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Research Article

Unique Positive Almost Periodic Solution for Discrete Nonlinear Delay Survival Red Blood Cells Model

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We obtain sufficient conditions which guarantee the global attractivity of solutions for nonlinear delay survival red blood cells model. Then, some criteria are established for the existence, uniqueness and global attractivity of positive almost periodic solutions of the almost periodic system.

1. Introduction

Recently, Saker [1] studied the existence and global attractivity of positive periodic solution for the following discrete nonlinear delay survival red blood cells model:

$$x(n+1) - x(n) = -\delta(n)x(n) + P(n)e^{-q(n)x(n-\tau(n))}, \tag{1.1}$$

where $\delta(n)$, $P(n)$, $q(n)$, $\tau(n)$ are nonnegative bounded sequence, $\tau(n) \in \mathbf{N}$ for all $n \in \mathbf{Z}$ and $0 < \delta(n) < 1$. The dynamic behavios of (1.1) was investigated by many authors (see [1–18]) because of its biological and ecological significance. When $\delta(n)$, $P(n)$, $q(n)$ and $\tau(n)$ are positive constants, the global attractivity of the positive equilibrium for equation

$$x(n+1) - x(n) = -\delta x(n) + Pe^{-qx(n-\tau)} \tag{1.2}$$

has been investigated by some authors (see [3, 15–18] and reference therein). Recently, Ma and Yu [17] obtained the following Theorem A.

Theorem A. *Suppose that $\delta \in (0,1)$, $P, q \in [0,\infty)$ and τ are nonnegative integer. If*

$$q\overline{x}\left(1-(1-\delta)^{\tau+1}\right) \leq 1, \tag{1.3}$$

then the unique positive equilibrium $\overline{x}$ of (1.2) *is a global attractor of all positive solutions of* (1.2).

Equation (1.2) is the discrete analogue of equation

$$x^{'}(t) = -\delta x(t) + p\exp(-qx(t-\tau)), \tag{1.4}$$

which was first used by Ważewska-Czyżewska and Lasota as a model for the survival of red blood cells in an animal in [19], see also [20]. Here, $x(t)$ denotes the number of red blood cells at time t, δ is the probability of death of a red blood cell, p and q are positive constants related to the production of red blood cells per unit time and τ is the time required to produce a red blood cell. Researching the behavior of the solution of (1.4) and its analogue was posed as open problems by Kocic and Ladas [10] as well as Györi and Ladas [6].

Because of seasonal variation (1.1) needs not to be exactly periodic but almost periodic instead. It is natural to ask if the results in [1] hold for the almost periodic case. It is a difficult problem in which significant difference appears in comparison with the periodic case, for example, contrary to periodic functions, there exists an almost periodic function $x(t)$ such that $x(t) > 0$ for all $t \in \mathbb{R}$ and $\inf_{t\in\mathbb{R}} x(t) = 0$.

One purpose of the present paper is to extend Theorem A to (1.1). The other purpose is to extend some results in [1] to the almost periodic case. In [1], the author use a fixed theorem to obtain the existence of positive periodic solution. The operator used in [1] depends on the period of system (1.1), therefore we cannot apply these topological tools to the almost periodic case. An important notion in almost periodic differential theory is the hull (cf. [21–23]). In this paper, we use essentially this notion to establish our results. We begin with some notations.

Let $\tau = \max_{n\in\mathbb{Z}}\tau(n)$, $I = \{-\tau, -\tau+1, \ldots, -1, 0\}$, $C = \{\varphi : I \to \mathbb{R}\}$ and $C^+ = \{\varphi \in C : \varphi \geq 0, \varphi(0) > 0\}$. For each $\varphi \in C$, we define the norm of φ as $\|\varphi\| = \max_{s\in I}|\varphi(s)|$. Denote x_n the element of C with $x_n = x(n+s)$ for all $s \in I$. For any bounded sequence $e(n)$, denote $e_* = \liminf_{n\to\infty} e(n)$ and $e^* = \limsup_{n\to\infty} e(n)$, $\prod_{j=l}^{k} e(j) = e(l)e(l+1)\cdots e(k)$ if $k \geq l$ and $\prod_{j=l}^{k} e(j) = 1$ if $k < l$. It is easy to see that, for any $\varphi \in C^+$, there is a unique solution $x(n, 0, \varphi)$ of (1.1) with $x_0 = \varphi$ and $x(n, 0, \varphi) > 0$ for all $n \in \mathbb{N}$. Now, we give some definitions.

