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Research Article

New Exact Solutions for the $(2+1)$-Dimensional Broer-Kaup-Kupershmidt Equations

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We investigate the (2 + 1)-dimensional Broer-Kaup-Kupershmidt equations. Some explicit expressions of solutions for the equations are obtained by using the bifurcation method and qualitative theory of dynamical systems. These solutions contain kink-shaped solutions, blow-up solutions, periodic blow-up solutions, and solitary wave solutions. Some previous results are extended.

1. Introduction

Consider the $(2+1)$-dimensional Broer-Kaup-Kupershmidt (BKK) equations [1–10]

$$\begin{aligned} u_{ty} - u_{xxy} + 2(uu_x)_y + 2v_{xx} &= 0, \\ v_t + v_{xx} + 2(uv)_x &= 0. \end{aligned} \tag{1.1}$$

These equations have been widely applied in many branches of physics like plasma physics, fluid dynamics, nonlinear optics, and so forth. So a good understanding of more solutions of the $(2+1)$-dimensional BKK equations (1.1) might be very helpful, especially for coastal and civil engineers to apply the non-linear water models in a harbor and coastal design.

Recently, the $(2+1)$-dimensional BKK equations have been studied by many authors. Yomba [1, 2] used the modified extended Fan subequation method to obtain soliton-like solutions, triangular-like solutions, and single and combined nondegenerate Jacobi elliptic wave function-like solutions of (1.1). Zhang and Xia [3] used the further improved extended Fan sub-equation method to obtain soliton-like solutions, triangular-like solutions, single and combined non-degenerate Jacobi elliptic wave function-like solutions, and Weierstrass elliptic doubly-like periodic solutions of (1.1). Abdou and Soliman [4] obtained some traveling wave solutions of (1.1) by using the modified extended tanh-function method. Song et al. [5] used the new extended Riccati equation rational expansion method to study multiple exact solutions of (1.1). Zhang [6] used the Exp-function method to seek generalized exact solutions with three arbitrary functions of (1.1). El-Wakil and Abdou [7] obtained exact travelling wave solutions by using improved tanh-function method. Lu et al. [8] obtained some exact traveling wave solutions of (1.1) by using the first integral method. Davodi et al. [9] obtained some generalized solitary solutions of (1.1). Bai and Zhao [10] used the Repeated General Algebraic Method to obtain exact solutions of (1.1).

In this paper, we employ the bifurcation method and qualitative theory of dynamical systems [11–19] to investigate the (2 + 1)-dimensional BKK equations (1.1), and we obtain some explicit expressions of solutions for (1.1). These solutions contain kink-shaped solutions, blow-up solutions, periodic blow-up solutions, and solitary wave solutions, most of which are new by comparing with the solutions of the references [1–10].

The remainder of this paper is organized as follows. In Section 2, we present our main results. Section 3 gives the theoretical derivation for our main results. A short conclusion will be given in Section 4.

2. Main Results

In this section, we state our main results. For ease of exposition, we have omitted the expressions of v with $v_n(x,y,t) = \phi_n(\xi) = 1/2(c\varphi + \varphi' - \varphi^2)$, $n = 1,2,\ldots,10$, in the entire process.

Proposition 2.1. *For given constants c and g_0, which will be given later in* (3.10)*, the* (2 + 1)*-dimensional BKK equations have the following exact solutions.*

(1) If $g = 0$, we get two kink solutions:

$$u_1(x,y,t) = \frac{\delta c(1+\tanh(c/2)(x+y-ct))}{\delta - 1 + (\delta+1)\tanh(c/2)(x+y-ct)},$$
$$u_2(x,y,t) = \frac{c(-1+\tanh(c/2)(x+y-ct))}{\eta - 1 + (\eta+1)\tanh(c/2)(x+y-ct)}, \tag{2.1}$$

where δ and η are constants, two blow-up solutions

$$u_{3\pm}(x,y,t) = \frac{c}{2}\left(1 \pm \coth\frac{c}{2}(x+y-ct)\right), \tag{2.2}$$

and four periodic blow-up solutions

$$u_{4\pm}(x,y,t) = \frac{c}{2}\left(1 \pm \sqrt{2}\csc\frac{\sqrt{2}}{2}c(x+y-ct)\right),$$
$$u_{5\pm}(x,y,t) = \frac{c}{2}\left(1 \pm \sqrt{2}\sec\frac{\sqrt{2}}{2}c(x+y-ct)\right). \tag{2.3}$$

