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Research Article

Strong Convergence of a Hybrid Projection Algorithm for Equilibrium Problems, Variational Inequality Problems and Fixed Point Problems in a Banach Space

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We introduce and study a new hybrid projection algorithm for finding a common element of the set of solutions of an equilibrium problem, the set of common fixed points of relatively quasi-nonexpansive mappings, and the set of solutions of the variational inequality for an inverse-strongly-monotone operator in a Banach space. Under suitable assumptions, we show a strong convergence theorem. Using this result, we obtain some applications in a Banach space. The results obtained in this paper extend and improve the several recent results in this area.

1. Introduction

Let E be a real Banach space with norm $\|\cdot\|$ and E^* the dual space of E. Let C be a nonempty closed and convex subset of E, and A a monotone operator of C into E^*. A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| < \|x - y\|$ for all $x, y \in C$. We denote by $F(T) = \{x \in E : Tx = x\}$ the set of fixed points of T. The classical variational inequality problem [1, 2], denoted by $VI(C, A)$, is to find $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0 \tag{1.1}$$

for all $v \in C$. One can see that the variational inequality problem (1.1) is connected with the convex minimization problem, the complementarity problem, the problem of finding a point $u \in E$ satisfying $0 = Au$, and so on.

Recall that an operator A is called monotone if $\langle Ax - Ay, x - y\rangle \geq 0$ for all $x, y \in C$. An operator A of C into E^* is said to be α-inverse-strongly-monotone [3–5] if each $x, y \in C$. We have $\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2$, for a constant $\alpha > 0$.

Assume that the following hold:

(C1) A is α-inverse-strongly-monotone,

(C2) $VI(C, A) \neq \emptyset$,

(C3) $\|Ay\| \leq \|Ay - Au\|$ for all $y \in C$ and $u \in VI(C, A)$.

For finding a solution of the variational inequality problem for an operator A that satisfies conditions (C1)–(C3) in a 2-uniformly convex and uniformly smooth Banach space E, Iiduka and Takahashi [6] introduced and studied the following algorithm: $x_1 = x \in C$, define a sequence $\{x_n\}$ by

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n) \tag{1.2}$$

for every $n = 1, 2, \ldots$, where J is the duality mapping from E into E^*, Π_C is the generalized projection from E onto C, and $\{\lambda_n\}$ is a sequence of positive real numbers. They proved that under certain appropriate conditions imposed on $\{\lambda_n\}$, and J is weakly sequentially continuous, the sequence $\{x_n\}$ generated by (1.2) converges weakly to some element z in $VI(C, A)$, where $z = \lim_{n\to\infty}\Pi_{VI(C,A)}(x_n)$.

In 2004, Matsushita and Takahashi [7] introduced the following iteration: $x_0 \in C$ chosen arbitrarily,

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JTx_n), \tag{1.3}$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$, T is a relatively nonexpansive mapping, and Π_C denotes the generalized projection from E onto a closed convex subset C of E. They prove that the sequence $\{x_n\}$ generated by (1.3) converges weakly to a fixed point of T.

The problem of finding a common element of the set of the variational inequalities for monotone mappings in the framework of Hilbert spaces and Banach spaces has been intensively studied by many authors; see, for instance, [3–5] and the references cited therein.

Let f be a bifunction of $C\times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $f : C \times C \to \mathbb{R}$ is to find $\hat{x} \in C$ such that

$$f(\hat{x}, y) \geq 0 \tag{1.4}$$

for all $y \in C$. The set of solutions of (1.4) is denoted by $EP(f)$, that is, $EP(f) = \{\hat{x} \in C : f(\hat{x}, y) \geq 0 \text{ for all } y \in C\}$. Given a mapping $T : C \to E^*$, let $f(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then $\hat{x} \in EP(f)$ if and only if $\langle T\hat{x}, y - \hat{x}\rangle \geq 0$ for all $y \in C$, that is, $\hat{x}$ is a solution of the variational inequality. Many problems in physics, optimization, and economics reduce to finding a solution of (1.4). Equilibrium problems have been studied extensively; see, for instance, [8, 9]. Combettes and Hirstoaga [8] introduced an iterative scheme for finding the best approximation to the initial data when $EP(f)$ is nonempty and proved a strong convergence theorem.

For solving the equilibrium problem, let us assume that a bifunction f satisfies the following conditions:

(A1) $f(x,x) = 0$ for all $x \in C$;

(A2) f is monotone, that is, $f(x,y) + f(y,x) \leq 0$ for all $x, y \in C$;

(A3) for all $x, y, z \in C$,

$$\limsup_{t \downarrow 0} f(tz + (1-t)x, y) \leq f(x,y); \tag{1.5}$$

(A4) for all $x \in C$, $f(x,\cdot)$ is convex and lower semicontinuous.

The problem of finding a common element of the set of fixed points and the set of solutions of an equilibrium problem in the framework of Hilbert spaces and Banach spaces has been studied by many authors; see [7, 8, 10–18].

In 2008, Takahashi and Zembayashi [15] introduced the shrinking projection method which is the modification of (1.3) for a relatively nonexpansive mapping. It is given as follows:

$$\begin{gathered} x_0 = x \in C, \qquad C_0 = C, \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T x_n), \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{ z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) \}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0 \end{gathered} \tag{1.6}$$

for every $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on $E, \{\alpha_n\} \subset [0,1]$ satisfying $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\{r_n\} \subset [s, \infty)$ for some $s > 0$. They proved that the sequence $\{x_n\}$ generated by (1.6) converges strongly to $q = \Pi_{F(T)\cap EP(f)} x_0$, where $\Pi_{F(T)\cap EP(f)}$ is the generalized projection of E onto $F(T) \cap EP(f)$.

In the same year, Qin et al. [19] extend the iteration process (1.6) from a single relatively nonexpansive mapping to two relatively quasi-nonexpansive mappings: $x_0 \in E$, $C_1 = C$ and $x_1 = \Pi_{C_1} x_0$,

$$\begin{gathered} y_n = J^{-1}(\alpha_n J x_n + \beta_n J T x_n + \gamma_n J S x_n), \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{ z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) \}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0 \end{gathered} \tag{1.7}$$

for every $n \in \mathbb{N} \cup \{0\}$. Under appropriate conditions imposed on $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, and $\{r_n\}$, they obtain that the sequence $\{x_n\}$ generated by (1.7) converges strongly to $q = \Pi_{F(T)\cap F(S)\cap EP(f)} x_0$.

