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Research Article

On the Norm of Certain Weighted Composition Operators on the Hardy Space

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We obtain a representation for the norm of certain compact weighted composition operator $C_{\psi,\varphi}$ on the Hardy space H^2, whenever $\varphi(z) = az + b$ and $\psi(z) = az - b$. We also estimate the norm and essential norm of a class of noncompact weighted composition operators under certain conditions on φ and ψ. Moreover, we characterize the norm and essential norm of such operators in a special case.

1. Introduction

Let D denote the open unit disk in the complex plane. The Hardy space H^2 is the space of analytic functions on D whose Taylor coefficients, in the expansion about the origin, are square summable. Also we recall that H^∞ is the space of all bounded analytic function defined on D. For $\alpha \in D$, the reproducing kernel at α for H^2 is defined by $K_\alpha(z) = 1/(1-\overline{\alpha}z)$. An easy computation shows that $\langle f, K_\alpha\rangle = f(\alpha)$ whenever $f \in H^2$. For any analytic self-map φ of D, the composition operator C_φ on H^2 is defined by the rule $C_\varphi(f) = f \circ \varphi$. Every composition operator is bounded, with

$$\sqrt{\frac{1}{1-|\varphi(0)|^2}} \leq \left\| C_\varphi : H^2 \longrightarrow H^2 \right\| \leq \sqrt{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}} \tag{1.1}$$

(see [1]). We see from expression (1.1) that $\|C_\varphi\| = 1$ whenever $\varphi(0) = 0$. There are few other cases for which the exact value of the norm has been known for many years. For example, the norm of C_φ was obtained by Nordgren in [2], whenever φ is an inner function. In [3] this norm was determined, when $\varphi(z) = az + b$, with $|a| + |b| \le 1$, and if $0 < s < 1$ and $0 \le r \le 1$ the norm was found in [4] for $\varphi(z) = ((r+s)z + (1-s))/(r(1-s)z + (1+rs))$.

In 2003, Hammond [5] obtained exact values for the norms of composition operators C_φ for certain linear fractional maps φ. In [6], Bourdon et al. determined the norm of C_φ for a large class of linear-fractional maps, including those of the form $\varphi(z) = b/(d - z)$, where $0 < b < d - 1$. The connection between the norm of certain composition operators C_φ with linear-fractional symbol acting on the Hardy space and the roots of associated hypergeometric functions was first made by Basor and Retsek [7]. It was later refined by Hammond [8]. In [9] Effinger-Dean et al. computed the norms of composition operators with rational symbols that satisfy certain properties. Their work is based on the initial work of Hammond [5]. Some other recent results regarding the calculation of the operator norm of some composition operators on the other spaces can be found in [10–14].

If ψ is a bounded analytic function on D and φ is an analytic map from D into itself, the weighted composition operator $C_{\psi,\varphi}$ is defined by $C_{\psi,\varphi}(f)(z) = \psi(z)f(\varphi(z))$. The map φ is called the composition map and ψ is called the weight. If ψ is a bounded analytic function on D, then the operator can be rewritten as $C_{\psi,\varphi} = M_\psi C_\varphi$, where M_ψ is a multiplication operator and C_φ is a composition operator. Recall that if φ is an analytic self-map of D, then the composition operator C_φ on H^2 is bounded, hence in this case $C_{\psi,\varphi}$ is bounded, but in general every weighted composition operator $C_{\psi,\varphi}$ on H^2 is not bounded. If $C_{\psi,\varphi}$ is bounded, then $C_{\psi,\varphi}(1) = \psi$ belongs to H^2. These operators come up naturally. In 1964, Forelli [15] showed that every isometry on H^p for $1 < p < \infty$ and $p \neq 2$ is a weighted composition operator. Recently there has been a great interest in studying weighted composition operators in the unit disk, polydisk, or the unit ball; see [12, 16–27], and the references therein. In this paper we investigate the norm of certain bounded weighted composition operators $C_{\psi,\varphi}$ on H^2.

2. Norm Calculation

In this section we obtain a representation for the norm of a class of compact weighted composition operators $C_{\psi,\varphi}$ on the Hardy space H^2, whenever $\varphi(z) = az + b$, $\psi(z) = az - b$, $|b|^2 \ge 1/2$, and $2|a|^2 + |b|^2 \le 2/3$. Also we give the norm and essential norm inequality for a class of noncompact weighted composition operators $C_{\psi,\varphi}$ on H^2 when $\varphi(z) = az^n + b$, for some $n \in \mathbb{N}$, $|a| + |b| = 1$, and ψ is a bounded analytic map on D such that the radial limit of $|\psi|$ at one of the nth roots of $b|a|/a|b|$ is the supremum of $|\psi|$ on D. Also, when $n = 1$ we obtain the norm and essential norm of such operators.

The following lemma was inspired by a similar result for unweighted composition operators [28, Theorem 1.4]. See [29] for a similar proof.

Lemma 2.1. *Let K_w be the reproducing kernel at w. Then*

$$C^*_{\psi,\varphi} K_w = \overline{\psi(w)} K_{\varphi(w)}. \tag{2.1}$$

In the next proposition we generalize the lower bound in (1.1).

