f : [0,T] \to D$ is a continuous function, each operator $A(s)$ is closed, and*

$$C\int_0^t K(r+s,t-s)f(s)ds \in D, \quad 0 < t \le T, \tag{3.9}$$

then the initial value equation (3.8) *admits a unique solution*

$$x(t) = K(r,t)Cx_0 + \int_0^t K(r+s,t-s)Cf(s)ds. \tag{3.10}$$

Proof. For the existence of the solution, it is enough to show that $x(t)$ in (3.10) is continuously differentiable and satisfies (3.8).

Trivially $x(0) = Cx_0$. We know that $y(t) = K(r,t)Cx_0$ is a solution of (3.1) by Theorem 3.1. Define

$$g(t) = \int_0^t K(r+s,t-s)Cf(s)ds, \tag{3.11}$$

which is in D by our hypothesis. We have

$$\begin{aligned}\frac{g(t+h)-g(t)}{h} &= \frac{1}{h}\left[\int_0^{t+h} K(r+s,t+h-s)Cf(s)ds - \int_0^t K(r+s,t-s)Cf(s)ds\right]\\ &= \frac{1}{h}\left[\int_0^t K(r+s,t+h-s)Cf(s)ds - \int_0^t K(r+s,t-s)Cf(s)ds\right.\\ &\qquad \left.+\int_t^{t+h} K(r+s,t+h-s)Cf(s)ds\right].\end{aligned} \tag{3.12}$$

On the other hand, the C-quasi-semigroup properties imply that

$$\begin{aligned}K(r+s,t+h-s)Cf(s) &= K(r+s+t+h-s-h,h)K(r+s,t+h-s-h)f(s)\\ &= K(r+t,h)K(r+s,t-s)f(s).\end{aligned} \tag{3.13}$$

So

$$\begin{aligned}\frac{g(t+h)-g(t)}{h} &= \frac{1}{h}\left[\int_0^t K(r+t,h)K(r+t,t-s)f(s)ds\right.\\ &\qquad \left.-\int_0^t K(r+s,t-s)Cf(s)ds + \int_t^{t+h} K(r+s,t+h-s)Cf(s)ds\right]\\ &= \int_0^t K(r+t,t-s)\left(\frac{K(r+t,h)f(s)-Cf(s)}{h}\right)ds\\ &\qquad + \frac{1}{h}\int_t^{t+h} K(r+s,t+h-s)Cf(s)ds.\end{aligned} \tag{3.14}$$

Since the range of f is in D, passing to the limit when $h \to 0$, and using Theorem 2.6(v), we have

$$\begin{aligned}\dot{g}(t) &= \int_0^t K(r+s,t-s)CA(r+t)f(s)ds + K(r+t,t-t)Cf(t)\\ &= \int_0^t K(r+s,t-s)CA(r+t)f(s)ds + C^2 f(t).\end{aligned} \tag{3.15}$$

Therefore, $\dot{g}(t)$ exists. Also by our hypothesis $A(r+t)$ is closed, and $\int_0^t K(r+s,t-s)Cf(s)ds \in D$, thus

$$\int_0^t K(r+s,t-s)CA(r+t)f(s)ds = A(r+t)\int_0^t K(r+s,t-s)Cf(s)ds. \tag{3.16}$$

Consequently,

$$\dot{g}(t) = A(r+t)g(t) + C^2 f(t), \quad t \geq 0. \tag{3.17}$$

Hence

$$\begin{aligned} \dot{x}(t) &= \frac{\partial}{\partial t}K(r,t)Cx_0 + A(r+t)\int_0^t K(r+s,t-s)Cf(s)ds + C^2 f(t) \\ &= A(r+t)\left(K(r,t)Cx_0 + \int_0^t K(r+s,t-s)Cf(s)ds\right) + C^2 f(t) \\ &= A(r+t)x(t) + C^2 f(t). \end{aligned} \tag{3.18}$$

This completes the proof. □

We conclude this section with two simple perturbation theorems and some examples, as applications of our discussion.

