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Research Article

Convergence Theorems for a Maximal Monotone Operator and a V-Strongly Nonexpansive Mapping in a Banach Space

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Let E be a smooth Banach space with a norm $\|\cdot\|$. Let $V(x,y)=\|x\|^2+\|y\|^2-2\langle x,Jy\rangle$ for any $x,y\in E$, where $\langle\cdot,\cdot\rangle$ stands for the duality pair and J is the normalized duality mapping. With respect to this bifunction $V(\cdot,\cdot)$, a generalized nonexpansive mapping and a V-strongly nonexpansive mapping are defined in E. In this paper, using the properties of generalized nonexpansive mappings, we prove convergence theorems for common zero points of a maximal monotone operator and a V-strongly nonexpansive mapping.

1. Introduction

Let E be a smooth Banach space with a norm $\|\cdot\|$ and let C be a nonempty, closed and convex subset of E. We use the following bifunction $V(\cdot,\cdot)$ studied by Alber [1], as well as Kamimura and Takahashi [2]. Let $V(\cdot,\cdot):E\times E\to[0,\infty)$ be defined by $V(x,y)=\|x\|^2-2\langle x,Jy\rangle+\|y\|^2$ for any $x,y\in E$, where $\langle\cdot,\cdot\rangle$ stands for the duality pair and J is the normalized duality mapping. Note that the duality mapping is single valued in a smooth Banach space (see [3]). From the definition of $V(\cdot,\cdot)$ the following properties are trivial.

Lemma 1.1. (a) *For all* $a,b,c\in E$,

$$V(a,b)\le V(a,b)+V(b,c)=V(a,c)-2\langle a-b,Jb-Jc\rangle. \tag{1.1}$$

(b) *If a sequence* $\{x_n\}\subset E$ *satisfies* $\lim_{n\to\infty}V(x_n,p)<\infty$ *for some* $p\in E$, *then* $\{x_n\}$ *is bounded.*

Let $F(T)$ be the fixed points set of T. Ibaraki and Takahashi defined a generalized nonexpansive mapping in a Banach space (see [4]).

Definition 1.2. A mapping $T : C \to C$ is said to be generalized nonexpansive if $F(T) \neq \emptyset$ and $V(Tx, p) \leq V(x, p)$ for all $x \in C$ and $p \in F(T)$.

In this paper, we prove strong convergence theorem for finding common fixed points of a family of generalized nonexpansive mappings. In addition, we prove strong convergence theorem for finding zeroes of a generalized nonexpansive mapping and a maximal monotone operator. Now, we define a V-strongly nonexpansive mapping as follows.

Definition 1.3. A mapping $T : C \to E$ is called V-strongly nonexpansive if there exists a constant $\lambda > 0$ such that

$$V(Tx, Ty) \leq V(x, y) - \lambda V((I - T)x, (I - T)y), \tag{1.2}$$

for all $x, y \in C$, where I is the identity mapping on E. More explicitly, if (1.2) holds, then T is said to be V-strongly nonexpansive with λ.

If T is V-strongly nonexpansive with λ, then T is V-strongly nonexpansive with any $\gamma \in (0, \lambda]$. It is trivial that a V-strongly nonexpansive mapping is generalized nonexpansive if $F(T) \neq \emptyset$. In the following section, we show that in a Hilbert space H a firmly nonexpansive mapping is V-strongly nonexpansive with $\lambda = 1$ and a V-strongly nonexpansive mapping is strongly nonexpansive if $F(T) \neq \emptyset$. Motivated by the results of Manaka and Takahashi [5], we prove weak convergence theorem for common zero points of a maximal monotone operator and a V-strongly nonexpansive mapping in a Banach space.

2. Preliminaries

Let D be a nonempty subset of a Banach space E. A mapping $R : E \to D$ is said to be sunny, if for all $x \in E$ and $t \geq 0$,

$$R(Rx + t(x - Rx)) = Rx. \tag{2.1}$$

A mapping $R : E \to D$ is called a retraction if $Rx = x$ for all $x \in D$ (see [6]). It is known that a generalized nonexpansive and sunny retraction of E onto D is uniquely determined if E is a smooth and strictly convex Banach space (cf., [7]). Ibaraki and Takahashi proved the following results in [4].

Lemma 2.1 (cf., [4]). *Let E be a reflexive, strictly convex, and smooth Banach space, and let T be a generalized nonexpansive mapping from E into itself. Then there exists a sunny and generalized nonexpansive retraction on $F(T)$.*

Lemma 2.2 (cf., [4]). *Let D be a nonempty subset of a reflexive, strictly convex, and smooth Banach space E. Let R be a retraction from E onto D. Then R is sunny and generalized nonexpansive if and only if*

$$\langle x - Rx, JRx - Jy \rangle \geq 0, \tag{2.2}$$

for all $x \in E$ and $y \in D$.

A generalized resolvent J_r of a maximal monotone operator $B \subset E^* \times E$ is defined by $J_r = (I + rBJ)^{-1}$ for any real number $r > 0$. It is well known that $J_r : E \to E$ is single valued if E is reflexive, smooth, and strictly convex (see [8]). It is also known that J_r satisfies

$$\langle x - J_r x - (y - J_r y), JJ_r x - JJ_r y \rangle \geq 0, \quad \forall x, y \in E. \tag{2.3}$$

This implies that

$$\langle x - J_r x, JJ_r x - Jp \rangle \geq 0, \quad \forall x \in E,\ p \in F(J_r). \tag{2.4}$$

Therefore, from Lemma 1.1(a), we obtain the following proposition.

Proposition 2.3. (a) *If a sunny retraction R is generalized nonexpansive, then R satisfies*

$$\begin{aligned} V(x, Rx) + V(Rx, y) &= V(x, y) - 2\langle x - Rx, JRx - Jy \rangle \\ &\leq V(x, y), \quad \forall x, y \in D. \end{aligned} \tag{2.5}$$

(b) *For each $r > 0$, a generalized resolvent J_r satisfies*

$$V(x, J_r x) + V(J_r x, p) \leq V(x, p), \quad \forall x \in E,\ p \in F(J_r). \tag{2.6}$$

Remark 2.4. The property in Proposition 2.3(b) means that J_r is generalized nonexpansive for any $r > 0$.

We recall some nonlinear mappings in Banach spaces (see, e.g., [9–12]).