Definition 1.1 (see [22, 23]). A sequence $f(n)$ is said to be almost periodic, if for any $\epsilon > 0$, there is a constant $l(\epsilon) > 0$ such that in any interval of length $l(\epsilon)$ there exists $\overline{\tau} \in \mathbb{Z}$ such that the inequality

$$\left|f(n+\overline{\tau}) - f(n)\right| < \epsilon \tag{1.5}$$

is satisfied for all $n \in \mathbb{Z}$.

Denote set

$$H(f) = \left\{g : \text{there is } \{n_k\} \subset \mathbb{Z} \text{ such that } \lim_{n\to\infty} f(n+n_k) = g(n)\ \forall n \in \mathbb{Z}\right\} \tag{1.6}$$

the hull of f. It is easy to see that if $f(n)$ is almost periodic, then for all $f_1 \in H(f)$,

$$\liminf_{n\to\infty} f(n) = \inf_{n\in\mathbb{Z}} f(n) = \inf_{n\in\mathbb{Z}} f_1(n),$$
$$\limsup_{n\to\infty} f(n) = \sup_{n\in\mathbb{Z}} f(n) = \sup_{n\in\mathbb{Z}} f_1(n). \tag{1.7}$$

Definition 1.2. Let $f : \mathbb{Z} \times C \to \mathbb{R}$. Then f is said to be almost periodic in $n \in \mathbb{Z}$ uniformly on compact set of C, if $f(n,\cdot)$ is continuous for each $n \in \mathbb{Z}$, and for any $\epsilon > 0$ and every compact set $K \subset C$, there is a constant $l(\epsilon, K) > 0$ such that in any interval of length $l(\epsilon, K)$ there exists $\overline{\tau}$ such that the inequality

$$\left| f(n+\overline{\tau}, \varphi) - f(n, \varphi) \right| < \epsilon \tag{1.8}$$

is satisfied for all $n \in \mathbb{Z}$ and $\varphi \in K$.

Definition 1.3. The positive solution $N(n)$ to (1.1), in the sense that $N(n) > 0$ for all $n \geq 0$, is said to be global attractive, if for any $\varphi \in C^+$,

$$\lim_{n\to\infty} \left(x(n,0,\varphi) - N(n) \right) = 0. \tag{1.9}$$

The remainder of this paper is organized as follows. The main results are given in Section 2 while the proofs are left to Section 4. In Section 3, we will give some lemmas needed in the proofs of main results.

2. Main Results

Theorem 2.1. *Assume that* $0 < \delta_* < 1$ *and* $P_* > 0$. *Then, for any* $\varphi \in C^+$, *the solution* $x(n,0,\varphi)$ *of* (1.1) *satisfies*

$$\overline{u} \leq \liminf_{n\to\infty} x(n,0,\varphi) \leq \limsup_{n\to\infty} x(n,0,\varphi) \leq \overline{v}, \tag{2.1}$$

where $(\overline{u}, \overline{v})$ *is the limit of* $\{(u_n, v_n)\}$ *with* $u_0 = 0$ *and*

$$v_k = \frac{P^*}{\delta_*} e^{-q_* u_{k-1}}, \qquad k = 1, 2, \ldots,$$
$$u_k = \frac{P_*}{\delta^*} e^{-q^* v_k}, \qquad k = 1, 2, \ldots. \tag{2.2}$$

The following corollary follows from [24, Theorems 1 and 2].

Corollary 2.2. *Assume that the conditions of Theorem 2.1 are satisfied, and* $\delta(n)$, $P(n)$, $q(n)$ *and* $\tau(n)$ *are* ω*-periodic with* $\omega > 0$. *Then* (1.1) *admits a positive* ω*-periodic solution.*

Remark 2.3. When $\tau(n) \equiv \omega$, under the conditions of Corollary 2.2, Saker in [1, Theorem 2.1] proved that the conclusion of Corollary 2.2. Thus Corollary 2.2 extends Theorem 2.1 in [1].