Theorem 3.3. (a) *Suppose B is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ and $\{A(s)\}_{s\geq 0}$ with domain D is the generator of a regularized C-quasi-semigroup $\{K(s,t)\}_{s,t\geq 0}$, which commutes with $\{T(t)\}_{t\geq 0}$. Then $\{A(s)+B\}_{s\geq 0}$ with domain $D\cap D(B)$ is the infinitesimal generator of a regularized C-quasi-semigroup.*

(b) *Suppose B is the infinitesimal generator of an exponentially bounded C-semigroup $\{T(t)\}_{t\geq 0}$ and $\{A(s)\}_{s\geq 0}$ with domain D is the generator of a quasi-semigroup (resp., regularized C'-quasi-semigroup) $\{K(s,t)\}_{s,t\geq 0}$, which commutes with $\{T(t)\}_{t\geq 0}$. Then $\{A(s)+B\}_{s\geq 0}$ with domain $D\cap D(B)$ is the infinitesimal generator of a C-regularized quasi-semigroup (resp., regularized CC'-quasi-semigroup).*

Proof. In (a) and (b), define

$$R(s,t) = T(t)K(s,t). \tag{3.19}$$

One can see that $\{R(s,t)\}_{s,t\geq 0}$ is a C-regularized quasi-semigroup (in (b), resp., regularized CC'-quasi-semigroup). We just prove that $\{A(s)+B\}_{s\geq 0}$ is its generator. In (a), let $\{B(s)\}_{s\geq 0}$ be the infinitesimal generator of $\{R(s,t)\}_{s,t\geq 0}$ and $x \in D\cap D(B)$. Hence

$$\lim_{t\to 0^+}\frac{T(t)x-x}{t}, \qquad \lim_{t\to 0^+}\frac{K(s,t)x-Cx}{t} \tag{3.20}$$

exist in X and the range of C, respectively. Now the fact that C commutes with $T(t)$ and strongly continuity of $T(t)$ implies that

$$\lim_{t\to 0^+} T(t)\frac{K(s,t)x - Cx}{t} \tag{3.21}$$

exists in the range of C. So

$$\lim_{t\to 0^+}\frac{R(s,t)x - Cx}{t} = \lim_{t\to 0^+}\frac{T(t)K(s,t)x - Cx}{t} = \lim_{t\to 0^+} T(t)\frac{K(s,t)x - Cx}{t} + C\lim_{t\to 0^+}\frac{T(t)x - x}{t} \tag{3.22}$$

exists in the range of C and

$$CB(s)x = \lim_{t\to 0^+}\frac{R(s,t)x - Cx}{t} = CA(s)x + CBx. \tag{3.23}$$

By injectivity of C, $B(s)x = A(s)x + Bx$.

The proof the other parts is similar. □

Theorem 3.4. *Let $K(s,t)$ be a C-quasi-semigroup of operator on Banach space X with the generator $\{A(s)\}$ on domain D. If $B \in B(X)$ commutes with $K(s,t)$, $s,t \geq 0$, and $B^2 = B$, then $\{BA(s)\}_{s\geq 0}$ is the infinitesimal generator of C-regularized quasi-semigroup*

$$R(s,t) = B(K(s,t) - C) + C. \tag{3.24}$$

Proof. The C-quasi-semigroup properties of $\{R(s,t)\}_{s,t\geq 0}$ can be easily verified. We just prove that its generator is $\{BA(s)\}_{s\geq 0}$. Let $x \in D$; we have

$$\frac{R(s,t)x - Cx}{t} = \frac{B(K(s,t) - C)x + Cx - Cx}{t} = B\frac{K(s,t)x - Cx}{t} \tag{3.25}$$

which tends to $BA(s)x$, as $t \to 0$. This completes the proof. □

Example 3.5. Let $r > 0$. Consider the following initial value problem:

$$\begin{gathered}\frac{\partial}{\partial t}x(t,\varepsilon) = 2(r+t)\frac{\partial}{\partial \varepsilon}x(t,\varepsilon) + \varepsilon x(t,\varepsilon),\\ x(0,\varepsilon) = e^{-4\varepsilon^2}x_0(\varepsilon), \quad \varepsilon, t \geq 0.\end{gathered} \tag{3.26}$$