In 2009, Wattanawitoon and Kumam [17] introduced the following iterative scheme which is the modification of (1.6) and (1.7) in a Banach space: $x_0 \in E$, $C_1 = C$, and $x_1 = \Pi_{C_1} x_0$,

$$\begin{gathered} y_n = J^{-1}(\delta_n J x_n + (1-\delta_n) J z_n), \\ z_n = J^{-1}(\alpha_n J x_n + \beta_n J T x_n + \gamma_n J S x_n), \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J z_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0 \end{gathered} \tag{1.8}$$

for finding a common element of the set of solutions of an equilibrium problem and the set of common fixed points of two relatively quasi-nonexpansive mappings in a Banach space. They proved that under certain appropriate conditions imposed on $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$, and $\{r_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ generated by (1.8) converge strongly to $q \in F(T)\cap F(S)\cap EP(f)$, where $q = \Pi_{F(T)\cap F(S)\cap EP(f)} x_0$.

For finding common elements of the set of the equilibrium problem, the set of the variational inequality problem for an inverse-strongly-monotone operator and the set of common fixed points for relatively quasi-nonexpansive mappings. Cholamjiak [20] introduced an iterative scheme by using the new hybrid method in a Banach space. The scheme is defined as follows: $x_0 \in E$, $C_1 = C$ and $x_1 \in \Pi_{C_1} x_0$,,

$$\begin{gathered} z_n = \Pi_C J^{-1}(J x_n - \lambda_n A x_n), \\ y_n = J^{-1}(\alpha_n J x_n + \beta_n J T x_n + \gamma_n J S z_n), \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0 \end{gathered} \tag{1.9}$$

for every $n \in \mathbb{N}$, where J is the duality mapping on E and $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $[0, 1]$. He proved that under certain appropriate conditions imposed on $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{r_n\}$, and $\{\lambda_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ generated by (1.9) converge strongly to $q = \Pi_{F(T)\cap F(S)\cap EP(f)\cap VI(C,A)} x_0$.

Motivated by the recent works, we introduce an iterative scheme by a new hybrid method as follows: $x_0 \in E, C_1 = C$ and $x_1 = \Pi_{C_1} x_0$,

$$\begin{aligned}
w_n &= \Pi_C J^{-1}(Jx_n - \lambda_n A x_n), \\
y_n &= J^{-1}(\delta_n J x_n + (1-\delta_n) J w_n), \\
z_n &= J^{-1}(\alpha_n J x_n + \beta_n J T x_n + \gamma_n J S y_n), \\
u_n \in C \text{ such that } & f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jz_n\rangle \geq 0, \quad \forall y \in C, \\
C_{n+1} &= \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\
x_{n+1} &= \Pi_{C_{n+1}} x_0
\end{aligned} \tag{1.10}$$

for every $n \in \mathbb{N}$, where J is the duality mapping on E, Π_C is the generalized projection from E onto a closed convex subset C of E, f is an equilibrium bifunction satisfying (A1)–(A4), A is an operator of C into E^* satisfying (C1)–(C3), $T, S : C \to C$ are two closed relatively quasi-nonexpansive mappings, $\{\alpha_n\}, \{\beta_n, \{\gamma_n\}$, and $\{\delta_n\}$ are sequences in $[0,1]$ such that $\alpha_n+\beta_n+\gamma_n = 1, \limsup_{n\to\infty} \delta_n < 1$, $\liminf_{n\to\infty} \alpha_n\beta_n > 0$, and $\liminf_{n\to\infty} \alpha_n\gamma_n > 0, \{r_n\} \subset [s,\infty)$ for some $s > 0$ and $\{\lambda_n\} \subset [a,b]$ for some a, b with $0 < a < b < c^2\alpha/2$, where $1/c$ is the 2-uniformly convexity constant of E. We prove that the sequences $\{x_n\}$ and $\{u_n\}$ generated by the above iterative scheme converge strongly to $q = \Pi_{F(T)\cap F(S)\cap EP(f)\cap VI(C,A)} x_0$.

2. Preliminaries

In this section, we recall some well know concepts and results.

Let E be a real Banach space with dimension $E \geq 2$. The modulus of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in E,\ \|x\| = \|y\| = 1, \|x-y\| \geq \varepsilon\right\}. \tag{2.1}$$

Banach space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$. Let p be a fixed real number with $p \geq 2$. A Banach space E is said to be *p-uniformly convex* if there exists a constant $c > 0$ such that $\delta(\varepsilon) \geq c\varepsilon^p$ for all $\varepsilon \in [0,2]$; see [21–23] for more details. A Banach space E is said to be *smooth* if the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.2}$$

exists for all $x, y \in U$, where U denotes the unit sphere of E (i.e., $U = \{x \in E : \|x\| = 1\}$). It is also said to be *uniformly smooth* if the limit (2.2) is attained uniformly for $x, y \in U$. One should note that no Banach space is p-uniformly convex for $1 < p < 2$; see [23]. It is well known that a Hilbert space is 2-uniformly convex, uniformly smooth. For each $p > 1$, the *generalized duality mapping* $J_p : E \to 2^{E^*}$ is defined by

$$J_p(x) = \left\{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1} \right\} \tag{2.3}$$

for all $x \in E$. In particular, $J = J_2$ is called the *normalized duality mapping*. If E is a Hilbert space, then $J = I$, where I is the identity mapping. It is also known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. See [24, 25] for more details.

Let E be a smooth, strictly convex, and reflexive Banach space and let C be a nonempty closed convex subset of E. We denote by ϕ the function defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \tag{2.4}$$

for all $x, y \in E$. Following Alber [26], the generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution to the minimization problem

$$\phi(\overline{x}, x) = \inf_{y \in C} \phi(y, x). \tag{2.5}$$

The existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J (see, e.g., [24, 26–29]. In Hilbert space, $\Pi_C = P_C$. It is obvious from the definition of function ϕ that

$$(\|y\| - \|x\|)^2 \leq \phi(y, x) = (\|y\| + \|x\|)^2 \tag{2.6}$$

for all $x, y \in E$. If E is a Hilbert space, then $\phi(x, y) = \|x - y\|^2$.

If E is a reflexive, strictly convex, and smooth Banach space, then for $x, y \in E, \phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$, then $x = y$. From (2.6), we have $\|x\| = \|y\|$. This implies that $\langle x, Jy \rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, one has $Jx = Jy$. Therefore, we have $x = y$; see [24, 28] for more details.

