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Research Article

q-Analogue of Wright Function

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We introduce a q-analogues of Wright function and its auxiliary functions as Barnes integral representations and series expansion. The relations between q-analogues of Wright function and some other functions are investigated.

1. Introduction

In the second half of the past century, the discoveries of new special functions and applications of special functions to new areas of mathematics have initiated a resurgence of interest in this field [1]. These discoveries include work in combinatories initiated by Schutzenberger and Foata. Moreover, in recent years, particular cases of long familiar special functions have been clearly defined and applied to orthogonal polynomials [1].

The Wright function is one of the special functions, which plays an important role in the solution of fractional differential equations. The Wright function is defined by the series representation for all complex variable z as [2]:

$$W_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{n!\Gamma(\alpha n+\beta)},\quad \alpha>-1,\ \beta\in C. \tag{1.1}$$

The Barnes integral representation of Wright function is defined by

$$W_{\alpha,\beta}(z)=\frac{1}{2\pi i}\int_D\frac{\Gamma(-s)}{\Gamma(\beta+\alpha s)}(-z)^s ds, \tag{1.2}$$

where D is a contour in the complex s-plane which runs from $s = -i\infty$ to $s = i\infty$, so that the points $s = n,\ n = 0, 1, 2, \ldots$ lie to the right of D. There are two auxiliary functions of Wright function defined as

$$M_\alpha(z) = W_{-\alpha,1-\alpha}(-z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n!\Gamma(1-\alpha(n+1))}, \quad 0 < \alpha < 1,$$
$$F_\alpha(z) = W_{-\alpha,0}(-z) = \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n!\Gamma(-\alpha n)}, \quad 0 < \alpha < 1. \tag{1.3}$$

Here, we listed some of special cases of Wright function and its auxiliary functions [2]:

$$\begin{gathered}
W_{0,1}(z) = e^z, \\
W_{-1/2,1}(z) = \operatorname{Erfc}\left(-\frac{z}{2}\right), \\
\left(\frac{z}{2}\right)^\nu W_{1,1+\nu}\left(-\frac{z^2}{4}\right) = J_\nu(z), \\
\left(\frac{z}{2}\right)^\nu W_{1,1+\nu}\left(\frac{z^2}{4}\right) = I_\nu(z), \\
W_{-1,\beta}(z) = \frac{1}{\Gamma(\beta)}(1+z)^{\beta-1}, \quad \beta \neq 0, \ -1, -2, \ldots, |z| < 1, \\
F_\alpha(z) = \alpha z M_\alpha(z), \\
M_{1/2}(z) = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{z^2}{4}\right), \\
M_{1/3}(z) = 3^{2/3} A\left(\frac{z}{3^{1/3}}\right),
\end{gathered} \tag{1.4}$$

where Erfc (z) is the complementary error function, $J_\nu(z),\ I_\nu(z)$ are the Bessel and modified Bessel functions, and $A(z)$ is Airy's function, defined as

$$A(z) = \frac{\sqrt{z}}{3}\left[I_{-1/3}\left(\frac{2}{3}z^{3/2}\right) - I_{1/3}\left(\frac{2}{3}z^{3/2}\right)\right]. \tag{1.5}$$

2. Preliminaries

2.1. q-hypergeometric series

A q-hypergeometric series is a power series in one complex variable z with power series coefficients which depend, apart from q, on r complex upper parameters $a_1, a_2, \ldots, a_r$ and s

complex lower parameters $b_1, b_2, \ldots, b_s$ as follows [3]:

$$\begin{aligned} {}_r\phi_s\left[{}^{a_1,a_2,\ldots,a_r}_{b_1,b_2,\ldots,b_s};q,z\right] &= {}_r\phi_s\left(a_1,a_2,\ldots,a_r;b_1,b_2,\ldots,b_s;q,z\right), \\ &:= \sum_{n=0}^{\infty}\frac{\left(a_1,a_2,\ldots,a_r,q\right)_n}{\left(b_1,b_2,\ldots,b_s,q\right)_n}\left((-1)^n q^{\binom{n}{2}}\right)^{s-r+1} z^k, \end{aligned} \tag{2.1}$$

where $q \neq 0$ when $r > s+1$.