Definition 2.5. Let D be a nonempty, closed, and convex subset of E. A mapping $T : D \to E$ is said to be firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \langle x - y, j(Tx - Ty) \rangle, \tag{2.7}$$

for all $x, y \in D$ and some $j(Tx - Ty) \in J(Tx - Ty)$.

In [12], Reich introduced a class of strongly nonexpansive mappings which is defined with respect to the Bregman distance $D(\cdot,\cdot)$ corresponding to a convex continuous function f in a reflexive Banach space E. Let S be a convex subset of E, and let $T : S \to S$ be a self-mapping of S. A point p in the closure of S is said to be an asymptotically fixed point of T if S contains a sequence $\{x_n\}$ which converges weakly to p and the sequence $\{x_n - Tx_n\}$ converges strongly to 0. $\widehat{F}(T)$ denotes the asymptotically fixed points set of T.

Definition 2.6. The Bregman distance corresponding to a function $f : E \to R$ is defined by

$$D(x,y) = f(x) - f(y) - f'(y)(x-y), \tag{2.8}$$

where f is the Gâteaux differentiable and $f'(x)$ stands for the derivative of f at the point x. We say that the mapping T is strongly nonexpansive if $\widehat{F}(T) \neq \emptyset$ and

$$D(p,Tx) \le D(p,x), \quad \forall p \in \widehat{F}(T),\ x \in S, \tag{2.9}$$

and if it holds that $\lim_{n\to\infty} D(Tx_n, x_n) = 0$ for a bounded sequence $\{x_n\}$ such that $\lim_{n\to\infty}(D(p,x_n) - D(p,Tx_n)) = 0$ for any $p \in \widehat{F}(T)$.

We remark that the symbols $x_n \to u$ and $x_n \rightharpoonup u$ mean that $\{x_n\}$ converges strongly and weakly to u, respectively. Taking the function $\|\cdot\|^2$ as the convex, continuous, and Gâteaux differentiable function f, we obtain the fact that the Bregman distance $D(\cdot,\cdot)$ coincides with $V(\cdot,\cdot)$. Especially in a Hilbert space, $D(x,y) = V(x,y) = \|x-y\|^2$. Bruck and Reich defined strongly nonexpansive mappings in a Hilbert space H as follows (cf., [10]).

Definition 2.7. A mapping $T : D \to H$ is said to be strongly nonexpansive if T is nonexpansive with $F(T) \neq \emptyset$ and if it holds that

$$(x_n - y_n) - (Tx_n - Ty_n) \longrightarrow 0 \tag{2.10}$$

when $\{x_n\}$ and $\{y_n\}$ are sequences in D such that $\{x_n - y_n\}$ is bounded and $\lim_{n\to\infty}(\|x_n - y_n\| - \|Tx_n - Ty_n\|) = 0$.

The relation among firmly nonexpansive mappings, strongly nonexpansive mappings and V-strongly nonexpansive mappings is shown in the following proposition.

Proposition 2.8. *In a Hilbert space H, the following hold.*

(a) *A firmly nonexpansive mapping is V-strongly nonexpansive with $\lambda = 1$.*

(b) *A V-strongly nonexpansive mapping T with $\widehat{F}(T) \neq \emptyset$ is strongly nonexpansive.*

Proof. (a) Suppose that T is firmly nonexpansive. Since $J = I$ in a Hilbert space, it holds that

$$2\langle x-y, Tx-Ty\rangle - \|Tx-Ty\|^2 = \|x-y\|^2 - \|(I-T)x-(I-T)y\|^2, \tag{2.11}$$

for all $x, y \in D$. Therefore, it is obvious that T is firmly nonexpansive if and only if T satisfies

$$\|x-y\|^2 - \|(T-I)x-(T-I)y\|^2 \geq \|Tx-Ty\|^2, \tag{2.12}$$

for all $x, y \in D$. Hence we obtain (a).

(b) Suppose that T is V-strongly nonexpansive with λ. Then, it is trivial that T is nonexpansive and (2.9) holds. Suppose that the sequences $\{x_n\}$ and $\{y_n\}$ satisfy the conditions in Definition 2.7. Then $\{Tx_n - Ty_n\}$ is also bounded. Since T is V-strongly nonexpansive with λ, we have that

$$\begin{aligned} 0 &\leq \lambda\|x_n - y_n - (Tx_n - Ty_n)\|^2 \\ &= \lambda V((I-T)x_n, (I-T)y_n) \\ &\leq V(x_n, y_n) - V(Tx_n, Ty_n) \\ &= \|x_n - y_n\|^2 - \|Tx_n - Ty_n\|^2 \\ &= (\|x_n - y_n\| + \|Tx_n - Ty_n\|)(\|x_n - y_n\| - \|Tx_n - Ty_n\|) \\ &\longrightarrow 0. \end{aligned} \tag{2.13}$$

Hence, $(x_n - y_n) - (Tx_n - Ty_n) \to 0$ for $\lambda > 0$. This means that T is strongly nonexpansive. □

In a Banach space, V-strongly nonexpansive mappings have the following properties.

Proposition 2.9. *In a smooth Banach space E, the following hold.*

(a) *For $c \in (-1, 1]$, $T = cI$ is V-strongly nonexpansive. For $c = 1$, $T = I$ is V-strongly nonexpansive for any $\lambda > 0$. For $c \in (-1, 1)$, $T = cI$ is V-strongly nonexpansive for any $\lambda \in (0, (1+c)/(1-c)]$.*

(b) *If T is V-strongly nonexpansive with λ, then, for any $\alpha \in [-1, 1]$ with $\alpha \neq 0$, αT is also V-strongly nonexpansive with $\alpha^2\lambda$.*

(c) *If T is V-strongly nonexpansive with $\lambda \geq 1$, then $A = I - T$ is V-strongly nonexpansive with λ^{-1}.*