(2) If $0 < g < g_0$, we get a solitary wave solution

$$u_6(x,y,t) = \frac{c\left(\sqrt{2(\alpha-1)}\beta \cosh\theta(x+y-ct)\right) + (\alpha-1)(1+\beta-\cosh 2\theta(x+y-ct))}{\alpha(-1+\alpha+\beta-(\alpha-1)\cosh 2\theta(x+y-ct))}, \tag{2.4}$$

and two blow-up solutions

$$u_{7\pm}(x,y,t) = \frac{c(\alpha(2+\beta)-2-2(\alpha-1)\cosh\theta(x+y-ct) \pm \beta^{(3/2)}\coth(\theta/2)(x+y-ct))}{2\alpha(-1+\alpha+\beta-(\alpha-1)\cosh\theta(x+y-ct))}, \tag{2.5}$$

where $\beta = 6-6\alpha+\alpha^2$ and $\theta = c\sqrt{\beta}/\alpha$.

(3) If $g = g_0$, we get three blow-up solutions as follows:

$$\begin{aligned} u_8(x,y,t) &= \frac{-12\sqrt{3}+\left(6+6\sqrt{3}\right)c(x+y-ct)-\left(3+\sqrt{3}\right)c^2(x+y-ct)^2}{6\left(2\sqrt{3}(x+y-ct)-c(x+y-ct)^2\right)}, \\ u_9(x,y,t) &= \frac{12\sqrt{3}+\left(6+6\sqrt{3}\right)c(x+y-ct)+\left(3+\sqrt{3}\right)c^2(x+y-ct)^2}{6\left(2\sqrt{3}(x+y-ct)+c(x+y-ct)^2\right)}, \\ u_{10}(x,y,t) &= \frac{-9c+9\sqrt{3}c+\left(3+\sqrt{3}\right)c^3(x+y-ct)^2}{-18+6c^2(x+y-ct)^2}. \end{aligned} \tag{2.6}$$

3. The Derivations of Main Results

In this section, we will give the derivations for our main results.

For given constant wave speed c, substituting $u = \varphi(\xi)$, $v = \phi(\xi)$ with $\xi = x+y-ct$ into the $(2+1)$-dimensional BKK equations (1.1), it follows that

$$\begin{cases} -c\varphi'' - \varphi''' + 2(\varphi\varphi')' + 2\phi'' = 0, \\ -c\phi' + \phi'' + 2(\varphi\phi)' = 0. \end{cases} \tag{3.1}$$

Integrating the first equation of (3.1) twice and letting integral constants be zero, we have

$$\phi = \frac{1}{2}\left(c\varphi + \varphi' - \varphi^2\right). \tag{3.2}$$

Integrating the second equation of (3.1) once, we have

$$-c\phi + \phi' + 2\varphi\phi = \frac{1}{2}g, \tag{3.3}$$

where $(1/2)g$ is integral constant.

Substituting (3.2) into (3.3), we get

$$\frac{1}{2}\varphi'' - \frac{1}{2}c^2\varphi + \frac{3}{2}c\varphi^2 - \varphi^3 = \frac{1}{2}g. \tag{3.4}$$

Letting $\psi = \varphi'$, we get the following planar system:

$$\begin{aligned}\frac{d\varphi}{d\xi} &= \psi,\\ \frac{d\psi}{d\xi} &= 2\varphi^3 - 3c\varphi^2 + c^2\varphi + g.\end{aligned} \tag{3.5}$$

Obviously, the above system (3.5) is a Hamiltonian system with Hamiltonian function

$$H(\varphi,\psi) = \frac{1}{2}\psi^2 - \frac{1}{2}\varphi^4 + c\varphi^3 - \frac{1}{2}c^2\varphi^2 - g\varphi. \tag{3.6}$$

Now, we consider the phase portraits of system (3.5). Set

$$\begin{aligned}f_0(\varphi) &= 2\varphi^3 - 3c\varphi^2 + c^2\varphi,\\ f(\varphi) &= 2\varphi^3 - 3c\varphi^2 + c^2\varphi + g.\end{aligned} \tag{3.7}$$