Theorem 2.4. *Assume that the conditions of Theorem 2.1 are satisfied and*

$$aq^* \le 1, \tag{2.3}$$

where

$$a = \limsup_{n\to\infty} \sum_{k=n}^{n+\tau} P(k) \prod_{i=k+1}^{n+\tau} (1-\delta(i)) e^{-q(k)\overline{x}(k-\tau(k))}, \tag{2.4}$$

and $\overline{x}(n)$ is a positive solution of (1.1). Then every positive solution $x(n)$ of (1.1) satisfies

$$\lim_{n\to\infty} [x(n) - \overline{x}(n)] = 0. \tag{2.5}$$

Corollary 2.5. *Assume that the conditions of Theorem 2.1 are satisfied, and*

$$q^* \limsup_{n\to\infty} \sum_{k=n}^{n+\tau} P(k) \prod_{i=k+1}^{n+\tau} (1-\delta(i)) e^{-q(k)\overline{u}} \le 1, \tag{2.6}$$

where $\overline{u}$ is the constant given in Theorem 2.1. Then every positive of (1.1) is globally attractive.

Remark 2.6. When $\delta(n) \equiv \delta$, $P(n) \equiv P$ and $q(n) \equiv q$, here δ, P and q are positive constants, we consider the global attractivity of the positive equilibrium $\overline{x}$ for (1.2). In this case, $\delta\overline{x} = Pe^{-q\overline{x}}$ and

$$\begin{aligned} a &= \limsup_{n\to\infty} \sum_{k=n}^{n+\tau} P(k) \prod_{i=k+1}^{n+\tau} (1-\delta(i)) e^{-q(k)\overline{x}(k-\tau(k))} \\ &= Pe^{-q\overline{x}} \frac{1-(1-\delta)^{\tau+1}}{\delta} \\ &= \overline{x}\left(1-(1-\delta)^{\tau+1}\right). \end{aligned} \tag{2.7}$$

We see that (2.3) changes to (1.3) and Theorem 2.4 reproduces Theorem A.

Theorem 2.7. *Assume that system (1.1) is almost periodic, that is, $\delta(t)$, $P(n)$, $q(n)$ and $\tau(n)$ are almost periodic, and the conditions of Corollary 2.5 are satisfied. Then there is a unique globally attractive positive almost periodic solution $p(n)$ for (1.1).*

Remark 2.8. Let $P(n) = 1/10+(1/20)\sin n$, $q(n) = \tau(n) = 2$ and $\delta(n) = \delta$ be a positive constant such that

$$\frac{3}{10}e^{-(1/10\delta)e^{-3/10\delta}}\frac{1-(1-\delta)^3}{\delta} \leq 1,$$
$$\frac{3(1-\delta)}{\delta} > 1. \tag{2.8}$$

Since $P^* = 3/20$, $P_* = 1/20$, $q^* = q_* = 2$, $v_1 = P^*/\delta = 3/20\delta$, $\overline{u} > u_1 = (P_*/\delta)e^{-q^*v_1} = (1/20\delta)e^{-3/10\delta}$, $\tau = \sup_{n\in\mathbf{Z}}\tau(n) = 2$ and

$$q^*\limsup_{n\to\infty}\sum_{k=n}^{n+\tau}P(k)\prod_{i=k+1}^{n+\tau}(1-\delta(i))e^{-q(k)\overline{u}} \leq \frac{3}{10}e^{-(1/10\delta)e^{-3/10\delta}}\frac{1-(1-\delta)^3}{\delta} \leq 1, \tag{2.9}$$

we can see that there is unique globally attractive positive almost periodic solution $p(n)$ for (1.1). Since

$$\frac{P^*q^*(1-\delta)}{\delta} = \frac{3(1-\delta)}{\delta} > 1, \tag{2.10}$$

we cannot obtain the existence of positive almost periodic solution of (1.1) by [25, Theorem 4.1]. Moreover, we should point out that (4.1) in [25] is not correct. To see this, we consider (1.2) with $P(1-\delta)/\delta < \ln P/q < 1$. Let $f(x) = -\delta x + Pe^{-qx}$. It is easy to see that the positive equilibrium $\overline{x}$ of (1.2) is the unique zero point of f. Since

$$f\left(\frac{\ln P}{q}\right) = 1-\delta\frac{\ln P}{q} > 1-\delta > 0, \tag{2.11}$$

we see that $\overline{x} > \ln P/q > P(1-\delta)/\delta$. Thus we cannot conclude that for each positive solution $x(n)$ of (1.2),

$$\limsup_{n\to\infty} x(n) \leq \alpha := \frac{P(1-\delta)}{\delta}. \tag{2.12}$$

Therefore the conclusions of [25, Theorem 4.1] may not hold.