2.2. q-exponential series

The q-analogues of the exponential functions are given by [1, 3]

$$\begin{aligned} E_q^z &:= {}_0\phi_0\left(-;-;q,-(1-q)z\right) = \sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}z^n}{[n]_q!} = \left(-(1-q)z,q\right)_\infty, \\ e_q^z &:= {}_1\phi_0\left(0;-;q,(1-q)z\right) = \sum_{n=0}^{\infty}\frac{z^n}{[n]_q!} = \frac{1}{\left((1-q)z,q\right)_\infty}. \end{aligned} \tag{2.2}$$

2.3. Jackson q-derivative

In 1908, Jackson reintroduced and started a systematic study of the q-difference operator [1, 3]:

$$\left(D_q f\right)(x) := \frac{f(x)-f(qx)}{(1-q)x}, \quad q \neq 1,\ x \neq 0, \tag{2.3}$$

which is now sometimes referred to as Euler-Jackson , Jackson q-difference operator, or simply the q-derivative. By definition, the limit as q-approaches to 1 is the ordinary derivative, that is,

$$\lim_{q\to 1}\left(D_q f\right)(x) = \frac{df}{dx}(x), \tag{2.4}$$

if f is differentiable at x. The two exponential functions have the q-derivative:

$$\begin{aligned} D_q E_q^{ax} &= a E_q^{qax}, \\ D_q e_q^{ax} &= a e_q^{ax}. \end{aligned} \tag{2.5}$$

2.4. Jackson q-integrals

Thomae (1869) and Jackson (1910) introduced the q-integral defined in [3]

$$\int_0^a f(x)\,d_q x := a(1-q)\sum_{n=0}^{\infty} f\left(aq^n\right)q^n, \tag{2.6}$$

Jackson also defined an integral from 0 to ∞ by

$$\int_0^{\infty} f(x)d_qx := (1-q)\sum_{n=-\infty}^{\infty} f(q^n)q^n, \tag{2.7}$$

provided the sums converge absolutely.

The q-Jackson integral in a generic interval $[a,b]$ is given by

$$\int_a^b f(x)d_qx := \int_0^b f(x)d_qx - \int_0^a f(x)d_qx. \tag{2.8}$$

The q-integration by parts is given for suitable functions f and g by [3]

$$\int_a^b f(x)D_qg(x)d_qx := f(b)g(b) - f(a)g(a) - \int_a^b g(qx)D_qf(x)d_qx. \tag{2.9}$$

2.5. q-Gamma function

Jackson (1910) defined the q-analogue of the gamma function by

$$\Gamma_q(x) := \frac{(q,q)_\infty}{(q^x,q)_\infty}(1-q)^{1-x}, \quad 0<q<1. \tag{2.10}$$

Moreover, it has the q-integral representations [1, 3]:

$$\Gamma_q(x) := \int_0^{1/1-q} t^{x-1}E_q^{-qt}d_qt, \quad 0<q<1,\ \Re(x)>0. \tag{2.11}$$

Let k denote a positive integer. In the same spirit we define

$$\begin{aligned}\Gamma_{q^k}(x) &:= \frac{(q^k,q^k)_\infty}{(q^{kx},q^k)_\infty}(1-q^k)^{1-x}, \quad 0<q<1,\ x\neq 0,\ -1,-2,\ldots,\\ &:= \int_0^{1/1-q^k} t^{x-1}E_{q^k}^{-q^kt}d_{q^k}t, \quad 0<q<1,\ \Re(x)>0.\end{aligned} \tag{2.12}$$

It is obvious from (2.10) that $\Gamma_q(z)$ has simple poles at $z=0,-1,-2,\ldots$. The residue at $z=-n$ is [3]

$$\lim_{z\to -n}(z+n)\Gamma_q(z) = \frac{(-1)^{n+1}q^{n(n+1)/2}(1-q)^{n+1}}{(q,q)_n\log q}. \tag{2.13}$$