Proposition 2.2. *Let φ be a nonconstant analytic self-map of D, and let ψ be a nonzero analytic map on D. If n is the smallest nonnegative integer such that $\psi^{(n)}(0) \neq 0$, then*

$$\|C_{\psi,\varphi}\| \geq \left|\frac{\psi^{(n)}(0)}{n!}\right| \frac{1}{\sqrt{1-|\varphi(0)|^2}}. \tag{2.2}$$

Proof. We note that if f is in H^2, then for every $n \in \mathbb{N} \cup \{0\}$ we have $|f^{(n)}(0)/n!| \leq \|f\|_2$. Hence we have

$$\begin{aligned} \|C_{\psi,\varphi}\| &\geq \frac{\|C_{\psi,\varphi}K_{\varphi(0)}\|}{\|K_{\varphi(0)}\|} \\ &= \frac{\|\psi.(K_{\varphi(0)} \circ \varphi)\|}{\|K_{\varphi(0)}\|} \\ &\geq \frac{|(\psi^{(n)}(0)/n!)(K_{\varphi(0)} \circ \varphi)(0)|}{\|K_{\varphi(0)}\|} \\ &= \left|\frac{\psi^{(n)}(0)}{n!}\right| \frac{1}{\sqrt{1-|\varphi(0)|^2}}. \end{aligned} \tag{2.3}$$

□

Let T be a bounded operator on a Hilbert space H. We recall that $\|T\|_e$, the essential norm of T, is the norm of its equivalence class in the Calkin algebra. Since the spectral radius of the operator T^*T equals $\|T^*T\| = \|T\|^2$, we study the spectrum of T^*T when trying to determine $\|T\|$. We say that the operator T is norm-attaining if there is a nonzero $h \in H$ such that $\|T(h)\| = \|T\|\|h\|$. We know that $\|T(h)\| = \|T\|\|h\|$ if and only if $T^*T(h) = \|T\|^2 h$. Moreover, if $\|T\|_e < \|T\|$, then the operator T is norm-attaining and so the quantity $\|T\|^2$ equals the largest eigenvalue of T^*T; see [5] for more details. If $\varphi(z) = az+b$, $\psi(z) = az-b$, $|b|^2 \geq 1/2$, and $2|a|^2 + |b|^2 \leq 2/3$, then the operator $C_{\psi,\varphi}$ is compact (see the proof of Proposition 2.5). Hence $0 = \|C_{\psi,\varphi}\|_e < \|C_{\psi,\varphi}\|$ and so $C_{\psi,\varphi}$ is norm-attaining.

Now our goal is to find a functional equation that relates an eigenvalue of $C^*_{\psi,\varphi}C_{\psi,\varphi}$ to the values of its eigenfunctions at particular points in the disk. In what follows we use the techniques used in [5, 6, 30] and present some results that help us to obtain the norm of $C_{\psi,\varphi}$.

Let φ be an analytic self-map of D and let ψ be a bounded analytic map on D. Then

$$(C_{\psi,\varphi})^* = (M_\psi C_\varphi)^* = C^*_\varphi M^*_\psi = C^*_\varphi T^*_\psi. \tag{2.4}$$

But if $\varphi(z) = az + b$ such that $|a| + |b| \leq 1$, then by [3] or [28]

$$(C_{\psi,\varphi})^* = T_g C_\sigma T^*_h T^*_\psi = T_g C_\sigma (T_{\psi h})^*, \tag{2.5}$$

where $h(z) = 1$, $g(z) = 1/-\bar{b}z + 1$, and $\sigma(z) = \bar{a}z/-\bar{b}z + 1$.

From now on, unless otherwise stated, we assume that $\psi(z) = cz + d, \varphi(z) = az + b$, and $|a| + |b| \leq 1$. Since T_z^* is the backward shift on H^2, we see that

$$\begin{aligned}
C_{\psi,\varphi}^* C_{\psi,\varphi} f(z) &= T_g C_\sigma T_\psi^* T_\psi C_\varphi f(z) \\
&= T_g C_\sigma T_{cz+d}^* \left(\psi \cdot f(\varphi(z))\right) \\
&= T_g C_\sigma \left(\overline{c} \left(\frac{\psi \cdot f(\varphi(z)) - \psi \cdot f(\varphi(0))}{z} \right) \right) + T_g C_\sigma \left(\overline{d} \psi(z) \cdot f(\varphi(z)) \right) \\
&= T_g \left(\overline{c} \left(\frac{\psi(\sigma(z)) \cdot f(\varphi(\sigma(z))) - \psi(0) \cdot f(\varphi(0))}{\sigma(z)} \right) \right) \\
&\quad + \overline{d} g(z) \psi(\sigma(z)) \cdot f(\varphi(\sigma(z))) \\
&= g(z) \left(\overline{c} \left(\frac{\psi(\sigma(z)) \cdot f(\varphi(\sigma(z))) - \psi(0) \cdot f(\varphi(0))}{\sigma(z)} \right) \right) \\
&\quad + \overline{d} g(z) \psi(\sigma(z)) \cdot f(\varphi(\sigma(z))) \\
&= \gamma(z) f(\tau(z)) + \chi(z) f(\varphi(0))
\end{aligned} \tag{2.6}$$

for all z in D not equal to 0, where

$$\begin{aligned}
\gamma(z) &= \frac{\left(\overline{c}\left(1 - \overline{b}z\right) + \overline{d}\overline{a}z\right)\left(d\left(1 - \overline{b}z\right) + \overline{a}cz\right)}{(\overline{a}z)\left(1 - \overline{b}z\right)^2}, \\
\tau(z) &= \frac{\left(|a|^2 - |b|^2\right)z + b}{-\overline{b}z + 1}, \qquad \chi(z) = \frac{-\overline{c}d}{\overline{a}z}.
\end{aligned} \tag{2.7}$$