Let $X = \mathrm{BUC}(\mathbb{R})$, with the supremum-norm. Define $C \in B(X)$ by $Cx(\varepsilon) = e^{-\varepsilon^2}x(\varepsilon)$, $x(\cdot) \in X$. Also define $B : D(B) \to X$ by $Bx(\varepsilon) = \varepsilon x(\varepsilon)$, where $D(B) = \{x \in X : Bx \in X\}$. It is well known that B is the infinitesimal generator of C-regularized semigroup $T(t)$, defined by $T(t)x(\varepsilon) = e^{-\varepsilon^2+\varepsilon t}x(\varepsilon)$. Now with $D = \{x \in X : \dot{x} \in X\}$, if $A(s) : D \to X$ is defined by $A(s)x = 2s\dot{x}$, then by Example 2.3, $\{A(s)\}_{s\geq 0}$ is the infinitesimal generator of the regularized

C^2-quasi-semigroup $K(s,t)x(\varepsilon) = e^{-\varepsilon^2}x(t^2 + 2st + \varepsilon)$. Using Theorem 3.3 and the fact that $T(t)K(s,r) = K(s,r)T(t)$, $s,t,t \geq 0$, we obtain that $\{A(s) + B\}$ is the infinitesimal generator of regularized C^2-quasi-semigroup $R(s,t) = T(t)K(s,t)$. Also using these operators, (3.26) can be written as

$$\begin{aligned} \dot{x}(t) &= (A(r+t) + B)x(t), \\ x(0) &= C^4 x_0. \end{aligned} \tag{3.27}$$

Thus by Theorem 3.1 for any $x_0 \in D \cap D(B)$, (3.26) has the unique solution

$$x(t,\varepsilon) = R(r,t)C^2 x_0(\varepsilon) = e^{-4\varepsilon^2 + \varepsilon t} x_0\left(t^2 + 2rt + \varepsilon\right). \tag{3.28}$$

Example 3.6. For a given sequence $(p_n)_{n\in\mathbb{N}}$ of complex numbers with nonzero elements and $(y_n)_{n\in\mathbb{N}}$, consider the following equation:

$$\begin{aligned} \frac{d}{dt}x_n(t) &= e^{in(t+1)}x_n(t) + p_n x_n(t), \\ x_n(0) &= p_n^2 y_n, \quad n \in \mathbb{N}. \end{aligned} \tag{3.29}$$

Let X be the space c_0, the set of all complex sequence with zero limit at infinity. For a bounded sequence $p = (p_n)_{n\in\mathbb{N}}$, define $A : D(A) :\to X$ and M_p on X by

$$A(x_n)_{n\in\mathbb{N}} = \left(e^{in}x_n\right)_{n\in\mathbb{N}}, \qquad M_p(x_n)_{n\in\mathbb{N}} = (p_n x_n). \tag{3.30}$$

One can easily see that $D(A) = \{(x_n)_{n\in\mathbb{N}} \in c_0 : (e^{\text{in}}x_n)_{n\in\mathbb{N}} \in c_0\}$ and M_p is a bounded linear operator which is injective. It is well known that A is the infinitesimal generator of strongly continuous semigroup

$$T(t)(x_n)_{n\in\mathbb{N}} = \left(e^{in}x_n\right)_{n\in\mathbb{N}}. \tag{3.31}$$

Thus by Example 2.4, $\{A(t)\}_{t\geq 0}$, defined by

$$A(t)(x_n)_{n\in\mathbb{N}} := AT(t)(x_n)_{n\in\mathbb{N}} = \left(e^{in(1+t)}x_n\right)_{n\in\mathbb{N}}, \tag{3.32}$$

is the infinitesimal generator of the M_p-quasi-semigroup

$$K(s,t) = M_p\left(e^{T(s+t)-T(s)}\right). \tag{3.33}$$

Using these operators, one can rewrite (3.29) as

$$\dot{x}(t) = (A(t) + M_P)x(t),$$
$$x(0) = M_p^2 y_0, \tag{3.34}$$

where $x_0 = (y_n)_{n\in\mathbb{N}}$. Trivially $T(t)$ commutes with $K(r,s)$, for any $r,s,t \geq 0$. Now using Theorem 3.3 we obtain that $\{A(t) + M_p\}_{t\geq 0}$ is the infinitesimal generator of of M_p-quasi-semigroup