A q-analogue of Legendre's duplication formula can be easily derived by

$$\Gamma_q(2z)\Gamma_{q^2}\left(\frac{1}{2}\right) = (1+q)^{2z-1}\Gamma_{q^2}(z)\Gamma_{q^2}\left(z+\frac{1}{2}\right). \tag{2.14}$$

Similarly, it can be shown that Gauss multiplication formula has a q-analogue of the form

$$\Gamma_q(kz)\Gamma_{q^k}\left(\frac{1}{k}\right)\Gamma_{q^k}\left(\frac{2}{k}\right)\cdots\Gamma_{q^k}\left(\frac{k-1}{k}\right) = \left(1+q+q^2+\cdots+q^{k-1}\right)^{kz-1}\Gamma_{q^k}(z)\Gamma_{q^k}\left(z+\frac{1}{k}\right)\Gamma_{q^k}\left(z+\frac{2}{k}\right)\cdots\Gamma_{q^k}\left(z+\frac{k-1}{k}\right). \tag{2.15}$$

Lemma 2.1. *Let k be a positive integer and n a nonnegative integer, then one has*

$$\begin{aligned}
\Gamma_{q^k}\left(\frac{l}{k}+n\right) &= \frac{(q^l,q^k)_n}{(1-q^k)^n}\Gamma_{q^k}\left(\frac{l}{k}\right) \quad l=1,2,\dots,\\
\Gamma_{q^k}\left(\frac{l}{k}-n\right) &= \frac{(1-q^k)^n q^{(n/2)(kn+k-2l)}\Gamma_{q^k}(l/k)}{(-1)^n(q^{k-l},q^k)_n}, \quad l=1,2,\dots,\ k\neq l,\\
\Gamma_q(kn+l) &= \frac{(1+q+q^2+\cdots+q^{k-1})^{kn+l-1}(q^l,q^k)_n(q^{l+1},q^k)_n\cdots(q^{k+l-1},q^k)_n}{\Gamma_{q^k}(l/k)\Gamma_{q^k}(2/k)\cdots\Gamma_{q^k}((k-l)/k)(1-q^k)^{nk}}\\
&\quad\times\Gamma_{q^k}\left(\frac{l}{k}\right)\Gamma_{q^k}\left(\frac{l+1}{k}\right)\cdots\Gamma_{q^k}\left(\frac{k+l}{k}\right), \quad l=1,2,\dots.
\end{aligned} \tag{2.16}$$

The proof of this lemma follows from the definition of q-analogue of gamma function.

2.6. Jackson's q-Bessel function

Jackson introduced in 1905 the following q-analogues of the Bessel functions [3]:

$$\begin{aligned}
J_\nu^{(1)}(z;q) &:= \frac{(q^{\nu+1},q)_\infty}{(q,q)_\infty}\left(\frac{z}{2}\right)^\nu {}_2\phi_1\left(0,0;q^{\nu+1};q,-\frac{z^2}{4}\right), \quad |z|<2,\\
J_\nu^{(2)}(z;q) &:= \frac{(q^{\nu+1},q)_\infty}{(q,q)_\infty}\left(\frac{z}{2}\right)^\nu {}_0\phi_1\left(-;q^{\nu+1};q,-q^{\nu+1}\frac{z^2}{4}\right),\\
J_\nu^{(3)}(z;q) &:= \frac{(q^{\nu+1},q)_\infty}{(q,q)_\infty}\left(\frac{z}{2}\right)^\nu {}_1\phi_1\left(0;q^{\nu+1};q,\frac{qz^2}{4}\right).
\end{aligned} \tag{2.17}$$

3. The q-analogue of error functions

Definition 3.1. One defines the q-analogues of error function and complementary error function, respectively, as

$$\mathrm{Erf}_q(x) = \frac{1+q}{\Gamma_{q^2}(1/2)}\int_0^x E_{q^2}^{-q^2t^2}\,d_qt, \tag{3.1}$$

$$\mathrm{Erfc}_q(x) = \frac{1+q}{\Gamma_{q^2}(1/2)}\int_x^{1/\sqrt{1-q^2}} E_{q^2}^{-q^2t^2}\,d_qt. \tag{3.2}$$