(d) *Suppose that T is V-strongly nonexpansive with λ and that $\alpha \in [-1, 1]$ satisfies $\alpha^2\lambda \geq 1$. Then $(I - \alpha T)$ is V-strongly nonexpansive with $(\alpha^2\lambda)^{-1}$. Moreover, if $T_\alpha = I - \alpha T$, then*

$$V(T_\alpha x, T_\alpha y) \leq V(x, y) - \lambda^{-1} V(Tx, Ty). \tag{2.14}$$

Proof. (a) Let $T = cI$ for any $c \in (-1,1]$, and denote $I_l = V(Tx,Ty)$ and $I_r = V(x,y) - \lambda V((I - T)x, (I - T)y)$. Since $J(cx) = cJx$, we have

$$\begin{aligned}
I_l &= V(Tx,Ty) = c^2\left\{\|x\|^2 + \|y\|^2 - 2\langle x, Jy\rangle\right\} = c^2V(x,y),\\
I_r &= V(x,y) - \lambda V((I-T)x,(I-T)y)\\
&= \|x\|^2 - \lambda\|(1-c)x\|^2 + \|y\|^2 - \lambda\|(1-c)y\|^2\\
&\quad - 2\langle x, Jy\rangle + 2\lambda\langle(1-c)x, J((1-c)y)\rangle\\
&= \left\{1-\lambda(1-c)^2\right\}\left(\|x\|^2 + \|y\|^2\right) - 2\left\{1-\lambda(1-c)^2\right\}\langle x, Jy\rangle\\
&= \left\{1-\lambda(1-c)^2\right\}\left(\|x\|^2 + \|y\|^2 - 2\langle x, Jy\rangle\right)\\
&= \left\{1-\lambda(1-c)^2\right\}V(x,y).
\end{aligned} \tag{2.15}$$

For $c = 1$, it holds that $I_l \le \mathrm{I}_r$ for all $\lambda > 0$. For $c \in (-1,1)$, we obtain

$$\begin{aligned}
I_l \le I_r &\Longleftrightarrow c^2 \le 1 - \lambda(1-c)^2 \Longleftrightarrow 0 < \lambda(1-c)^2 \le 1 - c^2\\
&\Longleftrightarrow 0 < \lambda \le \frac{(1-c)(1+c)}{(1-c)^2} = \frac{(1+c)}{(1-c)}.
\end{aligned} \tag{2.16}$$

Therefore, $T = cI$ is V-strongly nonexpansive for any $\lambda \in (0,(1+c)/(1-c)]$.

(b) If T is V-strongly nonexpansive with $\lambda > 0$, then, for $\alpha \in [-1,1]$ with $\alpha \ne 0$,

$$\begin{aligned}
V(\alpha Tx, \alpha Ty) &= \|\alpha Tx\|^2 + \|\alpha Ty\|^2 - 2\langle \alpha Tx, J(\alpha Ty)\rangle\\
&= \alpha^2\left\{\|Tx\|^2 + \|Ty\|^2 - 2\langle Tx, J(Ty)\rangle\right\}\\
&= \alpha^2 V(Tx,Ty)\\
&\le \alpha^2\{V(x,y) - \lambda V((I-T)x,(I-T)y)\}\\
&= V(x,y) - \left(1-\alpha^2\right)V(x,y) - \alpha^2\lambda V((I-T)x,(I-T)y)\\
&\le V(x,y) - \alpha^2\lambda V((I-T)x,(I-T)y).
\end{aligned} \tag{2.17}$$

This means that αT is V-strongly nonexpansive with $\alpha^2\lambda$.

(c) Suppose that T is V-strongly nonexpansive with $\lambda \geq 1$ and let $A = I - T$. Then we have that

$$\begin{aligned} V((I-A)x,(I-A)y) &= V(Tx,Ty) \\ &\leq V(x,y) - \lambda V((I-T)x,(I-T)y) \\ &= V(x,y) - \lambda V(Ax,Ay). \end{aligned} \tag{2.18}$$

This inequality implies that

$$\begin{aligned} V(Ax,Ay) &\leq \lambda^{-1}\{V(x,y) - V((I-A)x,(I-A)y)\} \\ &= \lambda^{-1}V(x,y) - \lambda^{-1}V((I-A)x,(I-A)y) \\ &\leq V(x,y) - \lambda^{-1}V((I-A)x,(I-A)y). \end{aligned} \tag{2.19}$$

Thus A is V-strongly nonexpansive with λ^{-1}.

(d) From (b) and the assumption, αT is V-strongly nonexpansive with $\alpha^2\lambda \geq 1$, and from (c) we have that $(I - \alpha T)$ is V-strongly nonexpansive with $(\alpha^2\lambda)^{-1}$. Furthermore we obtain that

$$\begin{aligned} V(T_\alpha x, T_\alpha y) &\leq V(x,y) - \left(\alpha^2\lambda\right)^{-1} V((I-T_\alpha)x,(I-T_\alpha)y) \\ &= V(x,y) - \left(\alpha^2\lambda\right)^{-1} V(\alpha Tx, \alpha Ty) \\ &= V(x,y) - \left(\alpha^2\lambda\right)^{-1} \alpha^2 V(Tx,Ty) \\ &= V(x,y) - \lambda^{-1}V(Tx,Ty). \end{aligned} \tag{2.20}$$

This completes the proof. □

In Banach spaces, we have the following example of V-strongly nonexpansive mappings.

Example 2.10. Let $E = \mathbf{R} \times \mathbf{R}$ be a Banach space with a norm $\|\cdot\|$ defined by

$$\|x\| = |x_1| + |x_2|, \quad \forall x = (x_1, x_2) \in E. \tag{2.21}$$

The normalized duality mapping J is given by

$$Jx = \|x\|\left(\frac{1}{|x_1|}x_1, \frac{1}{|x_2|}x_2\right), \quad \forall x \in E. \tag{2.22}$$

Hence, we have for $x, y \in E$ that

$$\begin{aligned} V(x,y) &= \|x\|^2 + \|y\|^2 - 2\langle x, Jy\rangle \\ &= \|x\|^2 + \|y\|^2 - 2\|y\|\left\{\frac{x_1y_1}{|y_1|} + \frac{x_2y_2}{|y_2|}\right\}. \end{aligned} \tag{2.23}$$

We define a mapping $T : E \to E$ as follows:

$$Tx = \begin{cases} x & \text{if } \|x\| \leq 1, \\ \dfrac{1}{\|x\|}x & \text{if } \|x\| \geq 1. \end{cases} \tag{2.24}$$

We will show that this mapping is V-strongly nonexpansive for any $\lambda \leq 1$.