Recall that a point p in a closed convex subset C of E is said to be an asymptotic fixed point of T [30] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n \to \infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed point of T will be denoted by $\widehat{F}(T)$. A mapping $T : C \to C$ is called relatively nonexpansive [11, 31–33] if T satisfies the following conditions:

(1) $F(T) \neq \emptyset$;

(2) $\phi(p, Tx) \leq \phi(p, x)$ for all $p \in F(T)$ and $x \in C$;

(3) $\widehat{F}(T) = F(T)$.

The asymptotic behavior of a relatively nonexpansive mapping was studied in [31–33].

A mapping T is said to be relatively quasi-nonexpansive if T satisfies conditions (1) and (2). It is easy to see that the class of relatively quasi-nonexpansive mappings is more general than the class of relatively nonexpansive mappings [11, 31–33] which requires the strong restriction: $F(T) = \widehat{F}(T)$.

We give some examples which are closed relatively quasi-nonexpansive; see [19].

Example 2.1. Let E be a uniformly smooth and strictly convex Banach space and $A \subset E \times E^*$ be a maximal monotone mapping such that its zero set $A^{-1}0 \neq \emptyset$. Then, $J_r = (J + rA)^{-1}J$ is a closed relatively quasi-nonexpansive mapping from E onto $D(A)$ and $F(J_r) = A^{-1}0$.

Example 2.2. Let Π_C be the generalized projection from a smooth, strictly convex, and reflexive Banach space E onto a nonempty closed convex subset C of E. Then, Π_C is a closed relatively quasi-nonexpansive mapping with $F(\Pi_C) = C$.

An operator A of C into E^* is said to be *hemicontinuous* if for all $x, y \in C$, the mapping F of $[0,1]$ into E^* defined by $F(t) = A(tx + (1-t)y)$ is continuous with respect to the weak* topology of E^*. We define by $N_C(v)$ the *normal cone* for C at a point $v \in C$, that is,

$$N_C(v) = \{x^* \in E^* : \langle v - y, x^* \rangle \geq 0 \ \forall y \in C\}. \tag{2.7}$$

Let C be a nonempty, closed convex subset of a Banach space E and A a monotone, hemicontinuous operator of C into E^*. Let $T_e \subset E \times E^*$ be an operator defined as follows:

$$T_e v = \begin{cases} Av + N_C(v), & v \in C, \\ \emptyset, & \text{otherwise}. \end{cases} \tag{2.8}$$

Then, T_e is maximal monotone and $T_e^{-1}0 = VI(C, A)$; see [34].

In the sequel, the following lemmas are needed to prove our main results.

Lemma 2.3 ([22, 35]). *Let p be a given real number with $p \geq 2$ and E a p-uniformly convex Banach space. Then, for all $x, y \in E$, $j_x \in J_p(x)$ and $j_y \in J_p(y)$,*

$$\langle x - y, j_x - j_y \rangle \geq \frac{c^p}{2^{p-2}p} \|x - y\|^p, \tag{2.9}$$

where J_p is the generalized duality mapping of E and $1/c$ is the p-uniformly convexity constant of E.

Lemma 2.4 ([29]). *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \to 0$.*

Lemma 2.5 ([26]). *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then $x_0 = \Pi_C x$ if and only if $\langle x_0 - y, Jx - Jx_0 \rangle \geq 0$ for all $y \in C$.*

Lemma 2.6 ([26]). *Let C be a nonempty closed convex subset of a reflexive, strictly convex, and smooth Banach space E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C. \tag{2.10}$$

Lemma 2.7 ([19]). *Let E be a uniformly convex, smooth Banach space, let C be a closed convex subset of E, let T be a closed and relatively quasi-nonexpansive mapping from C into itself. Then $F(T)$ is a closed convex subset of C.*

Lemma 2.8 ([36]). *Let E be a uniformly convex Banach space and $B_r(0)$ be a closed ball of E. Then there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\|\alpha x + \beta y + \gamma z\|^2 \leq \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta g(\|x - y\|) \tag{2.11}$$

for all $x, y, z \in B_r(0)$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$.

Lemma 2.9 ([10]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let f be a bifunction from $C \times C$ to R satisfying (A1)–(A4), and let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C. \tag{2.12}$$

Lemma 2.10 ([19]). *Let C be a closed convex subset of a uniformly smooth, strictly convex, and reflexive Banach space E, and let f be a bifunction from $C \times C$ to R satisfying (A1)–(A4). For all $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ as follows:*

$$T_r x = \left\{z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0,\ \forall y \in C\right\}. \tag{2.13}$$

Then, the following hold:

(1) T_r is single-valued;

(2) T_r is a firmly nonexpansive-type mapping [37], that is, for all $x, y \in E$,

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle; \tag{2.14}$$

(3) $F(T_r) = EP(f)$;

(4) $EP(f)$ is closed and convex.

Lemma 2.11 ([16]). *Let C be a closed convex subset of a smooth, strictly, and reflexive Banach space E, let f be a bifucntion from $C \times C$ to R satisfying (A1)–(A4), let $r > 0$. Then, for all $x \in E$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x). \tag{2.15}$$

We make use of the following mapping V studied in Alber [26]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2 \tag{2.16}$$

for all $x \in E$ and $x^* \in E^*$, that is, $V(x, x^*) = \phi(x, J^{-1}(x^*))$.

Lemma 2.12 ([26]). *Let E be a reflexive, strictly convex, smooth Banach space and let V be as in* (2.16). *Then*

$$V(x, x^*) + 2\left\langle J^{-1}(x^*) - x, y^* \right\rangle \leq V(x, x^* + y^*) \tag{2.17}$$

for all $x \in E$ and $x^*, y^* \in E^*$.

3. Main Results

In this section, we prove strong convergence theorems by hybrid methods which solves the problem of finding a common element of the set of solutions of an equilibrium problem, the set of common fixed points of relatively quasi-nonexpansive mappings and the set of solutions of the variational inequality of an α-inverse-strongly-monotone mapping in a 2-uniformly convex, uniformly smooth Banach space.