3. Some Lemmas

Lemma 3.1 (see [26]). *The following system of inequalities,*

$$y \leq e^{-x}-1, \qquad x \geq e^{-y}-1, \tag{3.1}$$

where $x \leq y$ are real numbers, have exactly one solution $x = y = 0$.

Lemma 3.2. *Assume that $f : \mathbb{Z} \times C \to \mathbb{R}$, is almost periodic sequence in n uniformly on compact set of C. If there is a solution $x(n)$ $(n \geq 0)$ to the equation*

$$x(n+1) = f(n, x_n), \tag{3.2}$$

such that $A \leq \liminf_{n\to\infty} x(t) \leq \limsup_{n\to\infty} x(t) \leq B$. then for each $g \in H(f)$, equation

$$x(n+1) = g(n, x_n) \tag{3.3}$$

has a solution $p(n)$ which is defined on $\mathbb{Z}$ such that

$$A \leq p(n) \leq B, \quad n \in \mathbb{Z}. \tag{3.4}$$

Proof. Let $D = \sup_{n\in\mathbb{N}} |x(n)|$ and $S = \{\varphi \in C : \|\varphi\| \leq D\}$. Then S is a compact subset of C. For each $g \in H(f)$, there is $\{n_k\} \subset \mathbb{N}$ such that $n_k \to \infty$ and $f(n + n_k, \varphi) \to g(n, \varphi)$ uniformly on $\mathbb{Z} \times S$. By the diagonal process, we can choose $\{n_{k_j}\} \subset \{n_k\}$ such that $x(n + n_{k_j}) \to p(n)$ as $j \to \infty$ and $p(n)$ satisfies (3.3) on $\mathbb{Z}$. We want to prove that (3.4) holds.

For each $\epsilon > 0$, there is a $T > 0$ such that

$$A - \epsilon \leq x(n) \leq B + \epsilon, \quad n \geq T. \tag{3.5}$$

For any fixed $n \in \mathbb{Z}$, there is $J > 0$ such that $n + n_{k_j} > T$ for all $j > J$. It follows from (3.5) that

$$A - \epsilon \leq x\left(n + n_{k_j}\right) \leq B + \epsilon, \quad j \geq J. \tag{3.6}$$

Setting $j \to \infty$ and $\epsilon \to 0$, we see that (3.4) holds. This completes the proof. □

Lemma 3.3. *Assume that $f : \mathbb{Z} \times C \to \mathbb{R}$, is almost periodic sequence in n uniformly on compact set of C. Let S be a compact set of $\mathbb{R}$. If for all $g \in H(f)$, equation*

$$x(n+1) = g(n, x_n) \tag{3.7}$$

admits a unique solution $p_g(n)$, defined on $\mathbb{Z}$, whose range is in S, then all these solutions $p_g(n)$ are almost periodic.

The proof is similar to that of [21, Theorem 2.10.1]. We omit it here.

4. The Proofs of Main Results

Proof of Theorem 2.1. Let $x(n) = x(n, 0, \varphi)$ for each $\varphi \in C^+$. From (1.1), we have

$$x(1) = (1 - \delta(0))x(0) + P(0)e^{-q(0)x(-\tau(0))} \geq (1 - \delta(0))x(0) > 0, \tag{4.1}$$

which prove that $x(n) > 0$ $(n = 1, 2, \ldots)$ by induction. For any $\epsilon \in (0, \delta_*)$, there is $N > 0$ such that for $n > N$,

$$\delta_* - \epsilon < \delta(n) < \delta^* + \epsilon, \qquad P_* - \epsilon < P(n) < P^* + \epsilon,$$
$$q_* - \epsilon < q(n) < q^* + \epsilon. \tag{4.2}$$

By the fact that $x(n) > 0$ we deduce that

$$x(n+1) \le (1 - \delta_* + \epsilon)x(n) + P^* + \epsilon, \quad n > N. \tag{4.3}$$

This implies

$$x(n) \le (1 - \delta_* + \epsilon)^{n-N-1} x(N+1) + \frac{P^* + \epsilon}{\delta_* - \epsilon}\left[\left(1 - (1 - \delta_* + \epsilon)^{n-N-1}\right], \quad n > N. \tag{4.4}$$