(a) Suppose that $x, y \in E$ with $\|x\| \leq 1$ and $\|y\| \geq 1$. Then, we have

$$\begin{aligned} V(Tx,Ty) &= V(x,Ty) \\ &= \|x\|^2 + \|Ty\|^2 - 2\|Ty\|\left\{\frac{x_1(Ty)_1}{|(Ty)_1|} + \frac{x_2(Ty)_2}{|(Ty)_2|}\right\} \\ &= \|x\|^2 + 1 - 2\left\{\frac{x_1y_1}{|y_1|} + \frac{x_2y_2}{|y_2|}\right\}. \end{aligned} \tag{2.25}$$

Since

$$y - Ty = \left(\frac{\|y\| - 1}{\|y\|}y_1, \ \frac{\|y\| - 1}{\|y\|}y_2\right), \tag{2.26}$$

we have that

$$\begin{aligned} V(x - Tx, y - Ty) = V(0, y - Ty) &= \|y - Ty\|^2 \\ &= \left\{\frac{(\|y\| - 1)}{\|y\|}(|y_1| + |y_2|)\right\}^2 \\ &= (\|y\| - 1)^2. \end{aligned} \tag{2.27}$$

Hence, we obtain that

$$\begin{aligned}
&V(x,y) - V(Tx,Ty) - \lambda V(x - Tx, y - Ty) \\
&\quad = \|x\|^2 + \|y\|^2 - 2\|y\|\left\{\frac{x_1y_1}{|y_1|} + \frac{x_2y_2}{|y_2|}\right\} - \|x\|^2 - 1 + 2\left\{\frac{x_1y_1}{|y_1|} + \frac{x_2y_2}{|y_2|}\right\} - \lambda(\|y\| - 1)^2 \\
&\quad = \|y\|^2 - 1 - 2(\|y\| - 1)\left\{\frac{x_1y_1}{|y_1|} + \frac{x_2y_2}{|y_2|}\right\} - \lambda(\|y\| - 1)^2 \\
&\quad \geq (\|y\| - 1)(\|y\| + 1) - 2(\|y\| - 1)\left\{\frac{|x_1y_1|}{|y_1|} + \frac{|x_2y_2|}{|y_2|}\right\} - \lambda(\|y\| - 1)^2 \\
&\quad = (\|y\| - 1)\{\|y\| + 1 - 2\|x\| - \lambda\|y\| + \lambda\} \\
&\quad \geq (\|y\| - 1)\{(1 - \lambda)\|y\| + 1 - 2 + \lambda\} \\
&\quad = (\|y\| - 1)\{(1 - \lambda)(\|y\| - 1)\} \\
&\quad = (1 - \lambda)(\|y\| - 1)^2 \geq 0,
\end{aligned} \tag{2.28}$$

for any $\lambda \in [0,1]$. This means that T is V-strongly nonexpansive for any $\lambda \in [0,1]$.

(b) Suppose that $x, y \in E$ with $\|x\| \geq 1$ and $\|y\| \leq 1$. Then, we have

$$\begin{aligned}
V(Tx,Ty) = V(Tx,y) &= 1 + \|y\|^2 - 2\frac{\|y\|}{\|x\|}\left\{\frac{x_1y_1}{|y_1|} + \frac{x_2y_2}{|y_2|}\right\}, \\
V(x - Tx, y - Ty) &= V\left(\frac{(\|x\| - 1)}{\|x\|}x, 0\right) = (\|x\| - 1)^2.
\end{aligned} \tag{2.29}$$

Hence, we have that

$$\begin{aligned}
&V(x,y) - V(Tx,Ty) - \lambda V(x - Tx, y - Ty) \\
&\quad = \|x\|^2 + \|y\|^2 - 2\|y\|\left\{\frac{x_1y_1}{|y_1|} + \frac{x_2y_2}{|y_2|}\right\} - 1 - \|y\|^2 + 2\frac{\|y\|}{\|x\|}\left\{\frac{x_1y_1}{|y_1|} + \frac{x_2y_2}{|y_2|}\right\} - \lambda(\|x\| - 1)^2 \\
&\quad = \|x\|^2 - 1 - 2\|y\|\left(1 - \frac{1}{\|x\|}\right)\left\{\frac{x_1y_1}{|y_1|} + \frac{x_2y_2}{|y_2|}\right\} \\
&\quad \geq (\|x\| - 1)(\|x\| + 1 - \lambda\|x\| + \lambda) - 2\|y\|\left(\frac{\|x\| - 1}{\|x\|}\right)\|x\|
\end{aligned}$$

$$\begin{aligned}
&= (\|x\|-1)\{(1-\lambda)\|x\|+1+\lambda-2\|y\|\}\\
&\geq (\|x\|-1)\{(1-\lambda)\|x\|+1+\lambda-2\}\\
&= (1-\lambda)(\|x\|-1)^2 \geq 0,
\end{aligned} \tag{2.30}$$

for any $\lambda \in [0,1]$. This means that T is V-strongly nonexpansive for any $\lambda \in [0,1]$.

(c) Suppose that $x, y \in E$ with $\|x\|, \|y\| \geq 1$. Then, we have

$$\begin{aligned}
V(Tx,Ty) &= 1+1-\frac{2}{\|x\|}\left\{\frac{x_1y_1}{|y_1|}+\frac{x_2y_2}{|y_2|}\right\}\\
&= 2-\frac{2}{\|x\|}\left\{\frac{x_1y_1}{|y_1|}+\frac{x_2y_2}{|y_2|}\right\},\\
V(x-Tx,y-Ty) &= (\|x\|-1)^2+(\|y\|-1)^2\\
&\quad -2(\|y\|-1)\frac{(\|x\|-1)}{\|x\|}\left\{\frac{x_1y_1}{|y_1|}+\frac{x_2y_2}{|y_2|}\right\}.
\end{aligned} \tag{2.31}$$

Hence, we have that

$$\begin{aligned}
&V(x,y)-V(Tx,Ty)-\lambda V(x-Tx,y-Ty)\\
&\quad= \|x\|^2+\|y\|^2-2\|y\|\left\{\frac{x_1y_1}{|y_1|}+\frac{x_2y_2}{|y_2|}\right\}-2+\frac{2}{\|x\|}\left\{\frac{x_1y_1}{|y_1|}+\frac{x_2y_2}{|y_2|}\right\}\\
&\qquad -\lambda\left\{(\|x\|-1)^2+(\|y\|-1)^2\right\}+2\lambda(\|y\|-1)\frac{(\|x\|-1)}{\|x\|}\left\{\frac{x_1y_1}{|y_1|}+\frac{x_2y_2}{|y_2|}\right\}\\
&\quad= (\|x\|-1)(\|x\|+1)+(\|y\|-1)(\|y\|+1)-\lambda(\|x\|-1)^2-\lambda(\|y\|-1)^2\\
&\qquad -\frac{2}{\|x\|}\left\{\frac{x_1y_1}{|y_1|}+\frac{x_2y_2}{|y_2|}\right\}\{\|x\|\|y\|-1-\lambda(\|y\|-1)(\|x\|-1)\}\\
&\quad= (\|x\|-1)\{\|x\|+1-\lambda(\|x\|-1)\}+(\|y\|-1)\{\|y\|+1-\lambda(\|y\|-1)\}\\
&\qquad -\frac{2}{\|x\|}\left\{\frac{x_1y_1}{|y_1|}+\frac{x_2y_2}{|y_2|}\right\}\{\|x\|\|y\|-1-\lambda(\|x\|-1)(\|y\|-1)\}.
\end{aligned} \tag{2.32}$$