Theorem 3.1. *Let E be a* 2*-uniformly convex, uniformly smooth Banach space, C a nonempty closed convex subset of E, f a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (A1)–(A4), A an operator of C into E^* satisfying (C1)–(C3), and T, S two closed relatively quasi-nonexpansive mappings from C into itself such that the set $F := F(T) \cap F(S) \cap EP(f) \cap VI(C, A) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, define sequences $\{x_n\}$ and $\{u_n\}$ of C as follows:*

$$\begin{gathered} w_n = \Pi_C J^{-1}(Jx_n - \lambda_n A x_n), \\ y_n = J^{-1}(\delta_n J x_n + (1-\delta_n) J w_n), \\ z_n = J^{-1}(\alpha_n J x_n + \beta_n J T x_n + \gamma_n J S y_n), \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J z_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{ z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) \}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0 \end{gathered} \tag{3.1}$$

for every $n \in \mathbb{N}$, where J is the duality mapping on E. Suppose that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, and $\{\delta_n\}$ are sequences in $[0,1]$ satisfying the restrictions:

(B1) $\alpha_n + \beta_n + \gamma_n = 1$;

(B2) $\limsup_{n \to \infty} \delta_n < 1$, $\liminf_{n \to \infty} \alpha_n \beta_n > 0$ *and* $\liminf_{n \to \infty} \alpha_n \gamma_n > 0$;

(B3) $\{r_n\} \subset [s, \infty)$ *for some* $s > 0$;

(B4) $\{\lambda_n\} \subset [a, b]$ *for some a, b with $0 < a < b < c^2\alpha/2$, where $1/c$ is the* 2*-uniformly convexity constant of E.*

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $q = \Pi_F x_0$.

Proof. We divide the proof into eight steps.

Step 1. Show that $\Pi_F x_0$ and $\Pi_{C_{n+1}} x_0$ are well defined.

It is obvious that $VI(C, A)$ is a closed convex subset of C. By Lemma 2.7, we know that $F(T) \cap F(S)$ is closed and convex. From Lemma 2.10 (4), we also have $EP(f)$ is closed and convex. Hence, $F := F(T) \cap F(S) \cap EP(f) \cap VI(C, A)$ is a nonempty, closed, and convex subset of C; consequently, $\Pi_F x_0$ is well defined.

Next, we show that C_n is closed and convex for all $n \in \mathbb{N}$.

It is obvious that, $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for some $k \in \mathbb{N}$. For all $z \in C_k$, we know that $\phi(z, u_k) \leq \phi(z, x_k)$ is equivalent to

$$2\langle z, Jx_k - Ju_k \rangle \leq \|x_k\|^2 - \|u_k\|^2. \tag{3.2}$$

So, C_{k+1} is closed and convex. Then, for any $n \in \mathbb{N}$, C_n is closed and convex. This implies that $\Pi_{C_{n+1}} x_0$ is well defined.

Step 2. We prove by induction that $F \subset C_n$ for all $n \in \mathbb{N}$.

Putting $v_n = J^{-1}(Jx_n - \lambda_n Ax_n)$. First, we observe that $u_n = T_{r_n} z_n$ for all $n \in \mathbb{N}$ and $F \subset C_1 = C$. Suppose that $F \subset C_k$ for some $k \in \mathbb{N}$. Let $u \in F \subset C_k$. From Lemmas 2.6 and 2.12, we have

$$\begin{aligned}
\phi(u, w_k) &= \phi(u, \Pi_C v_k) \\
&\leq \phi(u, v_k) \\
&= \phi\left(u, J^{-1}(Jx_k - \lambda_k Ax_k)\right) \\
&= V(u, Jx_k - \lambda_k Ax_k) \\
&\leq V(u, (Jx_k - \lambda_k Ax_k) + \lambda_k Ax_k) - 2\left\langle J^{-1}(Jx_k - \lambda_k Ax_k) - u, \lambda_k Ax_k \right\rangle \\
&= V(u, Jx_k) - 2\lambda_k \langle v_k - u, Ax_k \rangle \\
&= \phi(u, x_k) - 2\lambda_k \langle x_k - u, Ax_k \rangle + 2\langle v_k - x_k, -\lambda_k Ax_k \rangle.
\end{aligned} \tag{3.3}$$

Using (C1) and $u \in VI(C, A)$, we have

$$\begin{aligned}
-2\lambda_k \langle x_k - u, Ax_k \rangle &= -2\lambda_k \langle x_k - u, Ax_k - Au + Au \rangle \\
&= -2\lambda_k \langle x_k - u, Ax_k - Au \rangle - 2\lambda_k \langle x_k - u, Au \rangle \\
&\leq -2\alpha\lambda_k \|Ax_k - Au\|^2.
\end{aligned} \tag{3.4}$$

By using Lemma 2.3 and (C3), we have

$$\begin{aligned}
2\langle v_k - x_k, -\lambda_k Ax_k \rangle &= 2\left\langle J^{-1}(Jx_k - \lambda_k Ax_k) - J^{-1}(Jx_k), -\lambda_k Ax_k \right\rangle \\
&\leq 2\left\| J^{-1}(Jx_k - \lambda_k Ax_k) - J^{-1}(Jx_k) \right\| \|\lambda_k Ax_k\| \\
&\leq \frac{4}{c^2} \left\| JJ^{-1}(Jx_k - \lambda_k Ax_k) - JJ^{-1}(Jx_k) \right\| \|\lambda_k Ax_k\|
\end{aligned}$$

$$
\begin{aligned}
&= \frac{4}{c^2}\|Jx_k - \lambda_k Ax_k - Jx_k\|\|\lambda_k Ax_k\| \\
&= \frac{4}{c^2}\lambda_k^2\|Ax_k\|^2 \\
&\leq \frac{4}{c^2}\lambda_k^2\|Ax_k - Au\|^2.
\end{aligned}
\tag{3.5}
$$

Replacing (3.4) and (3.5) into (3.3) and using (B4), we get

$$
\phi(u, w_k) \leq \phi(u, x_k) + 2a\left(\frac{2}{c^2}b - \alpha\right)\|Ax_k - Au\|^2 \leq \phi(u, x_k). \tag{3.6}
$$