Remark 3.2. If $x \to 1/\sqrt{1-q^2}$ or 0 in the above definitions, respectively, then we have

$$\operatorname{Erf}_q\left(\frac{1}{\sqrt{1-q^2}}\right) = \operatorname{Erfc}_q(0) = 1, \tag{3.3}$$

and we can deduce that

$$\operatorname{Erf}_q(x) = 1 - \operatorname{Erfc}_q(x). \tag{3.4}$$

Proof.

$$\begin{aligned}
\operatorname{Erf}_q\left(\frac{1}{\sqrt{1-q^2}}\right) &= \frac{1+q}{\Gamma_{q^2}(1/2)}\int_0^{1/\sqrt{1-q^2}} E_{q^2}^{-q^2t^2} d_q t \\
&= \frac{1+q}{\Gamma_{q^2}(1/2)} \frac{1-q}{\sqrt{1-q^2}} \sum_{n=0}^{\infty} q^n (q^{2n+2}, q^2)_\infty \\
&= \frac{\sqrt{1-q^2}}{\Gamma_{q^2}(1/2)} (q^2, q^2)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q^2, q^2)_n} \\
&= \frac{\sqrt{1-q^2}}{\Gamma_{q^2}(1/2)} \frac{(q^2, q^2)_\infty}{(q, q^2)_\infty} = 1.
\end{aligned} \tag{3.5}$$

The series representations of the q-error function are as follows:

$$\begin{aligned}
\operatorname{erf}_q(x) &= \frac{1+q}{\Gamma_{q^2}(1/2)} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)} x^{2n+1}}{[n]_{q^2}![2n+1]_q} \\
&= \frac{1+q}{\Gamma_{q^2}(1/2)} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)} (q, q^2)_n (1-q^2)^n x^{2n+1}}{(q^2, q^2)_n (q^3, q^2)_n} \\
&= \frac{(1+q)x}{\Gamma_{q^2}(1/2)} {}_1\phi_1(q; q^3; q^2, q^2(1-q^2)x^2).
\end{aligned} \tag{3.6}$$

□

4. q-analogue of Wright function

In this section, we introduce a definition of a q-analogue of Wright function ($W_{\alpha,\beta}(z, q^k)$) as a Barnes integral representations.

Definition 4.1. According to standard notation, one defines a q-analogue of Wright function as

$$W_{\alpha,\beta}(z, q^k) = \frac{\log q}{2\pi i(q-1)} \int_C \frac{\Gamma_q(-s)}{\Gamma_{q^k}(\beta + \alpha s)} (-z)^s ds, \tag{4.1}$$

where $k = 1, 2, \ldots, 0 < q < 1$, z is not equal to zero and

$$(-z)^s = \exp\left[s\left(\log|z| + i\arg(-z)\right)\right], \tag{4.2}$$

where C is a suitable path in the complex s-plane that runs from $s = -i\infty$ to $s = i\infty$, so the points $s = n$, $n = 0, 1, 2, \ldots$, lie to the right of the contour C.

4.1. Existence and representation of q-Wright function

Firstly, we rewrite the definition of q-Wright function as

$$W_{\alpha,\beta}(z, q^k) = \frac{(q,q)_\infty (1-q)}{(q^k,q^k)_\infty (1-q^k)^{1-\beta}} \frac{\log q}{2\pi i(q-1)} \int_C \frac{(q^{k\beta+ks\alpha}, q^k)_\infty}{(q^{-s},q)_\infty} \frac{(1-q^k)^{s\alpha}}{(1-q)^{-s}} (-z)^s ds. \tag{4.3}$$

Next, we consider a q-analogue of Wright function in the case that $0 < q < 1$. Let $q = e^{-\omega}$ ($\omega > 0$); using the triangle inequality, we get

$$\begin{gathered} \left|1 - |a| e^{-\omega \Re(s)}\right| \le \left|1 - a q^s\right| \le 1 + |a| e^{-\omega \Re(s)}, \\ \left|\frac{(q^{k\beta+ks\alpha}, q^k)_\infty}{(q^{-s},q)_\infty}\right| \le \prod_{r=0}^{\infty} \frac{1 + e^{-\pi[r+k(\Re(\beta)+\alpha\Re(s))]\omega}}{1 - e^{-\pi[r-\Re(s)]\omega}}, \end{gathered} \tag{4.4}$$

which is bounded on the contour C.