In particular, if g is an eigenfunction for $C_{\psi,\varphi}^* C_{\psi,\varphi}$ corresponding to an eigenvalue λ, then

$$\lambda g(z) = \gamma(z) g(\tau(z)) + \chi(z) g(\varphi(0)). \tag{2.8}$$

Formula (2.8) is essentially identical to [5, Formula (3.3)]. Using (2.8) we can find a set of conditions under which we determine $\|C_{\psi,\varphi}^* C_{\psi,\varphi}\|$. In the trivial case $a = 0$ we have $\|C_{\psi,\varphi}\| = \|\psi\|_2 (1/\sqrt{1 - |b|^2})$. Also if $d = 0$, then $\|C_{\psi,\varphi}\| = |c| \|C_\varphi\|$ and if $c = 0$, then $\|C_{\psi,\varphi}\| = |d| \|C_\varphi\|$. Therefore we assume that a, b, c, d are nonzero.

Throughout this paper, we write $\tau^{[j]}$ to denote the jth iterate of τ, that is, $\tau^{[0]}$ is the identity map on D and $\tau^{[j+1]} = \tau \circ \tau^{[j]}$.

By a similar argument as in the proof of [5, Proposition 5.1], we have the following lemma.

Lemma 2.3. *Let g be an eigenfunction for $C^*_{\psi,\varphi}C_{\psi,\varphi}$ corresponding to an eigenvalue λ, $z \in D$ and for each nonnegative integer j, $\tau^{[j]}(z) \neq 0$. Then one has*

$$\lambda^{j+1}g(z) = g\left(\tau^{[j+1]}(z)\right)\prod_{k=0}^{j}\left[\gamma\left(\tau^{[k]}(z)\right)\right] + \sum_{k=0}^{j}\left[g(\varphi(0))\chi\left(\tau^{[k]}(z)\right)\prod_{m=0}^{k-1}\left[\gamma\left(\tau^{[m]}(z)\right)\right]\right]\lambda^{j-k}, \tag{2.9}$$

where one takes $\prod_{m=0}^{-1}(\cdot) = 1$.

Lemma 2.4. *For each $n \in \mathbb{N}$, $\tau^{[n]}(0) = \alpha_n b$, where $\{\alpha_n\}$ is strictly increasing sequence such that $\alpha_n \geq 1$ for each $n \in \mathbb{N}$. Also $\alpha_{n+1} = 1 + \alpha_n|a|^2/(1-\alpha_n|b|^2)$.*

Proof. (By induction) Since $\tau(0) = b$ and $\tau^{[2]}(0) = (1 + |a|^2/(1-|b|^2))b$, the claim holds for $n = 1$. Assume the claim holds for $n-1$. We will prove it for n. We have

$$\tau^{[n]}(0) = \tau\left(\tau^{[n-1]}(0)\right) = \tau(\alpha_{n-1}b) = \left(1 + \frac{\alpha_{n-1}|a|^2}{1-\alpha_{n-1}|b|^2}\right)b. \tag{2.10}$$

Now if we set $\alpha_n = 1+(\alpha_{n-1}|a|^2)/(1-\alpha_{n-1}|b|^2)$, then $\tau^{[n]}(0) = \alpha_n b$. But by hypothesis $\alpha_{n-1} < \alpha_n$, so

$$1 + \frac{\alpha_{n-1}|a|^2}{1-\alpha_{n-1}|b|^2} < 1 + \frac{\alpha_n|a|^2}{1-\alpha_n|b|^2}, \tag{2.11}$$

which implies that $\alpha_n < \alpha_{n+1}$ also $\tau^{[n+1]}(0) = \tau(\alpha_n b) = (1 + \alpha_n|a|^2/(1-\alpha_n|b|^2))b$. Hence the proof is complete. □

Proposition 2.5. *Let $a = c$, $b = -d$ and let $\lambda = \|C_{\psi,\varphi}\|^2$. If $|b|^2 \geq 1/2$, and $2|a|^2 + |b|^2 \leq 2/3$, then for each $z \in D$ with the property that $\tau^{[j]}(z) \neq 0$ for every nonnegative integer j, one has*

$$g(z) = \sum_{k=0}^{\infty}\left[g(\varphi(0))\chi\left(\tau^{[k]}(z)\right)\prod_{m=0}^{k-1}\left[\gamma\left(\tau^{[m]}(z)\right)\right]\right]\frac{1}{\lambda^{k+1}}. \tag{2.12}$$