$f_0(\varphi)$ has three fixed points φ_0, φ_1, φ_2, and their expressions are given as follows:

$$\varphi_0 = 0, \qquad \varphi_1 = \frac{c}{2}, \qquad \varphi_2 = c. \tag{3.8}$$

It is easy to obtain the two extreme points of $f_0(\varphi)$ as follows:

$$\varphi_{\pm}^{*} = \frac{3c \pm \sqrt{3}c}{6}. \tag{3.9}$$

Let

$$g_0 = \left|f_0(\varphi_{\pm}^{*})\right| = \frac{c^3}{6\sqrt{3}}, \tag{3.10}$$

then it is easily seen that g_0 is the extreme values of $f_0(\varphi)$.

Let $(\varphi_i, 0)$ be one of the singular points of system (3.5). Then the characteristic values of the linearized system of system (3.5) at the singular points $(\varphi_i, 0)$ are

$$\lambda_{\pm} = \pm\sqrt{f'(\varphi_i)}. \tag{3.11}$$

From the qualitative theory of dynamical systems, we therefore know that,

(i) if $f'(\varphi_i) > 0$, $(\varphi_i, 0)$ is a saddle point;

(ii) if $f'(\varphi_i) < 0$, $(\varphi_i, 0)$ is a center point;

(iii) if $f'(\varphi_i) = 0$, $(\varphi_i, 0)$ is a degenerate saddle point.

Therefore, we obtain the phase portraits of system (3.5) in Figure 1.

Now, we will obtain the explicit expressions of solutions for the (2 + 1)-dimensional BKK equations (1.1).

(1) If $g = 0$, we will consider two kinds of orbits.

(i) First, we see that there are two heteroclinic orbits Γ_1 and Γ_2 connected at saddle points $(\varphi_0, 0)$ and $(\varphi_2, 0)$. In (φ, ψ)-plane, the expressions of the heteroclinic orbits are given as

$$\psi = \pm\varphi(\varphi - c). \tag{3.12}$$

Substituting (3.12) into $d\varphi/d\xi = \psi$ and integrating them along the heteroclinic orbits Γ_1 and Γ_2, it follows that

$$\begin{aligned} \int_{\varphi^*}^{\varphi} \frac{1}{s(c-s)} ds &= \int_0^{\xi} ds, \\ \int_{\varphi}^{\varphi^*} \frac{1}{s(s-c)} ds &= \int_{\xi}^{0} ds, \end{aligned} \tag{3.13}$$

where $\varphi^* \in (0, c)$ is constant and

$$\pm \int_{\varphi}^{+\infty} \frac{1}{s(s-c)} ds = \int_0^{\xi} ds. \tag{3.14}$$

From (3.13), we have

$$\begin{aligned} \varphi(\xi) &= \frac{\delta c (1 + \tanh(c/2)\xi)}{\delta - 1 + (\delta + 1)\tanh(c/2)\xi}, \\ \varphi(\xi) &= \frac{c(-1 + \tanh(c/2)\xi)}{\eta - 1 + (\eta + 1)\tanh(c/2)\xi}, \end{aligned} \tag{3.15}$$

where $\delta = \varphi^*/(\varphi^* - c), \eta = (\varphi^* - c)/\varphi^*$.

From (3.14), we have

$$\varphi(\xi) = \frac{c}{2}\left(1 \pm \coth \frac{c}{2}\xi\right). \tag{3.16}$$

Noting that $u = \varphi(\xi)$ and $\xi = x + y - ct$, we get two kink-shaped solutions $u_1(x, y, t)$, $u_2(x, y, t)$ and two blow-up solutions $u_{3^\pm}(x, y, t)$ as (2.1), and (2.2). (ii) From the phase portrait, we note