Now, we note that

$$\|x\|\|y\|-1-\lambda(\|x\|-1)(\|y\|-1) \geq 0, \tag{2.33}$$

for any $\|x\|, \|y\| \geq 1$ and for any $\lambda \in [0,1]$. Therefore, we obtain that

$$\begin{aligned} &V(x,y) - V(Tx,Ty) - \lambda V(x-Tx, y-Ty) \\ &\quad \geq (\|x\|-1)\{\|x\|+1-\lambda(\|x\|-1)\} + (\|y\|-1)\{\|y\|+1-\lambda(\|y\|-1)\} \\ &\qquad - \frac{2}{\|x\|}\|x\|\{\|x\|\|y\|-1-\lambda(\|x\|-1)(\|y\|-1)\} \\ &\quad = (1-\lambda)(\|x\|-\|y\|)^2 \geq 0, \end{aligned} \tag{2.34}$$

for any $\lambda \in [0,1]$. This means that T is V-strongly nonexpansive for any $\lambda \in [0,1]$.

It is clear that if $\|x\|, \|y\| \leq 1$ then T is V-strongly nonexpansive; therefore, from (a), (b), and (c), we obtain the conclusion that T is V-strongly nonexpansive with $\lambda \leq 1$.

Next, we present some lemmas which are used in the proofs of our theorems. Let $\mathbb{N}$ be the set of natural numbers.

Lemma 2.11. *Let $\{a_n\}$ and $\{t_n\}$ be sequences of nonnegative real numbers and satisfy the inequality $a_{n+1} \leq (1-t_n)a_n + t_n M$ for any $n \in \mathbb{N}$ and a constant $M > 0$. If $\sum_{n\in\mathbb{N}} t_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.*

Kamimura and Takahashi showed the following useful lemmas (see [2]).

Lemma 2.12. *Let E be a smooth and uniformly convex Banach space. Then, there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty)$ such that $g(0) = 0$, and for each real number $r > 0$,*

$$0 \leq g(\|x-y\|) \leq V(x,y), \tag{2.35}$$

for all $x, y \in B_r = \{z \in E : \|z\| \leq r\}$.

From this lemma, it is obvious that the following lemma holds.

Lemma 2.13. *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} V(x_n, y_n) = 0$, then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.*

We present the following lemma which plays an important role in our theorems (cf. Butnariu and Resmerita [13]).

Lemma 2.14. *Let E be a smooth and uniformly convex Banach space and C a nonempty, convex, and closed subset of E. Suppose that $T : C \to E$ satisfies*

$$V(Tx,Ty) \leq V(x,y), \quad \forall x,y \in C. \tag{2.36}$$

If a weakly convergent sequence $\{z_n\}_{n\in\mathbb{N}} \subset C$ satisfies that $\lim_{n\to\infty} V(Tz_n, z_n) = 0$, then $z_n \rightharpoonup z \in F(T)$.

3. Main Results

In this section, we prove three strong convergence theorems. In the first result, we prove strong convergence theorem for finding common fixed points of a family of generalized nonexpansive mappings. In the next result, we prove strong convergence theorem for finding zeroes of a generalized nonexpansive mapping and a maximal monotone operator. In the last result, we prove weak convergence theorem for finding zeroes of a maximal monotone operator and a V-strongly nonexpansive mapping. As consequence, we prove convergence theorem for common zeroes of a maximal monotone operator and a firmly nonexpansive mapping in a Hilbert space.

Theorem 3.1. *Let E be a reflexive, smooth, and strictly convex Banach space, and let $\{T_n\}_{n\in\mathbb{N}}$ be a family of generalized nonexpansive mappings. Suppose that $\bigcap_{n\in\mathbb{N}} F(T_n) = F \neq \emptyset$ and that R is a sunny and generalized nonexpansive retraction from E to F. Let a sequence $\{x_n\}$ be defined as follows. For any $x_1 = x \in E$,*

$$x_{n+1} = RT_n x_n, \quad \text{for any } n \in \mathbb{N}. \tag{3.1}$$

Then, $\{x_n\}$ converges strongly to a point x^* in F.

Proof. Since Rx_n is a point in F for all $n \in \mathbb{N}$, from Proposition 2.3(a), we have for all $n \in \mathbb{N}$ that

$$\begin{aligned} 0 &\leq V(x_{n+1}, Rx_{n+1}) \leq V(x_{n+1}, Rx_{n+1}) + V(Rx_{n+1}, Rx_n) \\ &\leq V(x_{n+1}, Rx_n) = V(RT_n x_n, Rx_n). \end{aligned} \tag{3.2}$$

Since R and T_n are generalized nonexpansive, we get that

$$V(RT_n x_n, Rx_n) \leq V(T_n x_n, Rx_n) \leq V(x_n, Rx_n). \tag{3.3}$$

Hence, we have that

$$0 \leq V(x_{n+1}, Rx_{n+1}) \leq V(x_n, Rx_n), \quad \forall n \in \mathbb{N}, \tag{3.4}$$

and therefore, $\lim_{n\to\infty} V(x_n, Rx_n) < \infty$. Furthermore, Proposition 2.3(a) implies that

$$V(x_{n+k}, Rx_{n+k}) + V(Rx_{n+k}, Rx_n) \leq V(x_{n+k}, Rx_n). \tag{3.5}$$

This is equivalent to

$$V(Rx_{n+k}, Rx_n) \leq V(x_{n+k}, Rx_n) - V(x_{n+k}, Rx_{n+k}). \tag{3.6}$$

Setting $m = n + k$ for all $n, k \in \mathbb{N}$, then we have that

$$\begin{aligned} V(Rx_m, Rx_n) &\le V(x_m, Rx_n) - V(x_m, Rx_m) \\ &\le V(RT_{m-1}x_{m-1}, Rx_n) - V(x_m, Rx_m) \\ &\le V(T_{m-1}x_{m-1}, Rx_n) - V(x_m, Rx_m) \\ &\le V(x_{m-1}, Rx_n) - V(x_m, Rx_m) \\ &\le \cdots \\ &\le V(x_n, Rx_n) - V(x_m, Rx_m) \longrightarrow 0, \quad \text{as } n, m \longrightarrow \infty. \end{aligned} \tag{3.7}$$