Proof. Since $2|a|^2 + |b|^2 \leq 2/3$, it is easy to see that $|a| + |b| = 1$ if and only if $|a| = 1/3$ and $|b| = 2/3$. By assumption $|b|^2 \geq 1/2$, so $|a| + |b| < 1$. Therefore C_φ is compact and, since $C_{\psi,\varphi} = M_\psi C_\varphi$, the operator $C_{\psi,\varphi}$ is compact. Now according to the paragraph after Proposition 2.2, there is function g in H^2 such that $C^*_{\psi,\varphi}C_{\psi,\varphi}g = \lambda g$. Let $z \in D$ and for each integer $j \geq 0$, $\tau^{[j]}(z) \neq 0$. By Lemma 2.3, we have

$$\begin{aligned} \lambda^{j+1}g(z) &= g\left(\tau^{[j+1]}(z)\right)\prod_{k=0}^{j}\left[\gamma\left(\tau^{[k]}(z)\right)\right] \\ &\quad + \sum_{k=0}^{j}\left[g(\varphi(0))\chi\left(\tau^{[k]}(z)\right)\prod_{m=0}^{k-1}\left[\gamma\left(\tau^{[m]}(z)\right)\right]\right]\lambda^{j-k}. \end{aligned} \tag{2.13}$$

Hence

$$\begin{aligned} g(z) &= g\left(\tau^{[j+1]}(z)\right)\prod_{k=0}^{j}\left[\frac{\gamma(\tau^{[k]}(z))}{\lambda}\right] \\ &\quad + \sum_{k=0}^{j}\left[g(\varphi(0))\chi\left(\tau^{[k]}(z)\right)\prod_{m=0}^{k-1}\left[\gamma\left(\tau^{[m]}(z)\right)\right]\right]\frac{1}{\lambda^{k+1}}. \end{aligned} \tag{2.14}$$

Now if w_0 is the Denjoy-Wolff point of τ, it suffices to show that

$$\left|\frac{\gamma(w_0)}{\lambda}\right| < 1. \tag{2.15}$$

Suppose the above inequality holds. Then we conclude that there is $0 < \beta < 1$ and $N \in \mathbb{N}$ such that for $k > N$ we have $|\gamma(\tau^{[k]}(z))/\lambda| < \beta < 1$. Now we break the proof into two parts.

(1) The Denjoy-Wolff point w_0 of τ lies inside D, then $g(\tau^{[j]}(z))$ converges to $g(w_0)$. Hence

$$\lim_{j\to\infty}\left|g\left(\tau^{[j+1]}(z)\right)\prod_{k=0}^{j}\left[\frac{\gamma(\tau^{[k]}(z))}{\lambda}\right]\right| \leq \lim_{j\to\infty} g\left(\tau^{[j+1]}(z)\right)\beta^{j-N}\left|\prod_{k=0}^{N}\left[\frac{\gamma(\tau^{[k]}(z))}{\lambda}\right]\right| = 0. \tag{2.16}$$

(2) The Denjoy-Wolff point w_0 of τ lies on ∂D, then by [31, Lemma 5.1] τ must be parabolic and by [6, Lemma 3.3] there is a constant C such that

$$\frac{1}{1-\left|\tau^{[j]}(z)\right|} \leq Cj. \tag{2.17}$$

Thus it follows that

$$\begin{aligned}\left|g\left(\tau^{[j]}(z)\right)\right| &= \left|\langle g, K_{\tau^{[j]}(z)}\rangle\right| \\ &\le \|g\| \cdot \|K_{\tau^{[j]}(z)}\| \\ &= \|g\| \cdot \sqrt{\frac{1}{1-\left|\tau^{[j]}(z)\right|^2}} \\ &\le \|g\| \cdot \sqrt{jC}.\end{aligned} \tag{2.18}$$

Hence

$$\begin{aligned}\lim_{j\to\infty}\left|g\left(\tau^{[j+1]}(z)\right)\prod_{k=0}^{j}\left[\frac{\gamma(\tau^{[k]}(z))}{\lambda}\right]\right| &\le \lim_{j\to\infty} g\left(\tau^{[j+1]}(z)\right)\beta^{j-N}\left|\prod_{k=0}^{N}\left[\frac{\gamma(\tau^{[k]}(z))}{\lambda}\right]\right| \\ &\le \lim_{j\to\infty}\|g\| \cdot \sqrt{(j+1)C} \cdot \beta^{j-N}\left|\prod_{k=0}^{N}\left[\frac{\gamma(\tau^{[k]}(z))}{\lambda}\right]\right| \\ &= 0.\end{aligned} \tag{2.19}$$

Now we show that $|\gamma(w_0)/\lambda| < 1$. Since $a = c$ and $b = -d$, we see that

$$\left|\frac{\gamma(w_0)}{\lambda}\right| = \left|\frac{\left(1-2\overline{b}w_0\right)\left(-b\left(1-\overline{b}w_0\right)+\overline{a}aw_0\right)}{\lambda(w_0)\left(1-\overline{b}w_0\right)^2}\right|. \tag{2.20}$$

By [30], we have

$$w_0 = \frac{1-|a|^2+|b|^2-\sqrt{\left(1-|a|^2+|b|^2\right)^2-4|b|^2}}{2\overline{b}}. \tag{2.21}$$