Theorem 4.2. *Let k be a positive integer, and let β be a complex number, then the q-Wright function is absolutely convergent for all complex variables z; if $\alpha > -\log(1-q)/\log(1-q^k)$ and if $\alpha = -\log(1-q)/\log(1-q^k)$, then $|z| < 1$.*

Proof. Consider the integral in (4.1) with the contour C replaced by the contour C_R consisting of a large clockwise-oriented semicircle of radius R and the center of the origin which lies to the right of the contour C is bounded away from the poles.

Let $s = \mathrm{R}e^{i\theta}$, then we have

$$\left|\frac{\Gamma_q(-s)}{\Gamma_{q^k}(\beta+\alpha s)}(-z)^s\right| = AB e^{R\cos\theta(\log(1-q)+\alpha\log(1-q^k))} |z|^{R\cos\theta} e^{-R\sin\theta\arg(-z)}, \tag{4.5}$$

where

$$A = \frac{(q,q)_\infty (1-q)}{(q^k,q^k)_\infty (1-q^k)^{1-\Re(\beta)}}, \qquad B = \left|\frac{(q^{k\beta+ks\alpha}, q^k)_\infty}{(q^{-s},q)_\infty}\right|, \tag{4.6}$$

as $R \to \infty$ on C_R, it follows that from (4.5) the integral (4.1) with C replaced by C_R tends to zero as $R \to \infty$ if and only if $\alpha > -\log(1-q)/\log(1-q^k)$ for all complex variable z and if $\alpha = -\log(1-q)/\log(1-q^k)$, then $|z| < 1$. □

Theorem 4.3 (explicit power series expansion). *Let k be a positive integer, let β be a complex number, and let either $\alpha > -\log(1-q)/\log(1-q^k)$ and $z \neq 0$ or $\alpha = -\log(1-q)/\log(1-q^k)$*

and $|z| < 1$. *Then the* q*-Wright function* (4.1) *has the power series expansion*

$$W_{\alpha,\beta}(z,q^k) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2} z^n}{[n]_q! \Gamma_{q^k}(\alpha n + \beta)}. \tag{4.7}$$

Proof. From the existence theorem after replacing s by $-s$, we can apply Cauchy's theorem (residues theorem) to the closed contour which is consisting of the contour C_R and that part of C terminated above and below by C_R as $R \to \infty$, we obtain that the q-analogue of Wright function (4.1) equals the sum of the residues of the integrand at $s = -n,\ n = 0, 1, 2, \ldots$. This completes the proof. □

Remark 4.4. If $q \to 1$, then $\alpha = -\log(1-q)/\log(1-q^k) \to -1$, and the q-Wright function (4.7) tends to the classical case (1.1).

Definition 4.5 (the auxiliary functions of q-Wright function). We introduce two (q-Wright-type) auxiliary functions $M_\alpha(z,q^k) = W_{-\alpha,1-\alpha}(-z,q^k)$ and $F_\alpha(z,q^k) = W_{-\alpha,0}(-z,q^k)$ with $0 < \alpha < \log(1-q)/\log(1-q^k)$. The two functions can be define for any order $\alpha \in (0, \log(1-q)/\log(1-q^k))$ and for all complex variable $z \neq 0$ by

$$M_\alpha(z,q^k) = W_{-\alpha,1-\alpha}(-z,q^k) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2} z^n}{[n]_q! \Gamma_{q^k}(1-\alpha(n+1))}, \tag{4.8}$$

$$F_\alpha(z,q^k) = W_{-\alpha,0}(-z,q^k) = \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2} z^n}{[n]_q! \Gamma_{q^k}(-\alpha n)}. \tag{4.9}$$