Since $V(x_{n+1}, p) = V(RT_n x_n, p) \le V(x, p)$ for any $p \in F$, Lemma 1.1(b) implies that $\{x_n\}$ is bounded. Thus, from Lemma 2.12, we can take the continuous and strictly increasing function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that

$$\begin{aligned} g(\|Rx_m - \mathrm{R}x_n\|) &\le V(Rx_m, Rx_n) \\ &\le V(x_n, Rx_n) - V(x_m, Rx_m) \longrightarrow 0, \quad \text{as } n, m \longrightarrow \infty. \end{aligned} \tag{3.8}$$

Since $Rx_n = x_n$ for all $n \ge 1$, we have $g(\|x_m - x_n\|) = g(\|Rx_m - Rx_n\|) \to 0$. Therefore, $\{x_n\}$ is a Cauchy sequence. Since E is complete and F is closed, this sequence $\{x_n\}$ converges strongly to point $x^* \in F$. □

Noting that the generalized resolvent $J_r = (I + rBJ)^{-1}$ of a maximal monotone operator B for $r > 0$ is a generalized nonexpansive mapping (see Remark 2.4), we obtain the following result.

Theorem 3.2. *Let E be a reflexive, smooth, and strictly convex Banach space. Let $T : E \to E$ be generalized nonexpansive and let $B \subset E^* \times E$ be a maximal monotone operator. Suppose that $F(T) \cap (BJ)^{-1}(0) \neq \emptyset$ and that R is a sunny and generalized nonexpansive retraction from E to $F = F(T) \cap (BJ)^{-1}(0)$. Let an iterative sequence $\{x_n\}$ be defined as follows: for any $x = x_1 \in E$,*

$$x_{n+1} = RTJ_{r_n}x_n, \quad \forall n \in \mathbb{N}, \tag{3.9}$$

where $\{r_n\}$ is a sequence of nonnegative real numbers. Then, the sequence $\{x_n\}$ converges strongly to a point x^* in $F(T) \cap (BJ)^{-1}(0)$.

Proof. From Propositions 2.3(a) and 2.3(b), we have for all $n \in N$ that

$$\begin{aligned} V(x_{n+1}, Rx_{n+1}) &\le V(x_{n+1}, Rx_{n+1}) + V(Rx_{n+1}, Rx_n) \\ &\le V(x_{n+1}, Rx_n) = V(RTJ_{r_n}x_n, Rx_n) \\ &\le V(TJ_{r_n}x_n, Rx_n) \\ &\le V(J_{r_n}x_n, Rx_n) \\ &\le V(x_n, Rx_n). \end{aligned} \tag{3.10}$$

Thus $\lim_{n\to\infty} V(x_n, Rx_n) < \infty$. Similarly, as in the proof of the previous theorem, we show that $\{x_n\}$ is a Cauchy sequence, and we obtain that $\{x_n\}$ converges strongly to point x^* in $F = F(T) \cap (BJ)^{-1}(0)$. □

The duality mapping J of a Banach space E with the Gâteaux differentiable norm is said to be weakly sequentially continuous if $x_n \rightharpoonup x$ in E implies that $\{Jx_n\}$ converges weak star to Jx in E^* (cf., [14]). This happens, for example, if E is a Hilbert space, finite dimensional and smooth, or l^p if $1 < p < \infty$ (cf., [15]). Next, we prove the main theorem.

Theorem 3.3. *Let E be a reflexive, smooth and strictly convex Banach space. Suppose that the duality mapping J of E is weakly sequentially continuous. Let C be a nonempty, closed, and convex subset of E. Let $B : E^* \to 2^E$ be a maximal monotone operator and let $J_{r_n} = (I + r_n BJ)^{-1}$ be a generalized resolvent of B for a sequence $\{r_n\} \subset (0,\infty)$. Suppose that $A : C \to E$ is a V-strongly nonexpansive mapping with $\lambda \geq 1$ such that $C_0 = A^{-1}(0) \cap (BJ)^{-1}(0) \neq \emptyset$ and that $R_C : E \to C$ is a sunny and generalized nonexpansive retraction. For an $\alpha \in [-1,1]$ such that $\alpha^2\lambda \geq 1$, let an iterative sequence $\{x_n\} \subset C$ be defined as follows: for any $x = x_1 \in C$ and $n \in \mathbb{N}$,*

$$\begin{aligned} y_n &= R_C(I - \alpha A)x_n, \\ x_{n+1} &= R_C\left(\beta_n x + (1-\beta_n) J_{r_n} y_n\right), \end{aligned} \tag{3.11}$$

where $\{\beta_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy that

$$\sum_{n\geq 1} \beta_n < \infty, \qquad \liminf_{n\to\infty} r_n > 0. \tag{3.12}$$

Then, there exists an element $u \in C_0$ such that

$$x_n \rightharpoonup u, \quad R_{C_0}(x_n) \longrightarrow u. \tag{3.13}$$

Proof. For simplicity, we denote R_C and R_{C_0} by R and R_0, respectively. Let $z_n = \beta_n x + (1 - \beta_n) J_{r_n} y_n$ for all $n \in \mathbb{N}$. Since R is generalized nonexpansive, we have for any $p \in C_0$ and all $n \in \mathbb{N}$ that

$$V(x_{n+1}, p) = V(Rz_n, p) \leq V(z_n, p). \tag{3.14}$$

The convexity of $\|\cdot\|^2$ implies that

$$\begin{aligned} V(z_n, p) &= V\left(\beta_n x + (1-\beta_n) J_{r_n} y_n, p\right) \\ &= \left\|\beta_n x + (1-\beta_n) J_{r_n} y_n\right\|^2 + \|p\|^2 - 2\langle \beta_n x + (1-\beta_n) J_{r_n} y_n, Jp\rangle \\ &\leq \beta_n \|x\|^2 + (1-\beta_n)\left\|J_{r_n} y_n\right\|^2 + \|p\|^2 - 2\beta_n\langle x, Jp\rangle - 2(1-\beta_n)\langle J_{r_n} y_n, Jp\rangle \\ &= \beta_n\left\{\|x\|^2 - 2\langle x, Jp\rangle + \|p\|^2\right\} + (1-\beta_n)\left\{\left\|J_{r_n} y_n\right\|^2 - 2\langle J_{r_n} y_n, Jp\rangle + \|p\|^2\right\} \\ &= \beta_n V(x, p) + (1-\beta_n) V\left(J_{r_n} y_n, p\right). \end{aligned} \tag{3.15}$$