By the convexity of $\|\cdot\|^2$ and (3.6), for each $u \in F \subset C_k$, we have

$$
\begin{aligned}
\phi(u, y_k) &= \phi\left(u, J^{-1}(\delta_k Jx_k + (1-\delta_k)Jw_k)\right) \\
&= V(u, \delta_k Jx_k + (1-\delta_k)Jw_k) \\
&= \|u\|^2 - 2\langle u, \delta_k Jx_k + (1-\delta_k)Jw_k\rangle + \|\delta_k Jx_k + (1-\delta_k)Jw_k\|^2 \\
&\leq \|u\|^2 - 2\delta_k\langle u, Jx_k\rangle - 2(1-\delta_k)\langle u, Jw_k\rangle \\
&\quad + \delta_k\|Jx_k\|^2 + (1-\delta_k)\|Jw_k\|^2 \\
&= (\delta_k + (1-\delta_k))\|u\|^2 - 2\delta_k\langle u, Jx_k\rangle - 2(1-\delta_k)\langle u, Jw_k\rangle \\
&\quad + \delta_k\|Jx_k\|^2 + (1-\delta_k)\|Jw_k\|^2 \\
&= \delta_k\left(\|u\|^2 - 2\langle u, Jx_k\rangle + \|Jx_k\|^2\right) + (1-\delta_k)\left(\|u\|^2 - 2\langle u, Jw_k\rangle + \|Jw_k\|^2\right) \\
&= \delta_k\phi(u, x_k) + (1-\delta_k)\phi(u, w_k) \\
&\leq \delta_k\phi(u, x_k) + (1-\delta_k)\phi(u, x_k) \\
&= \phi(u, x_k),
\end{aligned}
\tag{3.7}
$$

and so

$$
\begin{aligned}
\phi(u, u_k) &= \phi(u, T_{r_k}z_k) \\
&\leq \phi(u, z_k) \\
&= \phi\left(u, J^{-1}(\alpha_k Jx_k + \beta_k JTx_k + \gamma_k JSy_k)\right) \\
&= V(u, \alpha_k Jx_k + \beta_k JTx_k + \gamma_k JSy_k) \\
&= \|u\|^2 - 2\alpha_k\langle u, Jx_k\rangle - 2\beta_k\langle u, JTx_k\rangle - 2\gamma_k\langle u, JSy_k\rangle + \left\|\alpha_k Jx_k + \beta_k JTx_k + \gamma_k JSy_k\right\|^2 \\
&\leq \|u\|^2 - 2\alpha_k\langle u, Jx_k\rangle - 2\beta_k\langle u, JTx_k\rangle - 2\gamma_k\langle u, JSy_k\rangle \\
&\quad + \alpha_k\|Jx_k\|^2 + \beta_k\|JTx_k\|^2 + \gamma_k\left\|JSy_k\right\|^2
\end{aligned}
$$

$$\begin{aligned}
&= (\alpha_k + \beta_k + \gamma_k)\|u\|^2 - 2\alpha_k\langle u, Jx_k\rangle - 2\beta_k\langle u, JTx_k\rangle - 2\gamma_k\langle u, JSy_k\rangle \\
&\quad + \alpha_k\|Jx_k\|^2 + \beta_k\|JTx_k\|^2 + \gamma_k\|JSy_k\|^2 \\
&= \alpha_k\left(\|u\|^2 - 2\langle u, Jx_k\rangle + \|Jx_k\|^2\right) + \beta_k\left(\|u\|^2 - 2\langle u, JTx_k\rangle + \|JTx_k\|^2\right) \\
&\quad + \gamma_k\left(\|u\|^2 - 2\langle u, JSy_k\rangle + \|JSy_k\|^2\right) \\
&= \alpha_k\phi(u, x_k) + \beta_k\phi(u, Tx_k) + \gamma_k\phi(u, Sy_k) \\
&\le \alpha_k\phi(u, x_k) + \beta_k\phi(u, x_k) + \gamma_k\phi(u, y_k) \\
&\le \alpha_k\phi(u, x_k) + \beta_k\phi(u, x_k) + \gamma_k\phi(u, x_k) \\
&= (\alpha_k + \beta_k + \gamma_k)\phi(u, x_k) \\
&= \phi(u, x_k).
\end{aligned} \tag{3.8}$$

This shows that $u \in C_{k+1}$ and hence $F \subset C_{k+1}$. This implies that $F \subset C_n$ for all $n \in \mathbb{N}$.

Step 3. Show that $\lim_{n\to\infty}\phi(x_n, x_0)$ exists.

From $x_n = \Pi_{C_n}x_0$ and $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we have

$$\phi(x_n, x_0) \le \phi(x_{n+1}, x_0), \quad \forall n \in \mathbb{N}. \tag{3.9}$$

Therefore, $\{\phi(x_n, x_0)\}$ is nondecreasing. From Lemma 2.6, we have

$$\phi(x_n, x_0) = \phi(\Pi_{C_n}x_0, x_0) \le \phi(u, x_0) - \phi(u, x_n) \le \phi(u, x_0). \tag{3.10}$$

Then the sequence $\{\phi(x_n, x_0)\}$ is bounded. It follows that $\lim_{n\to\infty}\phi(x_n, x_0)$ exists.

Step 4. Show that $\{x_n\}$ is a Cauchy sequence in C.

Since $x_m = \Pi_{C_m}x_0 \in C_m \subset C_n$ for any positive integer $m \ge n$, by Lemma 2.6, we also have

$$\begin{aligned}
\phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n}x_0) \\
&\le \phi(x_m, x_0) - \phi(\Pi_{C_n}x_0, x_0) \\
&= \phi(x_m, x_0) - \phi(x_n, x_0).
\end{aligned} \tag{3.11}$$

Letting $m, n \to \infty$ in (3.11), we have $\phi(x_m, x_n) \to 0$. It follows from Lemma 2.4 that $\|x_m - x_n\| \to 0$ as $m, n \to \infty$. Hence, $\{x_n\}$ is a Cauchy sequence. By the completeness of E and the closedness of C, one can assume that $x_n \to q \in C$ as $n \to \infty$. Further, we obtain

$$\lim_{n\to\infty}\phi(x_{n+1}, x_n) = 0. \tag{3.12}$$

Since $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1}$, we have

$$\begin{aligned} \phi(x_{n+1}, u_n) &\le \phi(x_{n+1}, x_n) \\ &\longrightarrow 0 \quad (\text{as} \quad n \longrightarrow \infty). \end{aligned} \tag{3.13}$$

Applying Lemma 2.4 to (3.12) and (3.13), we have

$$\begin{aligned} \|u_n - x_n\| &= \|u_n - x_{n+1} + x_{n+1} - x_n\| \\ &\le \|u_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\longrightarrow 0 \quad (\text{as} \quad n \longrightarrow \infty). \end{aligned} \tag{3.14}$$

This implies that $u_n \to q$ as $n \to \infty$. Since J is uniformly norm-to-norm continuous on bounded subsets of E, we also obtain

$$\lim_{n\to\infty} \|Ju_n - Jx_n\| = 0. \tag{3.15}$$

Step 5. Show that $x_n \to q \in F(T) \cap F(S)$.