Remark 4.6. An important relationship between auxiliary functions of q-Wright function as

$$F_\alpha(z,q^k) = \frac{z(1-q)q^{1-\alpha k}}{1-q^k} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2} (q^{1-\alpha k} z)^n}{[n]_q! \Gamma_{q^k}(1-\alpha(n+1))} \frac{1-q^{\alpha k(n+1)}}{1-q^{n+1}}; \tag{4.10}$$

when $\alpha k = 1$, we get

$$F_{1/k}(z,q^k) = \frac{z}{1+q+\cdots+q^{k-1}} M_{1/k}(z,q^k). \tag{4.11}$$

5. Relation with some known special functions

It follows from the definition of the q-analogue of Wright function as a series expansion (4.7) that

$$W_{0,1}(z,q^k) = E_q^{qz}. \tag{5.1}$$

The Jackson's third q-Bessel function and modified third q-Bessel function can be expressed in terms of q-Wright function as

$$\begin{aligned} \left(\frac{z}{2}\right)^{\nu} W_{1,\nu+1}\left(-\frac{z^2}{4},q\right) &= J_{\nu}^{(3)}(z(1-q),q), \\ \left(\frac{z}{2}\right)^{\nu} W_{1,\nu+1}\left(\frac{z^2}{4},q\right) &= I_{\nu}^{(3)}(z(1-q),q) \end{aligned} \tag{5.2}$$

The q-error function complement can also be expressed a particular case of q-Wright function as

$$W_{-1/2,1}(z,q^2) = \operatorname{Erfc}_q\left(-\frac{qz}{1+q}\right), \tag{5.3}$$

where $\operatorname{Erfc}_q(z)$ denotes the q-error function complement which is defined as in (3.2). To prove this formula, we use the definition of q-Wright function (4.7) and the identities of the q-gamma function:

$$\begin{aligned} W_{-1/2,1}(z,q^2) &= \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2} z^n}{[n]_q! \Gamma_{q^2}(1-n/2)} \\ &= 1 + \sum_{n=0}^{\infty} \frac{q^{(2n+1)(n+1)} z^{2n+1}}{[2n+1]_q! \Gamma_{q^2}(1/2-n)} \\ &= 1 - \frac{1+q}{\Gamma_{q^2}(1/2)} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)}}{[n]_{q^2}! [2n+1]_q} \left(-\frac{qz}{1+q}\right)^{2n+1} \\ &= 1 - \operatorname{Erf}_q\left(-\frac{qz}{1+q}\right) = \operatorname{Erfc}_q\left(-\frac{qz}{1+q}\right). \end{aligned} \tag{5.4}$$

Taking $\alpha = -1$ and $k = 1$ in the definition of q-Wright function, then we obtain

$$\begin{aligned} W_{-1,\beta}(z,q) &= \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2} z^n}{[n]_q! \Gamma_q(\beta-n)} \\ &= \frac{1}{\Gamma_q(\beta)} \sum_{n=0}^{\infty} \frac{(-1)^n z^n (q^{1-\beta},q)_n}{(q,q)_n} \\ &= \frac{1}{\Gamma_q(\beta)} \frac{(-zq^{1-\beta},q)_\infty}{(-z,q)_\infty}, \quad |z|<1. \end{aligned} \tag{5.5}$$

Explicit expressions of $F_{1/k}(z,q^k)$ and $M_{1/k}(z,q^k)$ in terms of known functions are expected for some particular values of $k \geq 2$. In the particular case $k = 2$, we find

$$\begin{aligned} M_{1/2}(z,q^2) &= \frac{1}{\Gamma_{q^2}(1/2)} E_{q^2}^{-q^2 z^2/(1+q)^2}, \\ F_{1/2}(z,q^2) &= \frac{z}{(1+q)\Gamma_{q^2}(1/2)} E_{q^2}^{-q^2 z^2/(1+q)^2}. \end{aligned} \tag{5.6}$$