Applying the assumptions $|b|^2 \ge 1/2$ and $2|a|^2+|b|^2 \le 2/3$, an easy computation shows that

$$0 \le 2\overline{b}w_0 - 1 \le 1 - \overline{b}w_0. \tag{2.22}$$

Also by using Proposition 2.2, $1/\lambda < (1-|b|^2)/|b|^2$, and by Lemma 2.4, there is $\alpha_n \geq 1$ such that $\tau^{[n]}(0) = \alpha_n b$. Therefore

$$\begin{aligned}
\left|\frac{\gamma(w_0)}{\lambda}\right| &= \left|\frac{\left(1-2\bar{b}w_0\right)\left(-b\left(1-\bar{b}w_0\right)+\bar{a}aw_0\right)}{\lambda w_0\left(1-\bar{b}w_0\right)^2}\right| \\
&= \frac{\left(2\bar{b}w_0-1\right)\left|-b\left(1-\bar{b}w_0\right)+\bar{a}aw_0\right|}{\lambda|w_0|\left(1-\bar{b}w_0\right)^2} \\
&= \frac{\left(2\bar{b}w_0-1\right)\left|-b\left(1-\bar{b}\lim_{n\to\infty}\alpha_n b\right)+\bar{a}a\lim_{n\to\infty}\alpha_n b\right|}{\lambda|w_0|\left(1-\bar{b}w_0\right)\left(1-\bar{b}\lim_{n\to\infty}\alpha_n b\right)} \\
&\leq \frac{\left(2\bar{b}w_0-1\right)\left(\lim_{n\to\infty}|b|\left(1-\alpha_n\left(|b|^2+|a|^2\right)\right)\right)}{\lambda|b|\left(1-\bar{b}w_0\right)\left(\lim_{n\to\infty}1-\alpha_n|b|^2\right)} \\
&< \frac{\left(1-|b|^2\right)\left(2\bar{b}w_0-1\right)\left(\lim_{n\to\infty}\left(1-\alpha_n\left(|b|^2+|a|^2\right)\right)\right)}{|b|^2\left(1-\bar{b}w_0\right)\left(\lim_{n\to\infty}1-\alpha_n|b|^2\right)} \\
&\leq \frac{1-|b|^2}{|b|^2} \\
&\leq 1.
\end{aligned} \tag{2.23}$$

□

Proposition 2.6. *Let $a = c$, $b = -d$, $|b|^2 \geq 1/2$, and $2|a|^2 + |b|^2 \leq 2/3$. Then $\lambda = \|C_{\psi,\varphi}\|^2$ satisfies the equation*

$$1 = \sum_{k=0}^{\infty}\left[\chi\left(\tau^{[k+1]}(0)\right)\prod_{m=0}^{k-1}\left[\gamma\left(\tau^{[m+1]}(0)\right)\right]\right]\frac{1}{\lambda^{k+1}}. \tag{2.24}$$

Proof. Since for every integer $j \geq 0$, $\tau^{[k]}(\varphi(0)) \neq 0$, in Proposition 2.5 we set $z = \varphi(0)$, then we have

$$g(\varphi(0)) = \sum_{k=0}^{\infty}\left[g(\varphi(0))\chi\left(\tau^{[k]}(\varphi(0))\right)\prod_{m=0}^{k-1}\left[\gamma\left(\tau^{[m]}(\varphi(0))\right)\right]\right]\frac{1}{\lambda^{k+1}}. \tag{2.25}$$

Since $\varphi(0) = \tau(0)$, we see that

$$g(\varphi(0)) = \sum_{k=0}^{\infty}\left[g(\varphi(0))\chi\left(\tau^{[k+1]}(0)\right)\prod_{m=0}^{k-1}\left[\gamma\left(\tau^{[m+1]}(0)\right)\right]\right]\frac{1}{\lambda^{k+1}}. \tag{2.26}$$

But $g(\varphi(0)) \neq 0$, because otherwise Proposition 2.5 would dictate that the function $g(z)$ is identically 0. Thus eigenfunction g must have the property that $g(\varphi(0)) \neq 0$. Hence we have

$$1 = \sum_{k=0}^{\infty}\left[\chi\left(\tau^{[k+1]}(0)\right)\prod_{m=0}^{k-1}\left[\gamma\left(\tau^{[m+1]}(0)\right)\right]\right]\frac{1}{\lambda^{k+1}}. \tag{2.27}$$

□

We define

$$F(z) = \sum_{k=0}^{\infty}\left[\chi\left(\tau^{[k+1]}(0)\right)\prod_{m=0}^{k-1}\left[\gamma\left(\tau^{[m+1]}(0)\right)\right]\right]z^{k+1}. \tag{2.28}$$

Now we characterize the properties of F and by using these properties we obtain a formula for the norm of $C_{\psi,\varphi}$. The idea behind Proposition 2.7 is similar to the one found in [30].

Proposition 2.7. *Let $a = c$, $b = -d$, $|b|^2 \geq 1/2$, and $2|a|^2 + |b|^2 \leq 2/3$. Then $F(z)$ has the following properties.*

(a) *The power series that defines $F(z)$ has radius of convergence r_0 larger than $1/\lambda$.*