By the fact that ϵ is arbitrary, we obtain that

$$\limsup_{n\to\infty} x(n) \le \frac{P^*}{\delta_*} = v_1. \tag{4.5}$$

For each $\epsilon \in (0, \delta_*)$, there is $N_1 > N$ such that

$$x(n) < v_1 + \epsilon, \quad n \ge N_1. \tag{4.6}$$

By (1.1), (4.2), we obtain that

$$x(n+1) \ge (1 - \delta^* - \epsilon)x(n) + (P_* - \epsilon)e^{-(q^*+\epsilon)(v_1+\epsilon)}, \quad n \ge N_1 + \tau. \tag{4.7}$$

It follows that for $n \ge N_1 + \tau$

$$x(n) \ge (1 - \delta^* - \epsilon)^{n-N_1-\tau} x(N_1 + \tau) + \frac{P_* - \epsilon}{\delta^* + \epsilon} e^{-(q^*+\epsilon)(v_1+\epsilon)}\left[\left(1 - (1 - \delta^* - \epsilon)^{n-N_1-\tau}\right]. \tag{4.8}$$

This implies that

$$\liminf_{n\to\infty} x(n) \ge \frac{P_*}{\delta^*} e^{-q^* v_1} = u_1. \tag{4.9}$$

By induction we can see that

$$u_k \le \liminf_{n\to\infty} x(n) \le \limsup_{n\to\infty} x(n) \le v_k, \quad k = 1, 2, \ldots. \tag{4.10}$$

It is easy to see that $\{u_k\}$ increase and $\{v_k\}$ decrease. Thus the limit of $\{(u_n, v_n)\}$ exists. Therefore we have

$$\overline{u} \leq \liminf_{n\to\infty} x(n) \leq \limsup_{n\to\infty} x(n) \leq \overline{v}. \tag{4.11}$$

This completes the proof of Theorem 2.1. □

Proof of Theorem 2.4. Set $z(n) = x(n) - \overline{x}(n)$. Then $z(n)$ satisfies

$$z(n+1) - z(n) = -\delta(n)z(n) + P(n)e^{-q(n)\overline{x}(n-\tau(n))}\left(e^{-q(n)z(n-\tau(n))} - 1\right). \tag{4.12}$$

The proof will be accomplished by showing that

$$\lim_{n\to\infty} z(n) = 0. \tag{4.13}$$

We will prove that (4.13) holds in each of the following two cases.

Case 1. $\{z(n)\}$ is nonoscillatory. Suppose that $\{z(n)\}$ is eventually nonnegative. The case that $\{z(n)\}$ is eventually nonpositive is similar and will be omitted. By (4.12) we see that $\{z(n)\}$ is eventually decreasing. Thus the limit of $\{z(n)\}$ exists. Let $b = \lim_{n\to\infty} z(n)$. Then $b \geq 0$. We claim that $b = 0$. Otherwise, there would exist $N > 0$ such that

$$\frac{b}{2} < z(n) < \frac{3b}{2}, \quad n \geq N. \tag{4.14}$$

It follows from (4.12) that

$$z(n+1) - z(n) < -\frac{b}{2}\delta(n), \quad n \geq N + \tau. \tag{4.15}$$

This implies that

$$z(n) < -\frac{b}{2}\sum_{k=N+\tau}^{n} \delta(k) + z(N+\tau), \quad n \geq N + \tau. \tag{4.16}$$

By the fact that $\delta_* > 0$ we see that $\lim_{n\to\infty} z(n) = -\infty$, which contradicts the fact that $\{z(n)\}$ is eventually nonnegative. Thus $b = 0$. Therefore (4.13) holds in this case.

Case 2. $\{z(n)\}$ is oscillatory. Let $\lambda = \liminf_{n\to\infty} z(n)$ and $\mu = \limsup_{n\to\infty} z(n)$. Then

$$-\infty < \lambda \leq 0 \leq \mu < \infty. \tag{4.17}$$

There exist positive sequences $\{p_i\}$ and $\{q_i\}$ such that $\lim_{i\to\infty} p_i = \infty$,

$$p_i < q_i < p_{i+1}, \tag{4.18}$$

$z(p_i) < 0$ and $z(q_i) > 0$ for $i \in \mathbb{N}$, $z(k) \geq 0$ for $p_i < k \leq q_i$ and $z(k) \leq 0$ for $q_i < k \leq p_{i+1}$ and

$$\limsup_{i\to\infty} A_i = \mu, \qquad \liminf_{i\to\infty} a_i = \lambda, \tag{4.19}$$

where $A_i = \max\{z(j) \mid p_i \leq j \leq q_i\}$ and $a_i = \min\{z(j) \mid q_i \leq j \leq p_{i+1}\}$. Let $M_i = \min\{j \mid p_i \leq j \leq q_i, z(j) = A_i\}$ and $m_i = \min\{j \mid q_i \leq j \leq p_{i+1}, z(j) = a_i\}$. Since $z(M_i) > z(M_i - 1)$, by (4.12) we have

$$\delta(M_i - 1)z(M_i - 1) < P(M_i - 1)e^{-q(M_i-1)\overline{x}(M_i-1-\tau(M_i-1))}\left(e^{-q(n)z(M_i-1-\tau(M_i-1))} - 1\right). \tag{4.20}$$