Let $r = \sup_{n\ge1}\{\|x_n\|, \|Tx_n\|, \|Sw_n\|\}$. From (3.7) and Lemma 2.8, we obtain

$$\begin{aligned} \phi(u, u_n) &= \phi(u, T_{r_n} z_n) \\ &\le \phi(u, z_n) \\ &= \phi\Big(u, J^{-1}(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSy_n)\Big) \\ &= V(u, \alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSy_n) \\ &= \|u\|^2 - 2\alpha_n\langle u, Jx_n\rangle - 2\beta_n\langle u, JTx_n\rangle - 2\gamma_n\langle u, JSy_n\rangle \\ &\quad + \left\|\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSy_n\right\|^2 \\ &\le \|u\|^2 - 2\alpha_n\langle u, Jx_n\rangle - 2\beta_n\langle u, JTx_n\rangle - 2\gamma_n\langle u, JSy_n\rangle \\ &\quad + \alpha_n\|Jx_n\|^2 + \beta_n\|JTx_n\|^2 + \gamma_n\left\|JSy_n\right\|^2 - \alpha_n\beta_n g(\|Jx_n - JTx_n\|) \\ &= (\alpha_n + \beta_n + \gamma_n)\|u\|^2 - 2\alpha_n\langle u, Jx_n\rangle - 2\beta_n\langle u, JTx_n\rangle - 2\gamma_n\langle u, JSy_n\rangle \\ &\quad + \alpha_n\|Jx_n\|^2 + \beta_n\|JTx_n\|^2 + \gamma_n\left\|JSy_n\right\|^2 - \alpha_n\beta_n g(\|Jx_n - JTx_n\|) \\ &= \alpha_n\phi(u, x_n) + \beta_n\phi(u, Tx_n) + \gamma_n\phi(u, Sy_n) - \alpha_n\beta_n g(\|Jx_n - JTx_n\|) \\ &\le \alpha_n\phi(u, x_n) + \beta_n\phi(u, x_n) + \gamma_n\phi(u, y_n) - \alpha_n\beta_n g(\|Jx_n - JTx_n\|) \\ &\le \alpha_n\phi(u, x_n) + \beta_n\phi(u, x_n) + \gamma_n\phi(u, x_n) - \alpha_n\beta_n g(\|Jx_n - JTx_n\|) \\ &= \phi(u, x_n) - \alpha_n\beta_n g(\|Jx_n - JTx_n\|). \end{aligned} \tag{3.16}$$

This implies that

$$\begin{aligned}\alpha_n\beta_n g(\|Jx_n - JTx_n\|) &\le \phi(u,x_n) - \phi(u,u_n)\\ &= \|x_n\|^2 - \|u_n\|^2 - 2\langle u, Jx_n - Ju_n\rangle\\ &\le \|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|u\|\|Jx_n - Ju_n\|.\end{aligned} \tag{3.17}$$

It follows from (3.14), (3.15), and (B2) that

$$\lim_{n\to\infty} g(\|Jx_n - JTx_n\|) = 0. \tag{3.18}$$

Since g is strictly increasing and continuous at 0 with $g(0) = 0$, it follows that

$$\lim_{n\to\infty} \|Jx_n - JTx_n\| = 0. \tag{3.19}$$

Since J is uniformly norm-to-norm continuous on bounded sets, so is J^{-1}. Then

$$\lim_{n\to\infty}\|x_n - Tx_n\| = \lim_{n\to\infty}\left\|J^{-1}(Jx_n) - J^{-1}(JTx_n)\right\| = 0. \tag{3.20}$$

In the same manner, we can show that

$$\lim_{n\to\infty}\left\|x_n - Sy_n\right\| = 0. \tag{3.21}$$

In addition, $\phi(u,y_n) \le \delta_n\phi(u,x_n) + (1-\delta_n)\phi(u,w_n)$, using (3.6), we have

$$\begin{aligned}\phi(u,u_n) &\le \alpha_n\phi(u,x_n) + \beta_n\phi(u,x_n) + \gamma_n\phi(u,y_n)\\ &\le \alpha_n\phi(u,x_n) + \beta_n\phi(u,x_n) + \gamma_n\left(\delta_n\phi(u,x_n) + (1-\delta_n)\phi(u,w_n)\right)\\ &= \alpha_n\phi(u,x_n) + \beta_n\phi(u,x_n) + \gamma_n\delta_n\phi(u,x_n) + \gamma_n(1-\delta_n)\phi(u,w_n)\\ &\le \alpha_n\phi(u,x_n) + \beta_n\phi(u,x_n) + \gamma_n\delta_n\phi(u,x_n)\\ &\quad + \gamma_n(1-\delta_n)\left(\phi(u,x_n) + 2a\left(\frac{2}{c^2}b - \alpha\right)\|Ax_n - Au\|^2\right)\\ &= \alpha_n\phi(u,x_n) + \beta_n\phi(u,x_n) + \gamma_n\delta_n\phi(u,x_n)\\ &\quad + \gamma_n(1-\delta_n)\phi(u,x_n) + \gamma_n(1-\delta_n)2a\left(\frac{2}{c^2}b - \alpha\right)\|Ax_n - Au\|^2\\ &= \left(\alpha_n + \beta_n + \gamma_n\right)\phi(u,x_n) - \gamma_n(1-\delta_n)2a\left(\alpha - \frac{2}{c^2}b\right)\|Ax_n - Au\|^2\\ &= \phi(u,x_n) - \gamma_n(1-\delta_n)2a\left(\alpha - \frac{2}{c^2}b\right)\|Ax_n - Au\|^2,\end{aligned} \tag{3.22}$$

which leads to the following:

$$\gamma_n(1-\delta_n)2a\left(\alpha-\frac{2}{c^2}b\right)\|Ax_n-Au\|^2 \le \phi(u,x_n)-\phi(u,u_n). \tag{3.23}$$

Since $\limsup_{n\to\infty}\delta_n<1$, $0<\liminf_{n\to\infty}\alpha_n\gamma_n\le\liminf_{n\to\infty}\gamma_n$ and from (3.17), we observe

$$\lim_{n\to\infty}\left(\phi(u,x_n)-\phi(u,u_n)\right)=0, \tag{3.24}$$

which yields that

$$\lim_{n\to\infty}\|Ax_n-Au\|=0. \tag{3.25}$$

From Lemmas 2.6 and 2.12, and (3.5), we have

$$\begin{aligned}
\phi(x_n,w_n)=\phi(x_n,\Pi_C v_n)\le\phi(x_n,v_n)&\\
&=\phi\left(x_n,J^{-1}(Jx_n-\lambda_n Ax_n)\right)\\
&=V(x_n,Jx_n-\lambda_n Ax_n)\\
&\le V(x_n,(Jx_n-\lambda_n Ax_n)+\lambda_n Ax_n)\\
&\quad-2\left\langle J^{-1}(Jx_n-\lambda_n Ax_n)-x_n,\lambda_n Ax_n\right\rangle\\
&=\phi(x_n,x_n)+2\langle v_n-x_n,-\lambda_n Ax_n\rangle\\
&=2\langle v_n-x_n,-\lambda_n Ax_n\rangle\\
&\le\frac{4}{c^2}b^2\|Ax_n-Au\|^2\\
&\longrightarrow 0\quad(\text{as } n\longrightarrow\infty).
\end{aligned} \tag{3.26}$$