$$R(s,t) = T(t)K(s,t). \tag{3.35}$$

Also from Theorem 3.1, with $r = 0$, for any $y \in D(A)$, (3.34) has a unique solution

$$x(t) = R(0,t)M_p y = T(t)K(0,t)M_p^2 x_0. \tag{3.36}$$

But from definition of $K(s,t)$, for a given $(x_n)_{n\in\mathbb{N}} \in c_0$,

$$K(0,t)(x_n)_{n\in\mathbb{N}} = e^{T(t)-I} = e^{-1}\sum_{k=0}^{\infty}\frac{T^k(t)}{k!}(x_n)_{n\in\mathbb{N}} = e^{-1}\sum_{k=0}^{\infty}\left(\frac{e^{iknt}x_n}{k!}\right)_{n\in\mathbb{N}} = e^{-1}\left(\sum_{k=0}^{\infty}\frac{e^{iknt}x_n}{k!}\right)_{n\in\mathbb{N}}. \tag{3.37}$$

So the solution of (3.34) is

$$x(t) = R(0,t)M_p y = \left(\sum_{k=0}^{\infty}\frac{e^{ikt(n+1)-1}p_n^4 y_n}{k!}\right)_{n\in\mathbb{N}}, \tag{3.38}$$

or equivalently the solution of (3.29) is

$$x_n(t) = \sum_{k=0}^{\infty}\frac{e^{ikt(n+1)-1}p_n^4 y_n}{k!}, \quad n \in \mathbb{N}. \tag{3.39}$$

References

[1] D. Barcenas and H. Leiva, *Quasisemigroups, Evolutions Equation and Controllability*, Notas de Matemáticas no. 109, Universidad de Los Andes, Mérida, Venezuela, 1991.

[2] D. Bárcenas, H. Leiva, and A. Tineo Moya, "The dual quasisemigroup and controllability of evolution equations," *Journal of Mathematical Analysis and Applications*, vol. 320, no. 2, pp. 691–702, 2006.

[3] V. Cuc, "On the exponentially bounded C_0-quasisemigroups," *Analele Universităc tii de vest din Timişoara. Seria Matematică-Informatică*, vol. 35, no. 2, pp. 193–199, 1997.

[4] V. Cuc, "Non-uniform exponential stability of C_0-quasisemigroups in Banch space," *Radovi Matematichi*, vol. 10, pp. 35–45, 2001.

[5] V. Cuc, "A generalization of a theorem of Datko and Pazy," *New Zealand Journal of Mathematics*, vol. 30, no. 1, pp. 31–39, 2001.

[6] D. Bárcenas, H. Leiva, and A. Tineo Moya, "Quasisemigroups and evolution equations," *International Journal of Evolution Equations*, vol. 1, no. 2, pp. 161–177, 2005.

[7] E. B. Davies and M. M. H. Pang, “The Cauchy problem and a generalization of the Hille-Yosida theorem,” *Proceedings of the London Mathematical Society.*, vol. 55, no. 1, pp. 181–208, 1987.
[8] R. deLaubenfels, “C-semigroups and the Cauchy problem,” *Journal of Functional Analysis*, vol. 111, no. 1, pp. 44–61, 1993.
[9] Y.-C. Li and S.-Y. Shaw, “N-times integrated C-semigroups and the abstract Cauchy problem,” *Taiwanese Journal of Mathematics*, vol. 1, no. 1, pp. 75–102, 1997.
[10] Y.-C. Li and S.-Y. Shaw, “On characterization and perturbation of local C-semigroups,” *Proceedings of the American Mathematical Society*, vol. 135, no. 4, pp. 1097–1106, 2007.
[11] N. Tanaka and I. Miyadera, “Exponentially bounded C-semigroups and integrated semigroups,” *Tokyo Journal of Mathematics*, vol. 12, no. 1, pp. 99–115, 1989.
[12] N. Tanaka and I. Miyadera, “C-semigroups and the abstract Cauchy problem,” *Journal of Mathematical Analysis and Applications*, vol. 170, no. 1, pp. 196–206, 1992.
[13] A. Pazy, *Semigroups of Operators and Applications to Partial Differential Equations*, Springer, New York, NY, USA, 1983.
[14] K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, vol. 194 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 2000.