We claim that

$$M_i - 1 - p_i \leq \tau. \tag{4.21}$$

We first assume that $z(M_i - 1) < 0$. Therefore we have $M_i - 1 \leq p_i$. Thus (4.21) holds. Assume now that $z(M_i - 1) \geq 0$. By (4.20) we obtain that $z(M_i - 1 - \tau(M_i - 1)) < 0$. This implies that $M_i - 1 - p_i \leq \tau(M_i - 1) \leq \tau$. Thus (4.21) also holds.

Similarly, we can obtain that

$$m_i - 1 - q_i \leq \tau. \tag{4.22}$$

For each $\epsilon > 0$, there is $N > 0$ such that for $n > N$,

$$\begin{gathered} q_* - \epsilon < q(n) < q^* + \epsilon, \qquad \lambda - \epsilon < z(n) < \mu + \epsilon, \\ \sum_{k=n}^{n+\tau} P(k) \prod_{i=k+1}^{n+\tau} (1 - \delta(i))e^{-q(k)\overline{x}(k-\tau(k))} < a + \epsilon, \end{gathered} \tag{4.23}$$

Let $z(n) = y(n)\prod_{i=0}^{n-1}(1 - \delta(i))$. By (4.12) we obtain that

$$y(n+1) = y(n) + \left(\prod_{i=0}^{n}(1 - \delta(i))\right)^{-1} P(n)e^{-q(n)\overline{x}(n-\tau(n))}\left(e^{-q(n)z(n-\tau(n))} - 1\right). \tag{4.24}$$

This implies that

$$y(M_i) = y(p_i) + \sum_{n=p_i}^{M_i-1} \left(\prod_{j=0}^{n} (1-\delta(j)) \right)^{-1} P(n) e^{-q(n)\overline{x}(n-\tau(n))} \left(e^{-q(n)z(n-\tau(n))} - 1 \right). \tag{4.25}$$

It follows from (4.21) and (4.23)–(4.24) that for i large enough

$$\begin{aligned} z(M_i) &= \prod_{j=0}^{M_i-1} (1-\delta(j)) \\ &\quad \times \left\{ y(p_i) + \sum_{n=p_i}^{M_i-1} \left(\prod_{j=0}^{n} (1-\delta(j)) \right)^{-1} P(n) e^{-q(n)\overline{x}(n-\tau(n))} \left(e^{-q(n)z(n-\tau(n))} - 1 \right) \right\} \\ &\le \left(e^{-(q^*+\epsilon)(\lambda-\epsilon)} - 1 \right) \sum_{n=p_i}^{M_i-1} P(n) \prod_{j=n+1}^{M_i-1} (1-\delta(j)) e^{-q(n)\overline{x}(n-\tau(n))} \\ &\le (a+\epsilon) \left(e^{-(q^*+\epsilon)(\lambda-\epsilon)} - 1 \right). \end{aligned} \tag{4.26}$$

Using (4.19), then by ϵ being arbitrary, this implies that

$$\mu \le a\left(e^{-q^*\lambda} - 1 \right). \tag{4.27}$$

Now we will prove that

$$\lambda \ge a\left(e^{-q^*\mu} - 1 \right). \tag{4.28}$$

In fact, by (4.22)–(4.24), for i large enough we have

$$\begin{aligned} z(m_i) &= \prod_{j=0}^{m_i-1} (1-\delta(j)) \left\{ y(q_i) + \sum_{n=q_i}^{m_i-1} \left(\prod_{j=0}^{n} (1-\delta(j)) \right)^{-1} P(n) e^{-q(n)\overline{x}(n-\tau(n))} \left(e^{-q(n)z(n-\tau(n))} - 1 \right) \right\} \\ &\ge \left(e^{-(q^*+\epsilon)(\mu+\epsilon)} - 1 \right) \sum_{n=q_i}^{m_i-1} P(n) \prod_{j=n}^{m_i-1} (1-\delta(j)) e^{-q(n)\overline{x}(n-\tau(n))} \\ &\ge (a+\epsilon) \left(e^{-(q^*+\epsilon)(\mu+\epsilon)} - 1 \right). \end{aligned} \tag{4.29}$$