It follows from Lemma 2.4 and (3.25) that

$$\lim_{n\to\infty}\|x_n-w_n\|=0. \tag{3.27}$$

Hence $w_n\to q$ as $n\to\infty$ and

$$\lim_{n\to\infty}\|Jx_n-Jw_n\|=0. \tag{3.28}$$

By using (3.26), we have

$$\begin{aligned}
\phi(x_n, y_n) &= \phi\Big(x_n, J^{-1}(\delta_n J x_n + (1-\delta_n) J w_n)\Big) \\
&= V(u, \delta_n J x_n + (1-\delta_n) J w_n) \\
&= \|x_n\|^2 - 2\langle x_n, \delta_n J x_n + (1-\delta_n) J w_n\rangle + \|\delta_n J x_n + (1-\delta_n) J w_n\|^2 \\
&\le \|x_n\|^2 - 2\delta_n \langle x_n, J x_n\rangle - 2(1-\delta_n)\langle x_n, J w_n\rangle \\
&\quad + \delta_n \|J x_n\|^2 + (1-\delta_n)\|J w_n\|^2 \\
&= (\delta_n + (1-\delta_n))\|x_n\|^2 - 2\delta_n \langle x_n, J x_n\rangle - 2(1-\delta_n)\langle x_n, J w_n\rangle \\
&\quad + \delta_n \|J x_n\|^2 + (1-\delta_n)\|J w_n\|^2 \\
&= \delta_n \|x_n\|^2 + (1-\delta_n)\|x_n\|^2 - 2\delta_n \langle x_n, J x_n\rangle - 2(1-\delta_n)\langle x_n, J w_n\rangle \\
&\quad + \delta_n \|J x_n\|^2 + (1-\delta_n)\|J w_n\|^2 \\
&= \delta_n \phi(x_n, x_n) + (1-\delta_n)\phi(x_n, w_n) \\
&\le \phi(x_n, w_n) \\
&\longrightarrow 0 \quad (\text{as } n \longrightarrow \infty).
\end{aligned} \tag{3.29}$$

Applying Lemma 2.4, we get

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.30}$$

Hence, $y_n \to q$ as $n \to \infty$.

In addition,

$$\begin{aligned}
\|S y_n - y_n\| &= \|S y_n - x_n + x_n - y_n\| \\
&\le \|S y_n - x_n\| + \|x_n - y_n\|.
\end{aligned} \tag{3.31}$$

It follows from (3.21), (3.30), and (3.31) that

$$\lim_{n\to\infty} \|S y_n - y_n\| = 0. \tag{3.32}$$

From (3.20), (3.32) and by the closedness of T and S, we get $q \in F(T) \cap F(S)$.

Step 6. Show that $x_n \to q \in EP(f)$.

From (3.16), we have

$$\phi(u, z_n) \le \phi(u, x_n). \tag{3.33}$$

Note that $u_n = T_{r_n} z_n$. From (3.33) and Lemma 2.11, we have

$$\begin{aligned}\phi(u_n, z_n) = \phi(T_{r_n} z_n, z_n) &\le \phi(u, z_n) - \phi(u, T_{r_n} z_n) \\ &\le \phi(u, x_n) - \phi(u, T_{r_n} z_n) \\ &= \phi(u, x_n) - \phi(u, u_n).\end{aligned} \tag{3.34}$$

Using (3.24), we have $\lim_{n\to\infty}\phi(u_n, z_n) = 0$. By Lemma 2.4, we obtain

$$\lim_{n\to\infty} \|u_n - z_n\| = 0. \tag{3.35}$$

Since $r_n \ge s$, we have

$$\frac{\|Ju_n - Jz_n\|}{r_n} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.36}$$

From $u_n = T_{r_n} z_n$, we have

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jz_n\rangle \ge 0, \quad \forall y \in C. \tag{3.37}$$

By using (A2), we have

$$\begin{aligned}\|y - u_n\| \frac{\|Ju_n - Jz_n\|}{r_n} &\ge \frac{1}{r_n}\langle y - u_n, Ju_n - Jz_n\rangle \\ &\ge -f(u_n, y) \ge f(y, u_n), \quad \forall y \in C.\end{aligned} \tag{3.38}$$

From (A4) and $u_n \to q$, we get $f(y, q) \le 0$ for all $y \in C$. For t with $0 < t \le 1$ and $y \in C$, let $y_t = ty + (1-t)q$. Since $y \in C$ and $q \in C$, we have $y_t \in C$ and hence $f(y_t, q) \le 0$. So from (A1), we have $0 = f(y_t, y_t) \le tf(y_t, y) + (1-t)f(y_t, q) \le tf(y_t, y)$. That is, $f(y_t, y) \ge 0$. It follows from (A3) that $f(q, y) \ge 0$ for all $y \in C$ and hence $q \in EP(f)$.

Step 7. Show that $x_n \to q \in VI(C, A)$.

Define $T_e \subset E \times E^*$ be as in (2.8), which yields that T_e is maximal monotone and $T_e^{-1}0 = VI(C, A)$. Let $(v, w) \in G(T_e)$. Since $w \in T_e v = Av + N_C(v)$, we get $w - Av \in N_C(v)$. From $w_n \in C$, we have

$$\langle v - w_n, w - Av\rangle \ge 0. \tag{3.39}$$

On the other hand, from $w_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)$ and Lemma 2.5, we have $\langle v - w_n, Jw_n - (Jx_n - \lambda_n Ax_n)\rangle \ge 0$, and hence

$$\left\langle v - w_n, \frac{Jx_n - Jw_n}{\lambda_n} - Ax_n \right\rangle \le 0. \tag{3.40}$$