(b) *$F(x)$ is non-negative real number for all x in the interval $[0, r_0)$.*

(c) *$F'(x) > 0$ for all x in the interval $(0, r_0)$.*

Proof. (a) By Lemma 2.4, for each positive integer n there is $\alpha_n \geq 1$ such that $\tau^{[n]}(0) = \alpha_n b$, then $\chi(\tau^{[m+1]}(0)) = 1/\alpha_{m+1} \leq 1$. Also in the proof of Proposition 2.5 we have $|\gamma(w_0)/\lambda| < 1$, hence there is $0 < \beta < 1$ and $N \in \mathbb{N}$ such that if $n > N$, then

$$\left|\frac{\gamma(\tau^{[n]}(0))}{\lambda}\right| < \beta < 1. \tag{2.29}$$

Now let $\beta < \beta_1 < 1$ and $0 < \epsilon < \lambda(\beta_1 - \beta)/\beta_1$. Then if $n > N$ we have

$$\left|\frac{\gamma(\tau^{[n]}(0))}{\lambda}\right| < \left|\frac{\gamma(\tau^{[n]}(0))}{\lambda - \epsilon}\right| < \beta_1. \tag{2.30}$$

Therefore there is a constant C such that

$$\begin{aligned}\left|\sum_{k=0}^{\infty}\left[\chi\left(\tau^{[k+1]}(0)\right)\prod_{m=0}^{k-1}\left[\gamma\left(\tau^{[m+1]}(0)\right)\right]\right]\frac{1}{(\lambda-\epsilon)^{k+1}}\right| &\leq \sum_{k=0}^{\infty}\frac{1}{\lambda-\epsilon}\prod_{m=0}^{k-1}\left|\frac{\gamma(\tau^{[m+1]}(0))}{\lambda-\epsilon}\right| \\ &\leq C\sum_{k=0}^{\infty}\beta_1^k \\ &< \infty.\end{aligned} \tag{2.31}$$

By Lemma 2.4, there is strictly increasing sequence $\alpha_n \geq 1$ such that $\tau^{[n]}(0) = \alpha_n b$, and by hypothesis $|b| > \sqrt{2}/2$, hence $1 - 2\alpha_n|b|^2 < 1 - 2|b|^2 < 0$. Also we have $|a|^2 + |b|^2 \leq |b| \leq |b/w_0| < 1/\alpha_n$, so we conclude that $-(1 - \alpha_n|b|^2) + |a|^2\alpha_n < 0$. Therefore

$$\begin{aligned}
\gamma\left(\tau^{[m+1]}\right)(0) &= \gamma(\alpha_{m+1}b) \\
&= \frac{\left(1 - 2\alpha_{m+1}|b|^2\right)\left(-b\left(1 - \alpha_{m+1}|b|^2\right) + |a|^2\alpha_{m+1}b\right)}{\alpha_{m+1}b\left(1 - \alpha_{m+1}|b|^2\right)^2} \\
&= \frac{\left(1 - 2\alpha_{m+1}|b|^2\right)\left(-\left(1 - \alpha_{m+1}|b|^2\right) + |a|^2\alpha_{m+1}\right)}{\alpha_{m+1}\left(1 - \alpha_{m+1}|b|^2\right)^2} \\
&> 0.
\end{aligned} \tag{2.32}$$

Also it is obvious that

$$\chi\left(\tau^{[m+1]}(0)\right) = \frac{-\overline{c}d}{\overline{a}\alpha_{m+1}b} = \frac{1}{\alpha_{m+1}} > 0. \tag{2.33}$$

Hence the proof of part (b) is complete.

(c) Every coefficient of F is positive and so $F'(x) > 0$ for all x in the interval $(0, r_0)$. □

Now we find an equation that involves the norm of $C_{\psi,\varphi}$.

Theorem 2.8. *Let $a = c$, $b = -d$, $|b|^2 \geq 1/2$ and $2|a|^2 + |b|^2 \leq 2/3$. Then $\lambda = \|C_{\psi,\varphi}\|^2$ is the unique positive real solution of the equation*

$$1 = \sum_{k=0}^{\infty}\left[\chi\left(\tau^{[k+1]}(0)\right)\prod_{m=0}^{k-1}\left[\gamma\left(\tau^{[m+1]}(0)\right)\right]\right]\frac{1}{\lambda^{k+1}}. \tag{2.34}$$

Proof. By Propositions 2.6 and 2.7, there is exactly one positive real number λ which satisfies equation (2.34), and this number must be equal to $\|C_{\psi,\varphi}\|^2$. □

Corollary 2.9. *In Theorem 2.8 if one replaces a_0 with a and b_0 with b such that $|a| = |a_0|$, and $|b| = |b_0|$, then norm of $C_{\psi,\varphi}$ does not change.*

Proof. We have $\tau^{[n]}(0) = \alpha_n b$. But by Lemma 2.4, $\alpha_n = 1 + \alpha_{n-1}|a|^2/(1 - \alpha_{n-1}|b|^2)$. Hence if one replaces a_0 with a and b_0 with b such that $|a| = |a_0|$ and $|b| = |b_0|$, then α_n, $\gamma(\tau^{[m+1]}(0))$ and $\chi(\tau^{[m+1]}(0)) = 1/\alpha_{m+1}$ do not change. Hence by (2.34), the norm of $C_{\psi,\varphi}$ does not change. □

Example 2.10. Let $\varphi(z) = az + b$ and $\psi(z) = az - b$, where $|a| = 1/10$ and $|b| = 8/10$. Then we have

$$\chi(z) = \frac{4}{5z}, \qquad \tau(z) = \frac{63z - 80}{80z - 100}, \qquad \gamma(z) = \frac{(5 - 8z)(-16 + 13z)}{z(10 - 8z)^2}. \tag{2.35}$$