Note that ϵ is arbitrary. By (4.19) we can see that (4.28) holds. By Lemma 3.1 we see that $\lambda = \mu = 0$. Thus, (4.13) also holds in this case. This completes the proof. □

Proof of Theorem 2.7. We will prove our conclusion by applying Lemmas 3.1 and 3.2. For any $(\delta_1, P_1, q_1, \tau_1) \in H((\delta, P, q, \tau))$, consider equation

$$x(n+1) - x(n) = -\delta_1(n)x(n) + P_1(n)e^{-q_1(n)x(n-\tau_1(n))}. \tag{4.30}$$

It follows from Theorem 2.1 that for any $\varphi \in C^+$,

$$\overline{u} \le \liminf_{n\to\infty} x(n, 0, \varphi) \le \limsup_{n\to\infty} x(n, 0, \varphi) \le \overline{v}, \tag{4.31}$$

where $\overline{u}, \overline{v}$ are the constants given in Theorem 2.1, and $x(n, 0, \varphi)$ is the solution of (1.1) with $x_0 = \varphi$. By the almost periodicity of (1.1) and Lemma 3.2, we can find a solution $N(n)$ of (4.30) defined on $\mathbb{Z}$ whose range is in $[\overline{u}, \overline{v}]$. Now we will prove that the solution which possesses the above properties is unique. Suppose that $\overline{N}(n)$ is another solution of (4.30) defined on $\mathbb{Z}$ whose range is in $[\overline{u}, \overline{v}]$. Let $z(n) = N(n) - \overline{N}(n)$. Then $z(n)$ satisfies

$$z(n+1) - z(n) = -\delta_1(n)z(n) + P_1(n)e^{-q_1(n)\overline{N}(n-\tau(n))}\left(e^{-q_1(n)z(n-\tau_1(n))} - 1\right). \tag{4.32}$$

We claim that $z(n) \equiv 0$. Otherwise, we would have

$$z(\zeta) \neq 0 \tag{4.33}$$

for some $\zeta \in \mathbb{Z}$. Consider the following two cases.

Case 1. There is a $T < \zeta$ such that $z(n) \ge 0$ (or ≤ 0) for all $n \le T$. Without loss of generality, we suppose that $z(n) \ge 0$ for all $n \le T$. Then $z(n+1) \le z(n)$ for $n \le T$. Since $z(n)$ is bounded on $\mathbb{Z}$, we see that $\lim_{n\to-\infty} z(n)$ exists, denoted by l. We claim that $l = 0$. Otherwise, there would exist $N < T$ such that

$$\frac{l}{2} < z(n) < \frac{3l}{2}, \quad n < N. \tag{4.34}$$

Note that $\delta_{1*} = \delta_* = \inf_{n\in\mathbb{Z}} \delta(n)$. It follows from (4.32) that

$$z(n+1) - z(n) < -\frac{l}{2}\delta_*, \quad n < N. \tag{4.35}$$

This implies that

$$z(n) < z(n-k) - k\frac{l}{2}\delta_* \quad \forall k \in \mathbb{N},\ n < N. \tag{4.36}$$

Hence $\lim_{n\to-\infty} z(n) = +\infty$, which contradicts the fact that $z(n)$ is bounded on $\mathbb{Z}$. Thus $l = 0$. Therefore $z(n) = 0$ for all $n \le T$. It follows from (4.32) that $z(n) = 0$ for all $n \in \mathbb{Z}$, which contradicts (4.33).