It follows from (3.39) and (3.40) that

$$\begin{aligned}
\langle v - w_n, w\rangle &\geq \langle v - w_n, Av\rangle \\
&\geq \langle v - w_n, Av\rangle + \left\langle v - w_n, \frac{Jx_n - Jw_n}{\lambda_n} - Ax_n \right\rangle \\
&= \langle v - w_n, Av - Ax_n\rangle + \left\langle v - w_n, \frac{Jx_n - Jw_n}{\lambda_n} \right\rangle \\
&= \langle v - w_n, Av - Aw_n\rangle + \langle v - w_n, Aw_n - Ax_n\rangle \\
&\quad + \left\langle v - w_n, \frac{Jx_n - Jw_n}{\lambda_n} \right\rangle \\
&\geq -\|v - w_n\| \frac{\|w_n - x_n\|}{\alpha} - \|v - w_n\| \frac{\|Jx_n - Jw_n\|}{a} \\
&\geq -M\left(\frac{\|w_n - x_n\|}{\alpha} + \frac{\|Jx_n - Jw_n\|}{a} \right),
\end{aligned} \tag{3.41}$$

where $M = \sup_{n\geq 1}\{\|v - w_n\|\}$. By taking the limit as $n \to \infty$ and from (3.27) and (3.28), we obtain $\langle v - q, w\rangle \geq 0$. By the maximality of T_e, we have $q \in T_e^{-1}0$ and hence $q \in VI(C, A)$. That is, $q \in F$.

Step 8. Show that $q = \Pi_F x_0$.

From $x_n = \Pi_{C_n} x_0$, we have

$$\langle x_n - z, Jx_0 - Jx_n\rangle \geq 0, \quad \forall z \in C_n. \tag{3.42}$$

Since $F \subset C_n$, we also have

$$\langle x_n - u, Jx_0 - Jx_n\rangle \geq 0, \quad \forall u \in F. \tag{3.43}$$

By taking the limit in (3.43), we obtain that

$$\langle q - u, Jx_0 - Jq\rangle \geq 0, \quad \forall u \in F. \tag{3.44}$$

By Lemma 2.5, we can conclude that $q = \Pi_F x_0$. From (3.14), we have $\lim_{n\to\infty}\|u_n - x_n\| = 0$, and it follows that $\lim_{n\to\infty}\|u_n - q\| = 0$. This completes the proof. □

Finally, we prove two strong convergence theorems in a 2-uniformly convex, uniformly smooth Banach space by using Theorem 3.1.

First, we consider the problem of finding a zero point of an inverse-strongly-monotone operator of E into E^*. Assume that A satisfies the conditions:

(D1) A is α-inverse-strongly monotone,

(D2) $A^{-1}(0) = \{u \in E : \ Au = 0\} \neq \emptyset$.

Theorem 3.2. *Let E be a 2-uniformly convex, uniformly smooth Banach space, f a bifunction from $E \times E$ to $\mathbb{R}$ which satisfies (A1)–(A4), A an operator of E into E^* satisfying (D1)–(D2), and T, S two closed relatively quasi-nonexpansive mappings from E into itself such that $F := F(T) \cap F(S) \cap EP(f) \cap A^{-1}(0) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = E$, define sequences $\{x_n\}$ and $\{u_n\}$ as follows:*

$$
\begin{gathered}
w_n = J^{-1}(Jx_n - \lambda_n A x_n), \\
y_n = J^{-1}(\delta_n J x_n + (1-\delta_n) J w_n), \\
z_n = J^{-1}(\alpha_n J x_n + \beta_n J T x_n + \gamma_n J S y_n), \\
u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J z_n\rangle \geq 0, \quad \forall y \in E, \\
C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_0
\end{gathered}
\tag{3.45}
$$

for every $n \in \mathbb{N}$, where J is the duality mapping on E. Suppose that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, and $\{\delta_n\}$ are sequences in $[0,1]$ satisfying conditions (B1)–(B4) of Theorem 3.1.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $q = \Pi_F x_0$.

Proof. Putting $C = E$ in Theorem 3.1, we have $\Pi_E = I$. We also have $VI(E, A) = A^{-1}(0)$ and then condition (C3) of Theorem 3.1 holds for all $y \in E$ and $u \in A^{-1}(0)$. So, we obtain the desired result. □

Next, let K be a nonempty closed convex cone in E and A an operator of K into E^*. We define its *polar* in E^* to be the set

$$
K^* = \{y^* \in E^* : \langle x, y^*\rangle \geq 0 \quad \forall x \in K\}.
\tag{3.46}
$$

Then the element $u \in K$ is called a solution of the *complementarity problem* if

$$
Au \in K^*, \qquad \langle u, Au\rangle = 0.
\tag{3.47}
$$

The set of solutions of the complementarity problem is denoted by $C(K, A)$.

Assume that A is an operator satisfying the conditions:

(E1) A is α-inverse-strongly-monotone,

(E2) $C(K, A) \neq \emptyset$,

(E3) $\|Ay\| \leq \|Ay - Au\|$ for all $y \in K$ and $u \in C(K, A)$.

Theorem 3.3. *Let E be a 2-uniformly convex, uniformly smooth Banach space, K a nonempty closed convex cone in E, f a bifunction from $K \times K$ to $\mathbb{R}$ which satisfies (A1)–(A4), A an operator of K into E^* satisfying (E1)–(E3), and T, S two closed relatively quasi-nonexpansive mappings from K into itself* *such that* $F := F(T) \cap F(S) \cap EP(f) \cap C(K, A) \neq \emptyset$. *For an initial point* $x_0 \in E$ *with* $x_1 = \Pi_{C_1} x_0$ *and* $C_1 = K$, *define sequences* $\{x_n\}$ *and* $\{u_n\}$ *as follows:*

$$\begin{aligned}
w_n &= \Pi_K J^{-1}(Jx_n - \lambda_n A x_n), \\
y_n &= J^{-1}(\delta_n J x_n + (1-\delta_n) J w_n), \\
z_n &= J^{-1}(\alpha_n J x_n + \beta_n J T x_n + \gamma_n J S y_n), \\
u_n \in C \text{ such that } & f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J z_n\rangle \geq 0, \quad \forall y \in K, \\
C_{n+1} &= \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\
x_{n+1} &= \Pi_{C_{n+1}} x_0
\end{aligned} \tag{3.48}$$

for every $n \in \mathbb{N}$, *where* J *is the duality mapping on* E. *Suppose that* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, *and* $\{\delta_n\}$ *are sequences in* $[0, 1]$ *satisfying conditions (B1)–(B4) of Theorem 3.1.*

Then, $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $q = \Pi_F x_0$.

Proof. From [24, Lemma 7.1.1], we have $VI(K, A) = C(K, A)$. So by Theorem 3.1, we obtain the desired result. □

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