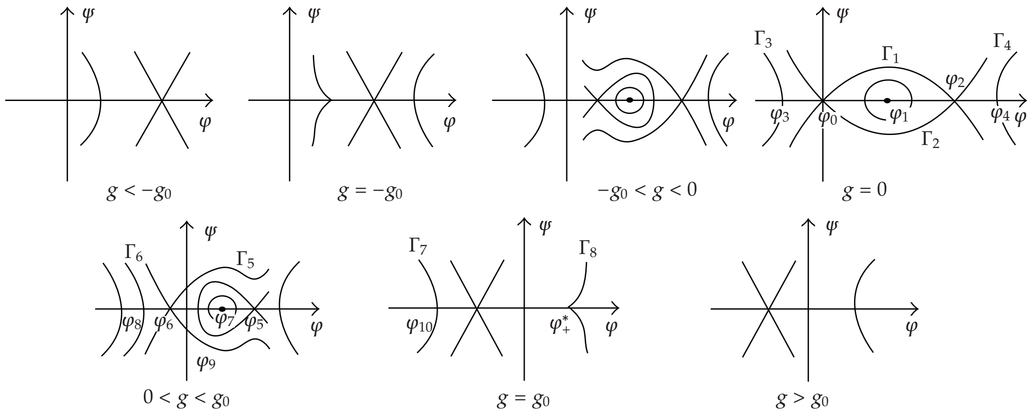

Figure 1: The phase portraits of system (3.5).

that there are two special orbits Γ_3 and Γ_4, which have the same Hamiltonian with that of the center point $(\varphi_1, 0)$. In (φ, ψ)-plane, the expressions of these two orbits are given as

$$\psi = \pm(\varphi - \varphi_1)\sqrt{(\varphi - \varphi_3)(\varphi - \varphi_4)}, \tag{3.17}$$

where

$$\begin{aligned} \varphi_3 &= \frac{1-\sqrt{2}}{2}c, \\ \varphi_4 &= \frac{1+\sqrt{2}}{2}c. \end{aligned} \tag{3.18}$$

Substituting (3.17) into $\mathrm{d}\varphi/\mathrm{d}\xi = \psi$ and integrating them along the two orbits Γ_3 and Γ_4, it follows that

$$\pm\int_{\varphi}^{+\infty} \frac{1}{(s-\varphi_1)\sqrt{(s-\varphi_3)(s-\varphi_4)}}ds = \int_0^{\xi} ds. \tag{3.19}$$

From (3.19), we have

$$\varphi(\xi) = \frac{c}{2}\left(1 \pm \sqrt{2}\csc\frac{\sqrt{2}}{2}c\xi\right). \tag{3.20}$$

At the same time, we note that if $u = \varphi(\xi)$ is a solution of system (3.5), then $u = \varphi(\xi+\gamma)$ is also a solution of system (3.5). Specially, when we take $\gamma = \pi/2$, we get other two solutions

$$\varphi(\xi) = \frac{c}{2}\left(1 \pm \sqrt{2}\sec\frac{\sqrt{2}}{2}c\xi\right). \tag{3.21}$$

Noting that $u = \varphi(\xi)$ and $\xi = x + y - ct$, we get four periodic blow-up solutions $u_{4^\pm}(x, y, t)$ and $u_{5^\pm}(x, y, t)$ as (2.3).

(2) If $0 < g < g_0$, we set the largest solution of $f(\varphi) = 0$ as $\varphi_5 = (c/\alpha)(1 < \alpha < 2)$, then we can get another two solutions of $f(v) = 0$ as follows:

$$\varphi_6 = \frac{c\left(-2 + 3\alpha - \sqrt{-12 + 12\alpha + \alpha^2}\right)}{4\alpha}, \qquad \varphi_7 = \frac{c\left(-2 + 3\alpha + \sqrt{-12 + 12\alpha + \alpha^2}\right)}{4\alpha}. \tag{3.22}$$

We see that there is a homoclinic orbit Γ_5, which passes the saddle point $(\varphi_5, 0)$. In (φ, ψ)-plane, the expressions of the homoclinic orbit are given as

$$\psi = \pm(\varphi - \varphi_5)\sqrt{(\varphi - \varphi_8)(\varphi - \varphi_9)}, \tag{3.23}$$

where

$$\varphi_8 = \frac{c(\alpha - 1) - c\sqrt{2(\alpha - 1)}}{\alpha}, \qquad \varphi_9 = \frac{c(\alpha - 1) + c\sqrt{2(\alpha - 1)}}{\alpha}. \tag{3.24}$$