Thus, we obtain that

$$V(z_n, p) \le \beta_n V(x, p) + (1 - \beta_n) V(J_{r_n} y_n, p), \tag{3.16}$$

and furthermore, since J_{r_n} is generalized nonexpansive, we have that

$$V(z_n, p) \le \beta_n V(x, p) + (1 - \beta_n) V(y_n, p). \tag{3.17}$$

Let $A_\alpha = (I - \alpha A)$. Then, from Proposition 2.9(d), A_α is V-strongly nonexpansive with $\alpha^2 \lambda$ and A_α is also generalized nonexpansive. Hence, we have that

$$V(y_n, p) = V(RA_\alpha x_n, p) \le V(A_\alpha x_n, p) \le V(x_n, p). \tag{3.18}$$

Thus, we have from (3.14), (3.16), (3.17), and (3.18) that

$$\begin{aligned} V(x_{n+1}, p) \le V(z_n, p) &\le \beta_n V(x, p) + (1 - \beta_n) V(J_{r_n} y_n, p) \\ &\le \beta_n V(x, p) + (1 - \beta_n) V(y_n, p) \\ &\le \beta_n V(x, p) + (1 - \beta_n) V(A_\alpha x_n, p) \\ &\le \beta_n V(x, p) + (1 - \beta_n) V(x_n, p). \end{aligned} \tag{3.19}$$

From Lemma 2.11, there exists $\alpha = \lim_{n\to\infty} V(x_n, p) < \infty$. Since $\lim_{n\to\infty} \beta_n = 0$, we have that

$$\alpha = \lim_{n\to\infty} V(x_n, p) = \lim_{n\to\infty} V(z_n, p), \tag{3.20}$$

$$= \lim_{n\to\infty} V(J_{r_n} y_n, p) = \lim_{n\to\infty} V(y_n, p) = \lim_{n\to\infty} V\ (A_\alpha x_n, p). \tag{3.21}$$

Hence, $\{x_n\}$, $\{z_n\}$, $\{J_{r_n} y_n\}$, $\{y_n\}$, and $\{A_\alpha x_n\}$ are bounded from Lemma 1.1(b). Since E is uniformly convex, the boundedness of $\{x_n\}$ implies that there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \rightharpoonup u \in C$. Moreover, we can take the index sequence $\{n_j\}_{j\ge 1}$ satisfies $\lim_{j\to\infty} r_{n_j - 1} > 0$. We will show that $u \in (BJ)^{-1}(0)$. From Proposition 2.3(a),

$$V(x_{n+1}, p) = V(Rz_n, p) \le V(z_n, Rz_n) + V(Rz_n, p) \le V(z_n, p), \tag{3.22}$$

and furthermore, from (3.16) and Proposition 2.3(b), we obtain that

$$\begin{aligned} V(z_n, p) &\le \beta_n V(x, p) + (1 - \beta_n) V(J_{r_n} y_n, p) \\ &\le \beta_n V(x, p) + (1 - \beta_n) \{ V(y_n, J_{r_n} y_n) + V(J_{r_n} y_n, p) \} \\ &\le \beta_n V(x, p) + (1 - \beta_n) V(y_n, p). \end{aligned} \tag{3.23}$$

These inequalities and (3.22) imply with $\lim_{n\to\infty}\beta_n = 0$ and (3.21) that

$$\begin{aligned}\alpha &\le \alpha + \lim_{n\to\infty} V(z_n, Rz_n) \\ &\le \alpha \le \lim_{n\to\infty} V(y_n, J_{r_n}y_n) + \alpha \le \alpha,\end{aligned} \tag{3.24}$$

that is,

$$\lim_{n\to\infty} V(z_n, x_{n+1}) = \lim_{n\to\infty} V(z_n, Rz_n) = \lim_{n\to\infty} V(y_n, J_{r_n}y_n) = 0. \tag{3.25}$$

Lemma 2.13 implies that

$$\lim_{n\to\infty}\|x_{n+1} - z_n\| = 0, \qquad \lim_{n\to\infty}\|J_{r_n}y_n - y_n\| = 0. \tag{3.26}$$

Furthermore, since

$$\begin{aligned}\lim_{n\to\infty}\|z_n - J_{r_n}y_n\| &= \lim_{n\to\infty}\|\beta_n x + (1-\beta_n)J_{r_n}y_n - J_{r_n}y_n\| \\ &= \lim_{n\to\infty}\beta_n\|x - J_{r_n}y_n\| = 0,\end{aligned} \tag{3.27}$$

we have from (3.26) and (3.27) that

$$\begin{aligned}\|x_{n+1} - J_{r_n}y_n\| &\le \|x_{n+1} - z_n\| + \|z_n - J_{r_n}y_n\| \\ &\longrightarrow 0 \quad (n \longrightarrow \infty).\end{aligned} \tag{3.28}$$

Hence, for an index sequence $\{n_j\}_{j\ge1}$ of $\{x_{n_j}\}$ such that $x_{n_j} \rightharpoonup u \in C$ and $\lim_{j\to\infty} r_{n_j-1} > 0$, we obtain that

$$J_{r_{n_j-1}}y_{n_j-1} \rightharpoonup u, \quad y_{n_j-1} \rightharpoonup u, \quad \text{as } j \rightharpoonup \infty. \tag{3.29}$$

Since $(1/r)(J_r^{-1} - I) = (1/r)(\mathrm{I} + rBJ - I) = BJ$, there exists $w_{n_j} \in BJ(J_{r_{n_j-1}}y_{n_j-1})$ such that

$$w_{n_j} = \frac{1}{r_{n_j-1}}\left(y_{n_j-1} - J_{r_{n_j-1}}y_{n_j-1}\right), \quad \text{for any } j \ge 1. \tag{3.30}$$

Since $\lim_{j\to\infty} r_{n_j-1} > 0$, (3.26) implies that

$$\lim_{j\to\infty}\|w_{n_j}\| = \lim_{j\to\infty}\frac{1}{r_{n_j-1}}\|J_{r_{n_j-1}}y_{n_j-1} - y_{n_j-1}\| = 0. \tag{3.31}$$

For $(p, q) \in BJ \subset E \times E$, the monotonicity of B implies that

$$\lim_{j \to \infty} \left\langle q - w_{n_j}, Jp - JJ_{r_{n_j-1}} y_{n_j-1} \right\rangle \geq 0, \tag{3.32}$$

and we have, since J is weakly sequentially continuous, that

$$\langle q, Jp - Ju \rangle \geq 0. \tag{3.33}$$

The maximality of B implies that $u \in (BJ)^{-1}(0)$.