Case 2. There exist positive sequences $\{p_i\}$ and $\{q_i\}$ such that $\lim_{i\to\infty} p_i = -\infty$,

$$p_{i+1} < q_i < p_i, \tag{4.37}$$

$z(p_i) < 0$ and $z(q_i) > 0$ for $i \in \mathbb{N}$, $z(k) \geq 0$ for $p_{i+1} < k \leq q_i$ and $z(k) \leq 0$ for $q_i < k \leq p_i$. Let $d = \sup_{n\leq p_1} z(n)$ and $c = \inf_{n\leq p_1} z(n)$. Then $d > 0$ and $c < 0$. For each $\epsilon \in (0, \min(-c, d))$, there are $n^0, n_0 \leq p_1$ such

$$z\left(n^0\right) > d - \epsilon, \qquad z(n_0) < c + \epsilon. \tag{4.38}$$

Suppose $n^0 \in [p_j, q_{j-1}]$ and $n_0 \in [q_i, p_i]$. Let $C = \max\{z(k) \mid p_j \leq k \leq q_{j-1}\}$ and $c = \min\{z(k) \mid q_i \leq k \leq p_i\}$. Set $M = \min\{k \mid p_j \leq k \leq q_{j-1}, z(k) = C\}$ and $m = \min\{k \mid q_i \leq k \leq p_i, z(k) = c\}$. Similar to the argument in the proof of Theorem 2.4, we can obtain that

$$\begin{gathered} M - 1 - p_j \leq \tau, \\ m - 1 - q_i \leq \tau. \end{gathered} \tag{4.39}$$

Let $\overline{a} = \limsup_{n\to\infty} \sum_{k=n+1}^{n+\tau} P(k) \prod_{i=k}^{n+\tau} (1 - \delta(i)) e^{-q(k)\overline{u}}$. By the almost periodicity of $\delta(n)$, $P(n)$ and $q(n)$, we have for all $n \in \mathbb{Z}$,

$$\begin{gathered} q_* \leq q_1(n) \leq q^*, \\ \sum_{k=n}^{n+\tau} P_1(k) \prod_{i=k+1}^{n+\tau} (1 - \delta_1(i)) e^{-q_1(k)\overline{u}} \leq \overline{a}. \end{gathered} \tag{4.40}$$

Let $z(n) = y(n) \prod_{i=0}^{n-1} (1 - \delta_1(i))$. By (4.32) we obtain that

$$y(n+1) = y(n) + \left(\prod_{i=0}^{n} (1 - \delta_1(i)) \right)^{-1} P_1(n) e^{-q_1(n)\overline{N}(n-\tau_1(n))} \left(e^{-q_1(n)z(n-\tau_1(n))} - 1 \right). \tag{4.41}$$

This implies that

$$y(M) = y(p_j) + \sum_{n=p_j}^{M_i-1} \left(\prod_{k=0}^{n} (1 - \delta_1(k)) \right)^{-1} P_1(n) e^{-q_1(n)\overline{N}(n-\tau_1(n))} \left(e^{-q_1(n)z(n-\tau_1(n))} - 1 \right). \tag{4.42}$$

Since $c \le z(n) \le d$ for all $n \le p_1$, it follows from (4.39)–(4.40) that

$$\begin{aligned}
d - \epsilon < z\left(n^0\right) &< z(M) \\
&= \prod_{k=0}^{M-1}(1-\delta_1(k)) \\
&\quad \times \left\{ y(p_j) + \sum_{n=p_j}^{M-1}\left(\prod_{k=0}^{n}(1-\delta_1(k))\right)^{-1} P_1(n)e^{-q_1(n)\overline{N}(n-\tau_1(n))}\left(e^{-q_1(n)z(n-\tau_1(n))}-1\right)\right\} \\
&\le \left(e^{-q^*c}-1\right)\sum_{n=p_j}^{M_i-1} P_1(n)\prod_{k=n+1}^{M-1}(1-\delta_1(k))e^{-q_1(n)\overline{u}} \\
&\le \overline{a}\left(e^{-q^*c}-1\right).
\end{aligned} \tag{4.43}$$

Thus, we have

$$d - \epsilon \le \overline{a}\left(e^{-q^*c}-1\right). \tag{4.44}$$

By the fact that ϵ is arbitrary, we obtain that

$$d \le \overline{a}\left(e^{-q^*c}-1\right). \tag{4.45}$$

Similarly, we can prove that

$$c \ge \overline{a}\left(e^{-q^*d}-1\right). \tag{4.46}$$

Equations (4.45) and (4.46) produce that $c = d = 0$. Therefore $z(n) = 0$ for all $n \le p_1$. It follows from (4.32) that $z(n) = 0$ for all $n \in \mathbb{Z}$, which also contradicts (4.33). By Lemma 3.3 we see that (1.1) has a unique almost periodic solution $p(n)$ whose range is in $[\overline{u}, \overline{v}]$. By Corollary 2.5 we can see that $p(n)$ is global attractivity. This also implies that the almost periodic solution of (1.1) is unique. This completes the proof. □

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