Substituting (3.23) into $d\varphi/d\xi = \psi$ and integrating them along the homoclinic orbit, it follows that

$$\pm\int_{\varphi_9}^{\varphi} \frac{1}{(s - \varphi_5)\sqrt{(s - \varphi_8)(s - \varphi_9)}}ds = \int_0^{\xi} ds, \qquad \pm\int_{\varphi}^{+\infty} \frac{1}{(s - \varphi_5)\sqrt{(s - \varphi_8)(s - \varphi_9)}}ds = \int_0^{\xi} ds. \tag{3.25}$$

From (3.25), we have

$$\varphi(\xi) = \frac{c\left(\sqrt{2(\alpha - 1)}\beta\cosh\theta\xi\right) + (\alpha - 1)(1 + \beta - \cosh 2\theta\xi)}{\alpha(-1 + \alpha + \beta - (\alpha - 1)\cosh 2\theta\xi)}, \qquad \varphi(\xi) = \frac{c\left(\alpha(2 + \beta) - 2 - 2(\alpha - 1)\cosh\theta\xi \pm \beta^{3/2}\coth(\theta/2)\xi\right)}{2\alpha(-1 + \alpha + \beta - (\alpha - 1)\cosh\theta\xi)}, \tag{3.26}$$

where $\beta = 6 - 6\alpha + \alpha^2$ and $\theta = (c\sqrt{\beta})/\alpha$.

Noting that $u = \varphi(\xi)$ and $\xi = x + y - ct$, we get a solitary wave solution $u_6(x, y, t)$ and two blow-up solutions $u_{7\pm}(x, y, t)$ as (2.4) and (2.5).

(3) If $g = g_0$, from the phase portrait, we see that there are two orbits Γ_7 and Γ_8, which have the same Hamiltonian with the degenerate saddle point $(\varphi_+^*, 0)$. In (φ, ψ)-plane, the expressions of these two orbits are given as

$$\psi = \pm(\varphi - \varphi_+^*)\sqrt{(\varphi - \varphi_+^*)(\varphi - \varphi_{12})}, \tag{3.27}$$

where

$$\varphi_{12} = \frac{1}{2}\left(c - \sqrt{3}c\right). \tag{3.28}$$

Substituting (3.27) into $d\varphi/d\xi = \psi$ and integrating them along these two orbits Γ_7 and Γ_8, it follows that

$$\begin{gathered}\int_\varphi^{+\infty} \frac{1}{(s - \varphi_+^*)\sqrt{(s - \varphi_+^*)(s - \varphi_{12})}} ds = \int_0^\xi ds,\\ \int_\varphi^{\varphi_{12}} \frac{1}{(s - \varphi_+^*)\sqrt{(s - \varphi_+^*)(s - \varphi_{12})}} ds = \int_0^\xi ds\end{gathered} \tag{3.29}$$

From (3.29), we have

$$\begin{gathered}\varphi = \frac{-12\sqrt{3} + \left(6 + 6\sqrt{3}\right)c\xi - \left(3 + \sqrt{3}\right)c^2\xi^2}{6\left(2\sqrt{3}\xi - c\xi^2\right)},\\ \varphi = \frac{12\sqrt{3} + \left(6 + 6\sqrt{3}\right)c\xi + \left(3 + \sqrt{3}\right)c^2\xi^2}{6\left(2\sqrt{3}\xi + c\xi^2\right)},\\ \varphi = \frac{-9c + 9\sqrt{3}c + \left(3 + \sqrt{3}\right)c^3\xi^2}{-18 + 6c^2\xi^2}.\end{gathered} \tag{3.30}$$

Noting that $u = \varphi(\xi)$ and $\xi = x + y - ct$, we get three blow-up solutions $u_8(x, y, t))$, $u_9(x, y, t)$, and $u_{10}(x, y, t)$ as (2.6). Thus, we obtain the results given in Proposition 2.1.

Remark 3.1. One may find that we only consider the case when $g \geq 0$ in Proposition 2.1. In fact, we may get exactly the same solutions in the opposite case.

4. Conclusion

In this paper, we have obtained many new solutions for the (2+1)-dimensional BKK equations (1.1) by employing the bifurcation method and qualitative theory of dynamical systems. The explicit expressions of the solutions have been given in Proposition 2.1. The method can be applied to many other nonlinear evolution equations, and we believe that many new results wait for further discovery by this method.

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