For positive integer k_0, let λ_{k_0} denote the positive solution of

$$1 = \sum_{k=0}^{k_0} \left[\chi\left(\tau^{[k+1]}(0)\right) \prod_{m=0}^{k-1} \left[\gamma\left(\tau^{[m+1]}(0)\right) \right] \right] \frac{1}{\lambda^{k+1}}. \tag{2.36}$$

Now by using numerical methods, we have

$$\begin{aligned} &\lambda_{10} \approx 1.796745850919, && \lambda_{20} \approx 1.797084678603, \\ &\lambda_{30} \approx 1.797084948747, && \lambda_{50} \approx 1.797084948963, \\ &\lambda_{70} \approx 1.797084948963, && \lambda_{100} \approx 1.797084948963. \end{aligned} \tag{2.37}$$

Hence we see that $\|C_{\psi,\varphi}\|^2 \approx 1.797084948$.

The hypotheses of Theorem 2.8 restrict us to considering the norms of compact operators. In the remainder of this section we investigate the norm and essential norm of a class of noncompact weighted composition operators.

Theorem 2.11. *Let $\varphi(z) = az^n + b$, for some $n \in \mathbb{N}$, where $|a| + |b| = 1$, $\psi \in H^\infty$,let α be one of the nth roots of $b|a|/a|b|$ such that ψ has radial limit at α, and let $|\psi|$ attains its supremum on $D \cup \{\alpha\}$ at α. Then*

$$\frac{1}{\sqrt{n|a|}} |\psi(\alpha)| \le \|C_{\psi,\varphi}\|_e \le \|C_{\psi,\varphi}\| \le \frac{1}{\sqrt{|a|}} |\psi(\alpha)|. \tag{2.38}$$

Proof. Let $0 < r < 1$. Taking $\beta = r\alpha$, by a similar proof for unweighted composition operators [28, Proposition 3.13], we have

$$\begin{aligned} \|C_{\psi,\varphi}\|_e^2 &\ge \lim_{r\to 1^-} \frac{\left\|C_{\psi,\varphi}^* K_\beta\right\|^2}{\left\|K_\beta\right\|^2} \\ &= \lim_{r\to 1^-} \left|\psi(\beta)\right|^2 \cdot \lim_{r\to 1^-} \frac{\left\|K_{\varphi(\beta)}\right\|^2}{\left\|K_\beta\right\|^2} \\ &= \left|\psi(\alpha)\right|^2 \cdot \lim_{r\to 1^-} \frac{1 - r^2}{1 - (r^n|a| + |b|)^2} \\ &= \frac{1}{n|a|(|a| + |b|)} \left|\psi(\alpha)\right|^2 \\ &= \frac{1}{n|a|} \left|\psi(\alpha)\right|^2. \end{aligned} \tag{2.39}$$

Therefore

$$\|C_{\psi,\varphi}\|_e \geq \frac{1}{\sqrt{n|a|}}|\psi(\alpha)|. \tag{2.40}$$

On the other hand, by [3], we have

$$\|C_{\psi,\varphi}\|_e \leq \|C_{\psi,\varphi}\| \leq \|M_\psi\| \|C_\varphi\| \leq \|\psi\|_\infty \|C_{az+b}\| = \frac{1}{\sqrt{|a|}}|\psi(\alpha)|. \tag{2.41}$$

Therefore

$$\frac{1}{\sqrt{n|a|}}|\psi(\alpha)| \leq \|C_{\psi,\varphi}\|_e \leq \|C_{\psi,\varphi}\| \leq \frac{1}{\sqrt{|a|}}|\psi(\alpha)|. \tag{2.42}$$

□

Corollary 2.12. *In Theorem 2.11 if $n = 1$, then*

$$\|C_{\psi,\varphi}\| = \|C_{\psi,\varphi}\|_e = \frac{1}{\sqrt{|a|}}|\psi(\alpha)|. \tag{2.43}$$

Example 2.13. (1) If $\varphi(z) = (1/2)z + 1/2$ and $\psi(z) = (z+1)/2$, then $\|C_{\psi,\varphi}\| = \sqrt{2}$.

(2) If $\varphi(z) = (1/3)z + (2/3)i$ and $\psi(z) = z^5 - 2z^3 + i$, then $\|C_{\psi,\varphi}\| = 4\sqrt{3}$.

(3) If $\varphi(z) = -(1/4)iz + 3/4$ and $\psi(z) = (7z^5 - 5z^3 + 2i)/(z^2+2)$, then $\|C_{\psi,\varphi}\| = 28$.