To prove the first formula and the second formula, we use the relationship between them (4.11). Using the definition of q-auxiliary Wright function (4.8) and the identities of the q-gamma function (2.16), we obtain the following:

$$\begin{aligned}
M_{1/2}(z,q^2) &= \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2} z^n}{[n]_q! \Gamma_{q^2}((1-n)/2)} \\
&= \sum_{n=0}^{\infty} \frac{q^{n(2n+1)/2} z^n}{\Gamma_q(2n+1)\Gamma_{q^2}(1/2-n)} \\
&= \frac{1}{\Gamma_{q^2}(1/2)} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)}}{[n]_{q^2}!} \left(\frac{q^2 z^2}{(1+q)^2} \right)^n \\
&= \frac{1}{\Gamma_{q^2}(1/2)} E_{q^2}^{-q^2 z^2/(1+q)^2}.
\end{aligned} \tag{5.7}$$

In the case of $k = 3$, we can deduce that

$$M_{1/3}(z,q^3) = (1+q+q^2)^{2/3} A_q\left(\frac{z}{(1+q+q^2)^{1/3}} \right), \tag{5.8}$$

where $A_q(z)$ is the q-analogue of the Airy function which is defined as

$$A_q(z) = \frac{\sqrt{qz}}{1+q+q^2} \left[I_{-1/3}^{(2)}\left(\frac{2}{1+q+q^2}(1-q^3)(qz)^{3/2} \right) - I_{1/3}^{(2)}\left(\frac{2}{1+q+q^2}(1-q^3)(qz)^{3/2} \right) \right], \tag{5.9}$$

where $I_\nu^{(2)}(z)$ is Jackson's modified second q-Bessel. To prove this formula, we use the definition of q-Wright function and the identities of the gamma function (2.16):

$$\begin{aligned}
M_{1/3}(z,q^3) &= \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2} z^n}{[n]_q! \Gamma_{q^3}((2-n)/3)} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n q^{3n(3n+1)/2} z^{3n}}{[3n]_q! \Gamma_{q^3}(2/3-n)} - \sum_{n=0}^{\infty} \frac{(-1)^n q^{(3n+1)(3n+2)/2} z^{3n}}{[3n+1]_q! \Gamma_{q^3}(1/3-n)} \\
&= \frac{1}{\Gamma_{q^3}(2/3)} \sum_{n=0}^{\infty} \frac{q^{3n(n-1)+5n} z^{3n} (1-q)^{3n}}{(q^2,q^3)_n (q^3,q^3)_n (1-q^3)^n} \\
&\quad - \frac{qz}{(1+q+q^2)\Gamma_{q^3}(4/3)} \sum_{n=0}^{\infty} \frac{q^{3n(n-1)+7n} z^{3n} (1-q)^{3n}}{(q^4,q^3)_n (q^3,q^3)_n (1-q^3)^n} \\
&= \frac{1}{\Gamma_{q^3}(2/3)} {}_0\phi_1\left(q^2; q^3, \frac{q^5 z^3 (1-q)^3}{1-q^3} \right) - \frac{qz\, {}_0\phi_1\left(q^4; q^3, q^7 z^3 (1-q)^3/(1-q^3)\right)}{(1+q+q^2)\Gamma_{q^3}(4/3)} \\
&= \frac{\sqrt{qz}}{(1+q+q^2)^{1/2}} \left[I_{-1/3}^{(2)}\left(\frac{2(1-q)(qz)^{3/2}}{(1+q+q^2)^{1/2}}, q^3 \right) - I_{1/3}^{(2)}\left(\frac{2(1-q)(qz)^{3/2}}{(1+q+q^2)^{1/2}}, q^3 \right) \right] \\
&= (1+q+q^2)^{2/3} A_q\left(\frac{z}{(1+q+q^2)^{1/3}} \right);
\end{aligned} \tag{5.10}$$

when $q \to 1$, we cover the classical results about the Wright function and its auxiliary functions.

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