Now, we will show that $u \in A^{-1}(0)$. From Proposition 2.9(d) and $p \in F(A_\alpha)$, we get that

$$\begin{aligned} V(A_\alpha x_n, p) &= V(A_\alpha x_n, A_\alpha p) \\ &\leq V(x_n, p) - \lambda^{-1} V(Ax_n, Ap). \end{aligned} \tag{3.34}$$

Thus, we have from (3.17) that

$$\begin{aligned} V(x_{n+1}, p) \leq V(z_n, p) &\leq \beta_n V(x, p) + (1 - \beta_n) V(y_n, p) \\ &= \beta_n V(x, p) + (1 - \beta_n) V(RA_\alpha x_n, p) \\ &\leq \beta_n V(x, p) + (1 - \beta_n) V(A_\alpha x_n, p) \\ &\leq \beta_n V(x, p) + (1 - \beta_n) \left\{ V(x_n, p) - \lambda^{-1} V(Ax_n, Ap) \right\}. \end{aligned} \tag{3.35}$$

This implies that

$$\begin{aligned} 0 &\leq (1 - \beta_n) \lambda^{-1} V(Ax_n, Ap) \\ &\leq \beta_n V(x, p) + (1 - \beta_n) V(x_n, p) - V(x_{n+1}, p) \\ &= \beta_n \{ V(x, p) - V(x_n, p) \} + V(x_n, p) - V(x_{n+1}, p) \\ &\longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{3.36}$$

Therefore we have that

$$\lim_{n \to \infty} V(Ax_n, Ap) = 0. \tag{3.37}$$

From Lemma 2.13, we get that $\lim_{n\to\infty}\|Ax_n - Ap\| = \lim_{n\to\infty}\|Ax_n\| = 0$ that is, $Ax_n \to 0$ as $n \to \infty$. From Lemma 1.1(a) and the boundedness of $\{A_\alpha x_n\}$, we have that

$$\begin{aligned}
0 &\le V(x_n, A_\alpha x_n) \\
&= V(x_n, p) - V(A_\alpha x_n, p) + 2\langle x_n - A_\alpha x_n, JA_\alpha x_n - Jp\rangle \\
&= V(x_n, p) - V(A_\alpha x_n, p) + 2\alpha\langle Ax_n, JA_\alpha x_n - Jp\rangle \\
&\le V(x_n, p) - V(A_\alpha x_n, p) + 2\alpha\|Ax_n\|\|JA_\alpha x_n - Jp\| \\
&\le V(x_n, p) - V(A_\alpha x_n, p) + 2\alpha\|Ax_n\|M,
\end{aligned} \tag{3.38}$$

for some $M > 0$. From (3.21), we have that $\lim_{n\to\infty}\{V(x_n, p) - V(A_\alpha x_n, p)\} = 0$, and we obtain that

$$\lim_{n\to\infty} V(x_n, A_\alpha x_n) = 0, \tag{3.39}$$

and this means that

$$\lim_{n\to\infty} \|x_n - A_\alpha x_n\| = 0. \tag{3.40}$$

From Lemma 2.14, we obtain $x_n \rightharpoonup u_0 \in F(A_\alpha)$. Since $x_{n_j} \rightharpoonup u$, this means that $x_{n_j} \rightharpoonup u_0 = u$; hence, we have $u \in F(A_\alpha)$; that is, $u \in A^{-1}(0)$. Therefore, we obtain that $u \in A^{-1}(0) \cap (BJ)^{-1}(0) = C_0$.

Let $u_n = R_0 x_n$ for any $n \in \mathbb{N}$. Since R_0 is a sunny generalized nonexpansive retraction,

$$\langle x_n - u_n, Ju_n - Jy\rangle \ge 0, \quad \forall y \in C_0. \tag{3.41}$$

Similarly as in the proof of Theorem 3.2, we can show that $\{u_n\}$ is a Cauchy sequence, and therefore there exists $u^* \in C_0$ such that $u_n \to u^*$. Set $y = u$ in (3.41). Since $x_n \rightharpoonup u$, we get that

$$\langle u - u^*, Ju^* - Ju\rangle \ge 0. \tag{3.42}$$

This means that $u = u^*$ by the strict convexity of J; that is, $R_0 x_n \to u$. This completes the proof. □

In a Hilbert space, we obtain the following theorem as a corollary of the main Theorem 3.3 by applying Proposition 2.8(a).

Corollary 3.4. *Let H be a Hilbert space, and let C be a nonempty, closed, and convex subset of H. Let $B : H \to 2^H$ be a maximal monotone operator, and let $J_{r_n} = (I + r_n B)^{-1}$ be a resolvent of B for a sequence $\{r_n\} \subset (0, \infty)$. Suppose that $A : C \to H$ is a firmly nonexpansive mapping with*

$C_0 = A^{-1}(0) \cap B^{-1}(0) \neq \emptyset$. Suppose that R_C is a sunny and generalized nonexpansive retraction to C. Let an iterative sequence $\{x_n\} \subset C$ be defined as follows: for any $x = x_1 \in C$ and $n \in \mathbb{N}$,

$$\begin{aligned} y_n &= R_C(I - \alpha A)x_n, \\ x_{n+1} &= R_C(\beta_n x + (1 - \beta_n) J_{r_n} y_n), \end{aligned} \tag{3.43}$$

where $\{\beta_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy that

$$\sum_{n\geq 1} \beta_n < \infty, \qquad \liminf_{n\to\infty} r_n > 0. \tag{3.44}$$

Then, there exists an element $u \in C_0$ such that

$$x_n \rightharpoonup u, \quad R_{C_0}(x_n) \longrightarrow u. \tag{3.45}$$

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