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References

[1] C. C. Cowen, "Composition operators on H^2," *Journal of Operator Theory*, vol. 9, no. 1, pp. 77–106, 1983.
[2] E. A. Nordgren, "Composition operators," *Canadian Journal of Mathematics*, vol. 20, pp. 442–449, 1968.
[3] C. C. Cowen, "Linear fractional composition operators on H^2," *Integral Equations and Operator Theory*, vol. 11, no. 2, pp. 151–160, 1988.
[4] C. C. Cowen and T. L. Kriete III, "Subnormality and composition operators on H^2," *Journal of Functional Analysis*, vol. 81, no. 2, pp. 298–319, 1988.
[5] C. Hammond, "On the norm of a composition operator with linear fractional symbol," *Acta Universitatis Szegediensis*, vol. 69, no. 3-4, pp. 813–829, 2003.
[6] P. S. Bourdon, E. E. Fry, C. Hammond, and C. H. Spofford, "Norms of linear-fractional composition operators," *Transactions of the American Mathematical Society*, vol. 356, no. 6, pp. 2459–2480, 2004.
[7] E. L. Basor and D. Q. Retsek, "Extremal non-compactness of composition operators with linear fractional symbol," *Journal of Mathematical Analysis and Applications*, vol. 322, no. 2, pp. 749–763, 2006.
[8] C. Hammond, "Zeros of hypergeometric functions and the norm of a composition operator," *Computational Methods and Function Theory*, vol. 6, no. 1, pp. 37–50, 2006.
[9] S. Effinger-Dean, A. Johnson, J. Reed, and J. Shapiro, "Norms of composition operators with rational symbol," *Journal of Mathematical Analysis and Applications*, vol. 324, no. 2, pp. 1062–1072, 2006.

[10] S. Stević, "Norm of weighted composition operators from Bloch space to H_μ^∞ on the unit ball," *Ars Combinatoria*, vol. 88, pp. 125–127, 2008.
[11] S. Stević, "Norms of some operators from Bergman spaces to weighted and Bloch-type spaces," *Utilitas Mathematica*, vol. 76, pp. 59–64, 2008.
[12] S. Stević, "Weighted composition operators from weighted Bergman spaces to weighted-type spaces on the unit ball," *Applied Mathematics and Computation*, vol. 212, no. 2, pp. 499–504, 2009.
[13] S. Stević, "Norm and essential norm of composition followed by differentiation from α-Bloch spaces to H_μ^∞," *Applied Mathematics and Computation*, vol. 207, no. 1, pp. 225–229, 2009.
[14] E. Wolf, "Weighted composition operators between weighted Bergman spaces and weighted Banach spaces of holomorphic functions," *Revista Matemática Complutense*, vol. 21, no. 2, pp. 475–480, 2008.
[15] F. Forelli, "The isometries of H^p," *Canadian Journal of Mathematics*, vol. 16, pp. 721–728, 1964.
[16] J. H. Clifford and M. G. Dabkowski, "Singular values and Schmidt pairs of composition operators on the Hardy space," *Journal of Mathematical Analysis and Applications*, vol. 305, no. 1, pp. 183–196, 2005.
[17] Z.-S. Fang and Z.-H. Zhou, "Differences of composition operators on the space of bounded analytic functions in the polydisc," *Abstract and Applied Analysis*, vol. 2008, Article ID 983132, 10 pages, 2008.
[18] X. Fu and X. Zhu, "Weighted composition operators on some weighted spaces in the unit ball," *Abstract and Applied Analysis*, vol. 2008, Article ID 605807, 8 pages, 2008.
[19] S. Li and S. Stević, "Weighted composition operators from H^∞ to the Bloch space on the polydisc," *Abstract and Applied Analysis*, vol. 2007, Article ID 48478, 13 pages, 2007.
[20] S. Li and S. Stević, "Weighted composition operators from Zygmund spaces into Bloch spaces," *Applied Mathematics and Computation*, vol. 206, no. 2, pp. 825–831, 2008.
[21] L. Luo and S. Ueki, "Weighted composition operators between weighted Bergman spaces and Hardy spaces on the unit ball of C^n," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 1, pp. 88–100, 2007.
[22] B. D. MacCluer and R. Zhao, "Essential norms of weighted composition operators between Bloch-type spaces," *The Rocky Mountain Journal of Mathematics*, vol. 33, no. 4, pp. 1437–1458, 2003.
[23] J. H. Shapiro and W. Smith, "Hardy spaces that support no compact composition operators," *Journal of Functional Analysis*, vol. 205, no. 1, pp. 62–89, 2003.
[24] S. Stević, "Weighted composition operators between mixed norm spaces and H_α^∞ spaces in the unit ball," *Journal of Inequalities and Applications*, vol. 2007, Article ID 28629, 9 pages, 2007.
[25] S. Stević, "Essential norms of weighted composition operators from the α-Bloch space to a weighted-type space on the unit ball," *Abstract and Applied Analysis*, vol. 2008, Article ID 279691, 11 pages, 2008.
[26] S.-I. Ueki and L. Luo, "Compact weighted composition operators and multiplication operators between Hardy spaces," *Abstract and Applied Analysis*, vol. 2008, Article ID 196498, 12 pages, 2008.
[27] X. Zhu, "Weighted composition operators from $F(p,q,s)$spaces to H_μ^∞ spaces," *Abstract and Applied Analysis*, vol. 2009, Article ID 290978, 14 pages, 2009.
[28] C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, Fla, USA, 1995.
[29] G. Gunatillake, *Weighted composition operators*, Ph.D. thesis, Purdue University, West Lafayette, Ind, USA, 2005.
[30] C. Hammond, "The norm of a composition operator with linear symbol acting on the Dirichlet space," *Journal of Mathematical Analysis and Applications*, vol. 303, no. 2, pp. 499–508, 2005.
[31] P. S. Bourdon, D. Levi, S. K. Narayan, and J. H. Shapiro, "Which linear-fractional composition operators are essentially normal?" *Journal of Mathematical Analysis and Applications*, vol. 280, no. 1, pp. 30–53, 2003.